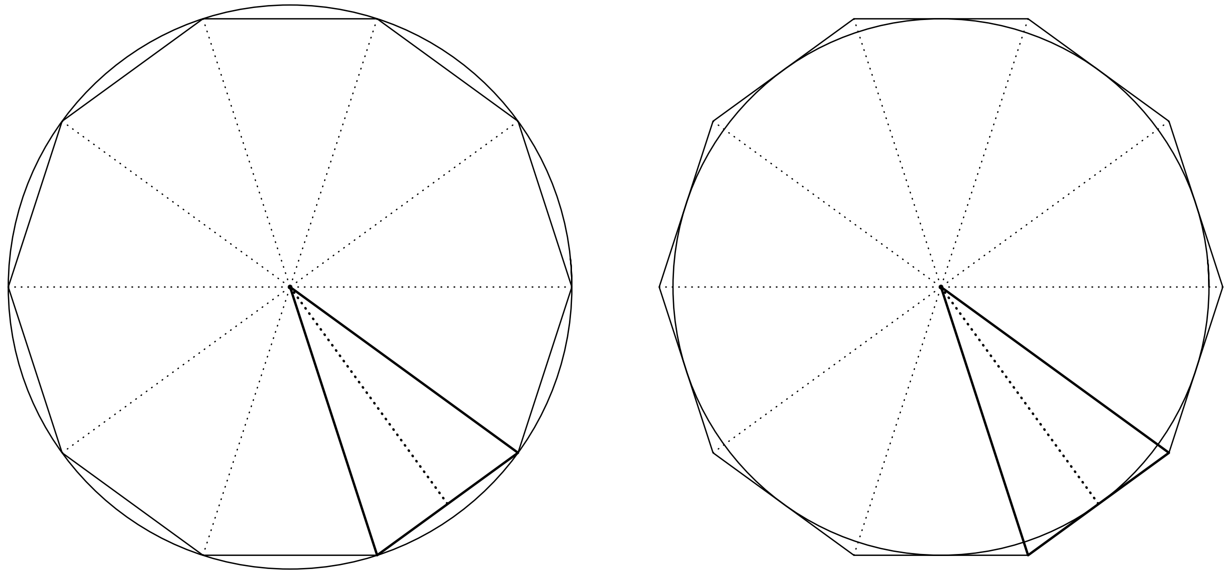
The Area of a Circle

You can underestimate the area of a circle by inscribing a regular n-sided polygon in it; for example, the figure on the left below shows $n = 10$. Let us call the underestimate U_n. The polygon consists of n isosceles triangles. Each such triangle has two sides of length r—the radius of the circle—and the angle between these two sides is $2\pi/n$ (why?). Thus each such triangle consists of two right-angled triangles, each of which has hypotenuse r, base $r\sin(\pi/n)$ and altitude $r\cos(\pi/n)$. Hence the area of each right-angled triangle is $\frac{1}{2} \cdot r\sin(\pi/n) \cdot r\cos(\pi/n)$; the area of each isosceles triangle is twice that amount; and there are n such triangles in all. Thus

$$U_n \;=\; n \cdot 2 \cdot \tfrac{1}{2} \cdot r\sin(\pi/n) \cdot r\cos(\pi/n) \;=\; \tfrac{1}{2} r^2 n \sin(2\pi/n),$$

on using the result that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. For example, if $r = 1$ then for $n = 10$ we have $U_{10} = 5\sin(\pi/5) \approx 2.939$.



Similarly, you can overestimate the area of a circle by inscribing it in a regular n-sided polygon; for example, the figure on the right above shows $n = 10$. Let us call the overestimate O_n. Again, the polygon consists of n isosceles triangles. Now, however, each such triangle has altitude r and base $2r\tan(\pi/n)$, and there are n such triangles. Thus

$$O_n \;=\; n \cdot 2 \cdot \tfrac{1}{2} \cdot r^2 \tan(\pi/n) \;=\; \frac{r^2 n \sin(2\pi/n)}{1 + \cos(2\pi/n)},$$

on using the result that $\cos(2\theta) = 2\cos^2(\theta) - 1$. For example, if $r = 1$ then for $n = 10$ we have $O_{10} = 10\sin(\pi/5)/\{1 + \cos(\pi/5)\} \approx 3.249$.

n	10	20	30	40	50	60	70	80	90	100
U_n	2.9389	3.0902	3.1187	3.1287	3.1333	3.1359	3.1374	3.1384	3.139	3.1395
O_n	3.2492	3.1677	3.1531	3.1481	3.1457	3.1445	3.1437	3.1432	3.1429	3.1426

If A is the area of the circle, then by definition of over- and under-estimate we have

$$U_n \;<\; A \;<\; O_n$$

for all values of n, no matter how large, as confirmed by the table above for $r = 1$. The bigger the value of n, the better the value of both our over- and our under-estimate, with A always sandwiched between. In the limit as $n \to \infty$, the above inequalities weaken, as U_n and O_n coalesce. That is, we have

$$\lim_{n\to\infty} U_n \;\le\; A \;\le\; \lim_{n\to\infty} O_n$$

and

$$\lim_{n\to\infty} U_n \;=\; \lim_{n\to\infty} O_n,$$

so that of necessity

$$\lim_{n\to\infty} U_n \;=\; A \;=\; \lim_{n\to\infty} O_n.$$

In other words, A is the limit as $n \to \infty$ of *either* the under- or the over-estimate.

From the underestimate, we have

$$\begin{aligned} A \;&=\; \lim_{n\to\infty} U_n \;=\; \lim_{n\to\infty} \tfrac{1}{2} r^2 n \sin(2\pi/n) \;=\; \lim_{n\to\infty} \pi r^2 \frac{\sin(2\pi/n)}{2\pi/n} \\ &=\; \pi r^2 \lim_{n\to\infty} \frac{\sin(2\pi/n)}{2\pi/n} \;=\; \pi r^2 \lim_{x\to 0+} \frac{\sin(x)}{x} \;=\; \pi r^2 \cdot 1 \;=\; \pi r^2, \end{aligned}$$

on using the substitution $x = 2\pi/n$. Equivalently, from the overestimate, we have

$$\begin{aligned} A \;&=\; \lim_{n\to\infty} O_n \;=\; \lim_{n\to\infty} \frac{r^2 n \sin(2\pi/n)}{1+\cos(2\pi/n)} \;=\; \lim_{n\to\infty} 2\pi r^2 \frac{\sin(2\pi/n)}{\{1+\cos(2\pi/n)\}\cdot 2\pi/n} \\ &=\; 2\pi r^2 \lim_{n\to\infty} \frac{\sin(2\pi/n)}{\{1+\cos(2\pi/n)\}\cdot 2\pi/n} \;=\; 2\pi r^2 \lim_{x\to 0+} \frac{\sin(x)}{\{1+\cos(x)\}x} \\ &=\; 2\pi r^2 \lim_{x\to 0+} \frac{\sin(x)}{x} \lim_{x\to 0+} \frac{1}{1+\cos(x)} \;=\; 2\pi r^2 \cdot 1 \cdot \frac{1}{1+\cos(0)} \;=\; \pi r^2 \end{aligned}$$

as before.*

*Or, if you prefer, use L'Hôpital's rule: $\lim_{x\to 0+} \frac{\sin(x)}{\{1+\cos(x)\}x} = \lim_{x\to 0+} \frac{\cos(x)}{-\sin(x)\cdot x + \{1+\cos(x)\}\cdot 1} = \frac{\cos(0)}{1+\cos(0)} = \frac{1}{2}$.