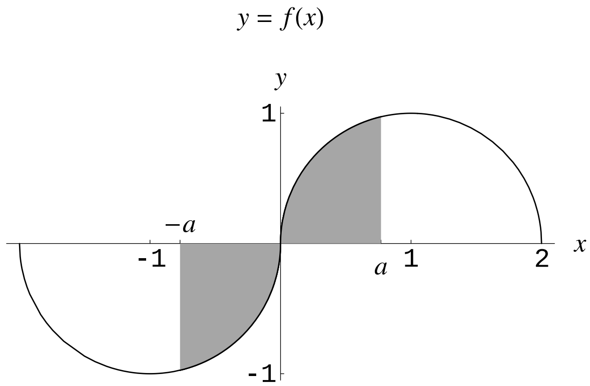
The Definite Integral

Intuitively, the definite integral of the (piecewise-continuous) function f over the interval $[a, b]$ is the signed area between the graph of f and segment $[a, b]$ of the horizontal axis, counted positively above that axis and negatively below it. Thus any definite integral depends on three things, namely, f, a and b: change any one of them, and (at least in general) you change the value of the integral. To emphasize this ternary dependence, we will denote the definite integral of f over the interval $[a, b]$ by $I(f, a, b)$, at least initially.



Suppose, for example, that f is the (continuous) join defined on $[-2, 2]$ by

$$f(x) \;=\; \begin{cases} -\sqrt{-x(2+x)} & \text{if} \quad -2 \le x < 0 \\ \sqrt{x(2-x)} & \text{if} \quad 0 \le x \le 2. \end{cases}$$

Then, because $y = \sqrt{-x(2+x)} \Longrightarrow (x+1)^2 + y^2 = 1^2$, which is the equation of a circle of radius 1 with center (-1,0), and because $y = \sqrt{x(2-x)} \Longrightarrow (x-1)^2 + y^2 = 1^2$, which is the equation of a circle of radius 1 with center (1,0), the graph of f consists of two semi-circles, as illustrated above. So, because the area of a semi-circle of radius 1 is $\frac{1}{2}\pi \cdot 1^2 = \frac{1}{2}\pi$ and the area of a quarter-circle of radius 1 is likewise $\frac{1}{4}\pi$, we find that

$$\begin{aligned} I(f,-2,-1) &= -\tfrac{1}{4}\pi & \qquad I(f,-2,1) &= -\tfrac{1}{2}\pi + \tfrac{1}{4}\pi = -\tfrac{1}{4}\pi \\ I(f,-2,0) &= -\tfrac{1}{2}\pi & I(f,-1,2) &= -\tfrac{1}{4}\pi + \tfrac{1}{2}\pi = \tfrac{1}{4}\pi \\ I(f,0,1) &= \tfrac{1}{4}\pi & I(f,-1,1) &= -\tfrac{1}{4}\pi + \tfrac{1}{4}\pi = 0 \\ I(f,0,2) &= \tfrac{1}{2}\pi & I(f,-2,2) &= -\tfrac{1}{2}\pi + \tfrac{1}{2}\pi = 0 \end{aligned}$$

