The derivative of an inverse function

f and g are inverse functions if

$$y = f(x) \iff x = g(y).$$

If either function is increasing or decreasing then so is the other, the domain of either is the other's range, and if one function is differentiable then so is the other, i.e.,

$$\frac{dy}{dx} = f'(x) \iff \frac{dx}{dy} = g'(y).$$

Note the extremely important point that $'$ denotes differentiation with respect to argument, i.e., with respect to x in the case of f but with respect to y in the case of g. Note also that it is traditional to denote an inverse relationship by writing $g = f^{-1} \Leftrightarrow f = g^{-1}$, though we shall largely refrain from doing so here.

Now, there exists a simple relationship between the derivatives of f and g, because $g(y) = x$ implies

$$g(f(x)) = x,$$

from which

$$\frac{d}{dx}\{g(f(x))\} = \frac{d}{dx}\{x\}$$

and hence

$$g'((f(x))\, f'(x) = 1$$

by the chain rule. So

$$g'(y)\, f'(x) = 1$$

or

$$\frac{dx}{dy}\frac{dy}{dx} = 1.$$

Therefore, if we know one derivative, then we know the other—but we mustn't forget that everything must ultimately be written solely in terms of the appropriate argument.

For example, suppose that $y = f(x) = \sqrt{x} = x^{1/2}$, implying $\frac{dy}{dx} = f'(x) = \frac{1}{2}x^{-1/2}$. Then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{1}{2}x^{-1/2}} = 2\sqrt{x} = 2y.$$

(Of course, it would have been far easier to observe that $y = f(x) = \sqrt{x}$ implies $x = g(y) = y^2$ and hence $\frac{dx}{dy} = 2y$, but that is largely beside the point.) Or suppose that $y = f(x) = e^x$, implying both that $x = g(y) = \ln(y)$ (by the definition of the natural logarithm as the inverse of the exponential function) and $\frac{dy}{dx} = f'(x) = e^x$ (from above). Then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{e^x} = \frac{1}{y}.$$

We have discovered a new result:

$$\frac{d}{dy}\ln(y) = \frac{1}{y}, \qquad y > 0$$

or—which is indubitably, inevitably, unavoidably, universally, exactly and precisely the very same thing—

$$\frac{d}{dx}\ln(x) \;=\; \frac{1}{x}, \qquad x > 0.$$

A brief digression. In writing the above result we naturally assume that $x > 0$: for negative x, $\ln(x)$ does not exist. On the other hand, for negative x, $\ln(-x)$ does exist, and we can use the chain rule with the substitution $u = -x$ (so that $u > 0$) to obtain

$$\frac{d}{dx}\ln(-x) \;=\; \frac{d}{dx}\ln(u) \;=\; \frac{d}{du}\ln(u)\,\frac{du}{dx} \;=\; \frac{1}{u}\frac{d}{dx}\{-x\} \;=\; \frac{1}{-x}\,(-1) \;=\; \frac{1}{x}, \qquad x < 0.$$

The last two results are often combined in the form

$$\frac{d}{dx}\ln|x| \;=\; \frac{1}{|x|}.$$

Now for a final example. Let us define the "hyperbolic sine and cosine" by

$$\sinh(x) \;=\; \tfrac{1}{2}e^{x} - \tfrac{1}{2}e^{-x}, \qquad \cosh(x) \;=\; \tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}$$

and note that

$$\cosh(x) + \sinh(x) \;=\; e^{x}, \qquad \cosh(x) - \sinh(x) \;=\; e^{-x}$$

so that

$$\cosh^2(x) - \sinh^2(x) \;=\; \{\cosh(x) + \sinh(x)\}\{\cosh(x) - \sinh(x)\} \;=\; e^{x}\,e^{-x} \;=\; 1.$$

Because $\frac{d}{dx}\{\sinh(x)\} = \frac{1}{2}\frac{d}{dx}\{e^{x}\} - \frac{1}{2}\frac{d}{dx}\{e^{-x}\} = \frac{1}{2}e^{x} - \frac{1}{2}\{-e^{-x}\} = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x} = \cosh(x)$ is strictly positive, the function sinh is increasing and therefore invertible. So if we define $y = f(x) = \sinh(x)$ then we also have $x = g(y) = f^{-1}(y) = \sinh^{-1}(y)$ and

$$\frac{dx}{dy} \;=\; \frac{1}{\frac{dy}{dx}} \;=\; \frac{1}{\cosh(x)} \;=\; \frac{1}{\sqrt{1+\sinh^2(x)}} \;=\; \frac{1}{\sqrt{1+y^2}}.$$

We have discovered another new result:

$$\frac{d}{dy}\sinh^{-1}(y) \;=\; \frac{1}{\sqrt{1+y^2}}$$

or—which of course is precisely the same—

$$\frac{d}{dx}\sinh^{-1}(x) \;=\; \frac{1}{\sqrt{1+x^2}}.$$