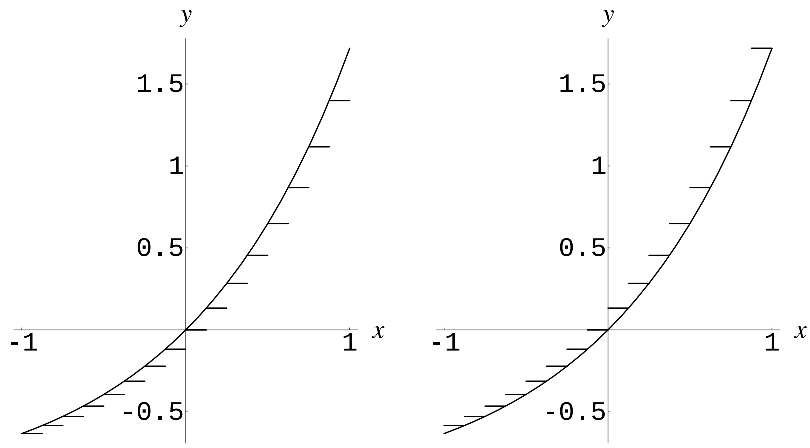
and, more generally, that

$$I(f, -a, a) \;=\; 0$$

for any a, because the shaded area to the left of the y-axis is counted negatively but is always identical to the shaded area to the right of the y-axis.* On the other hand, we cannot (at this stage) calculate $I(f, 0, a)$ unless $a = 1$ or $a = 2$, because those are the only values of a for which we know the shaded areas. In fact, the only functions for which we can always calculate $I(f, a, b)$ from merely knowing that I means signed area are piecewise

*More generally still, $I(f, -a, a) = 0$ whenever f is an odd function.

linear functions—of which piecewise-constant functions are an important special case.



The importance of piecewise-constant functions is that they are able to approximate any function as accurately as we please.† Consider, e.g., the function f defined by

$$f(x) \;=\; e^x - 1.$$

The diagram above shows two piecewise-constant approximations to f on the interval $[a,b] = [-1,1]$. The one on the left is the approximation f_{16} obtained by dividing $[a,b]$ into sixteen equal subintervals and insisting that the approximation is actually correct at the left-hand end of each subinterval; for that reason, we call f_{16} a left-hand (piecewise-constant) approximation. The one on the right is the approximation ϕ_{16} obtained by again dividing $[a,b]$ into sixteen equal subintervals but instead insisting that the approximation is actually correct at the right-hand end of each subinterval; for that reason, we call ϕ_{16} a (surprise, surprise!) right-hand approximation. More generally, for any f and with n equal subintervals, the n-th left- and right-hand approximations f_n and ϕ_n are defined by

$$f_n(x) \;=\; f\left(a + \tfrac{(i-1)(b-a)}{n}\right) \quad \text{when} \quad a + \tfrac{(i-1)(b-a)}{n} \le x < a + \tfrac{i(b-a)}{n} \quad \text{for} \quad i = 1, \ldots, n$$

and

$$\phi_n(x) \;=\; f\left(a + \tfrac{i(b-a)}{n}\right) \quad \text{when} \quad a + \tfrac{(i-1)(b-a)}{n} \le x < a + \tfrac{i(b-a)}{n} \quad \text{for} \quad i = 1, \ldots, n$$

respectively.

Now here's the thing: for any f, both $I(f_n, a, b)$ and $I(\phi_n, a, b)$ are easy to calculate, because the area between segment $[a,b]$ of the horizontal axis and the graph of any piecewise-constant function is always piecewise-rectangular. In fact, because the constituent rectangles of f_n and ϕ_n all have precisely the same width, namely, $(b-a)/n$, we readily obtain the n-th left-hand approximation

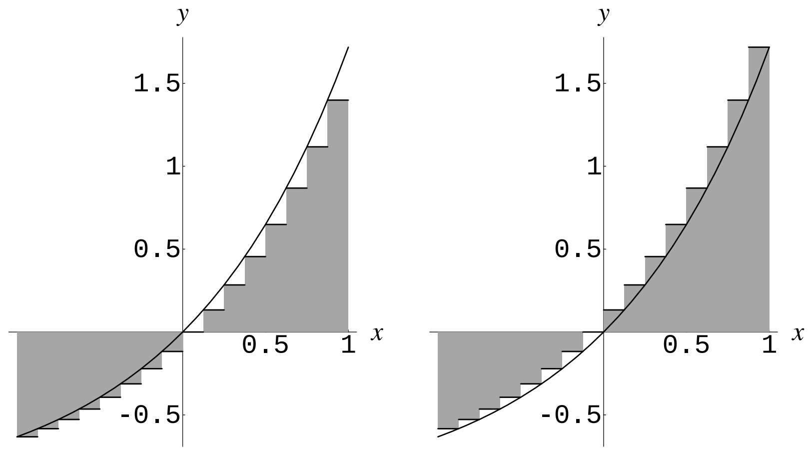
$$I(f_n, a, b) \;=\; \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + \tfrac{(i-1)(b-a)}{n}\right) \;=\; \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \tfrac{(i-1)(b-a)}{n}\right)$$

and the n-th right-hand approximation

$$I(\phi_n, a, b) \;=\; \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + \tfrac{i(b-a)}{n}\right) \;=\; \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \tfrac{i(b-a)}{n}\right).$$

† At least for the purpose of calculating a definite integral, which is all we care about here.

to $I(f, a, b)$. Note how the sign of f ensures that area is always counted positively above the axis and negatively below it.



If f is increasing then, as illustrated by the sketches above, we have

$$I(f_n, a, b) \;<\; I(f, a, b) \;<\; I(\phi_n, a, b)$$

for all values of n, no matter how large (and regardless of the sign of f—please check for yourself). The bigger the value of n, the better the value of both our over- and our under-estimate, with $I(f, a, b)$ always sandwiched between. In the limit as $n \to \infty$, the above inequalities weaken, as $I(f_n, a, b)$ and $I(\phi_n, a, b)$ coalesce. That is, we have

$$\lim_{n\to\infty} I(f_n, a, b) \;\le\; I(f, a, b) \;\le\; \lim_{n\to\infty} I(\phi_n, a, b)$$

and

$$\lim_{n\to\infty} I(f_n, a, b) \;=\; \lim_{n\to\infty} I(\phi_n, a, b),$$

so that of necessity

$$\lim_{n\to\infty} I(f_n, a, b) \;=\; I(f, a, b) \;=\; \lim_{n\to\infty} I(\phi_n, a, b).$$

In other words, $I(f, a, b)$ is the limit as $n \to \infty$ of *either* $I(f_n, a, b)$ or $I(\phi_n, a, b)$. Furthermore, the final assertion is always true, even if f is not an increasing function: if f is decreasing, then because $I(\phi_n, a, b) < I(f, a, b) < I(f_n, a, b)$ for all n, and more generally because $[a, b]$ can always be subdivided into intervals on which f is either increasing, decreasing or constant. Indeed the common limit as $n \to \infty$ of $I(f_n, a, b)$ and $I(\phi_n, a, b)$ is the legal definition of the definite integral of f over the interval $[a, b]$; however, we still tend to think of it as merely signed area.

Definite integrals have general properties that are useful in practice. Many of them follow more or less at once from thinking about signed area (and if necessary drawing a picture); and in any event, all follow readily from the legal definition. These general properties include the following (where f, g and h are any piecewise-continuous functions):

$$\begin{aligned} I(f,a,a) &= 0 \\ I(k_1 f + k_2 g, a, b) &= k_1 I(f,a,b) + k_2 I(g,a,b) \text{ where } k_1 \text{ and } k_2 \text{ are any constants} \\ a < c < b \implies I(f,a,c) + I(f,c,b) &= I(f,a,b) \\ f(x) \le g(x) \le h(x) \text{ for all } x \in [a,b] &\implies I(f,a,b) \le I(g,a,b) \le I(h,a,b) \\ I(f,b,a) &= -I(f,a,b). \end{aligned}$$

For example, to establish the last of these general properties, we use the result that

$$\sum_{i=1}^{n} \omega_i = \omega_1 + \omega_2 + \ldots + \omega_n = \omega_n + \omega_{n-1} + \ldots + \omega_1 = \sum_{i=1}^{n} \omega_{n+1-i}$$

for any ω_i, and hence, in particular for $\omega_i = f\left(b + \frac{(i-1)(a-b)}{n}\right)$:

$$\begin{aligned} I(f,b,a) &= \lim_{n\to\infty} I(f_n,b,a) = \lim_{n\to\infty} \frac{a-b}{n} \sum_{i=1}^{n} f\left(b + \tfrac{(i-1)(a-b)}{n}\right) \\ &= -\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(b + \tfrac{(i-1)(a-b)}{n}\right) \\ &= -\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(b + \tfrac{(\{n+1-i\}-1)(a-b)}{n}\right) \\ &= -\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(b + \tfrac{(n-i)(a-b)}{n}\right) \\ &= -\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \tfrac{i(b-a)}{n}\right) \\ &= -\lim_{n\to\infty} I(\phi_n,a,b) = -I(f,a,b). \end{aligned}$$

Note that the second general property makes I is a linear operator, and includes (as special cases for which $k_1 = -1, k_2 = 0$ and $k_1 = k_2 = 1$) that $I(-f,a,b) = -I(f,a,b)$ and $I(f+g,a,b) = I(f,a,b) + I(g,a,b)$. Note also that the third general property implies that if constants m, M are known with $m \le g(x) \le M$ for all $x \in [a,b]$ then

$$m(b-a) \le I(g,a,b) \le M(b-a).$$

Now let's stop talking about f in general and return to the particular f we began with, namely, f defined by $f(x) = e^x - 1$. We wish to calculate its definite integral over $[-1,1]$. First, let's use the third general property to split this definite integral into two:

$$I(f,-1,1) = I(f,-1,0) + I(f,0,1).$$

Then, for a bit of variety (you know, the spice of life), let's use a left-handed approach for $I(f,-1,0)$ and a right-handed approach for $I(f,0,1)$; and needless to say, whatever our answers, it is clear from the diagram that $I(f,-1,0)$ must be negative, $I(f,0,1)$ must be positive and $I(f,-1,1)$ must also be positive—but not quite as big. First we note a standard formula for the sum of a finite geometric series, namely,

$$\sum_{i=1}^{n} \omega^i = \omega + \omega^2 + \omega^3 + \ldots + \omega^n = \frac{\omega(1-\omega^n)}{1-\omega}$$

(which we use below with $\omega = e^{1/n}$). Now here goes:

$$\begin{aligned}
I(f,-1,0) &= \lim_{n\to\infty} \frac{0-(-1)}{n} \sum_{i=1}^{n} f\left(-1 + \tfrac{(i-1)(0-(-1))}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(-1 + \tfrac{i-1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left\{e^{-1+(i-1)/n} \quad - \quad 1\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left\{e^{-1}e^{i/n}e^{-1/n} \quad - \quad 1\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{\sum_{i=1}^{n} e^{-1}e^{i/n}e^{-1/n} \quad - \quad \sum_{i=1}^{n} 1\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{e^{-1}e^{-1/n} \sum_{i=1}^{n} e^{i/n} \quad - \quad \{1+1+\ldots+1\} \quad (n \text{ times})\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{e^{-(1+1/n)} \sum_{i=1}^{n} \left(e^{1/n}\right)^i \quad - \quad n\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{e^{-(1+1/n)}e^{1/n} \frac{1-\left(e^{1/n}\right)^n}{1-e^{1/n}} \quad - \quad n\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{e^{-1}e^{-1/n}e^{1/n} \frac{1-e^1}{1-e^{1/n}} \quad - \quad n\right\} \\
&= \lim_{n\to\infty} \left\{\frac{1}{n}e^{-1} \frac{1-e}{1-e^{1/n}} \quad - \quad 1\right\} \\
&= \lim_{n\to\infty} \frac{e^{-1}-1}{n(1-e^{1/n})} \quad - \quad 1 \quad = \quad \lim_{x\to 0+} \frac{x(e^{-1}-1)}{1-e^x} \quad - \quad 1 \\
&= \lim_{x\to 0+} \frac{(1-e^{-1})x}{e^x-1} \quad - \quad 1 \quad = \quad (1-e^{-1}) \lim_{x\to 0+} \frac{x}{e^x-1} \quad - \quad 1 \\
&= (1-e^{-1}) \lim_{x\to 0+} \frac{x}{1+x+xO[x]-1} \quad - \quad 1 \\
&= \quad (1-e^{-1}) \lim_{x\to 0+} \frac{1}{1+O[x]} \quad - \quad 1 \quad = \quad (1-e^{-1}) \frac{1}{1+0} - 1 \\
&= -e^{-1},
\end{aligned}$$

which is clearly negative. Note that we could have used L'Hôpital's rule instead to calculate the limit.

Similarly, using a *right*-handed approach:

$$
\begin{aligned}
I(f,0,1) &= \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} f\left(0+\tfrac{i(1-0)}{n}\right) \;=\; \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\tfrac{i}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left\{ e^{i/n} \quad - \quad 1 \right\} \;=\; \lim_{n\to\infty} \frac{1}{n} \left\{ \sum_{i=1}^{n} \left(e^{1/n}\right)^{i} \quad - \quad \sum_{i=1}^{n} 1 \right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{ e^{1/n} \frac{1-\left(e^{1/n}\right)^{n}}{1-e^{1/n}} \quad - \quad n \right\} \;=\; \lim_{n\to\infty} \left\{ \frac{1}{n} e^{1/n} \frac{1-e}{1-e^{1/n}} \quad - \quad 1 \right\} \\
&= (1-e) \lim_{n\to\infty} \frac{1}{n} \frac{e^{1/n}}{1-e^{1/n}} \quad - \quad 1 \;=\; (e-1) \lim_{n\to\infty} \frac{1}{n} \frac{e^{1/n}}{e^{1/n}-1} \quad - \quad 1 \\
&= (e-1) \lim_{x\to 0+} \frac{xe^{x}}{e^{x}-1} \quad - \quad 1 \\
&= (e-1) \lim_{x\to 0+} \frac{1\cdot e^{x} + x\cdot e^{x}}{e^{x}-0} \quad - \quad 1 \\
&= (e-1) \lim_{x\to 0+} \{1+x\} \quad - \quad 1 \;=\; (e-1)\cdot(1+0)-1 \;=\; e-2,
\end{aligned}
$$

on using L'Hôpital's rule. Thus

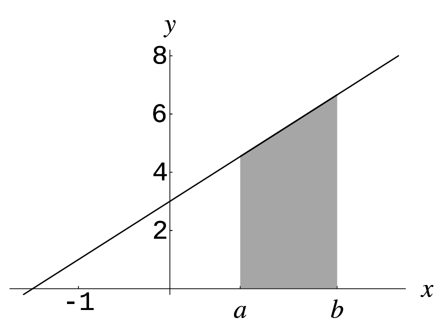
$$I(f,-1,1) \;=\; I(f,-1,0) \;+\; I(f,0,1) \;=\; -e^{-1}+e-2 \;=\; e-\frac{1}{e}-2 \;\approx\; 0.3504.$$

Not every definite integral, however, is this much trouble to calculate. For one thing, we have already agreed that definite integrals of piecewise-linear functions can be found without having to find any limit at all. So let's consider the linear function f defined by

$$f(x) \;=\; 2x+3,$$

for which I(f,a,b) is the difference in area between two triangles.

$y = 2x + 3$



From the diagram above, the larger[‡] triangle has base $b+\frac{3}{2}$ (why?) and height $2b+3$ (again, why?), and the smaller triangle has base $a+\frac{3}{2}$ and height $2a+3$. So

$$I(f,a,b) \;=\; \tfrac{1}{2}\left(b+\tfrac{3}{2}\right)(2b+3) - \tfrac{1}{2}\left(a+\tfrac{3}{2}\right)(2a+3) \;=\; (b-a)(b+a+3).$$

Notice two things. First, an anti-derivative of f defined by $f(x)=2x+3$ is F defined by

$$F(x) \;=\; x^2+3x.$$

Second, for f and F so defined, $F(b)-F(a) = b^2-a^2+3b-3a = (b-a)(b+a+3) = I(f,a,b)$.

[‡]at least when $b > a > 0$

How general is this relationship? Let's test it on our earlier results. For $f(x) = e^x - 1$ we have $I(f, -1, 1) = e - 1/e - 2$; and F defined by $F(x) = e^x - x$ is an anti-derivative of f. So $F(1) - F(-1) = e - 1 - \{e^{-1} - (-1)\} = I(f, -1, 1)$. Again, for

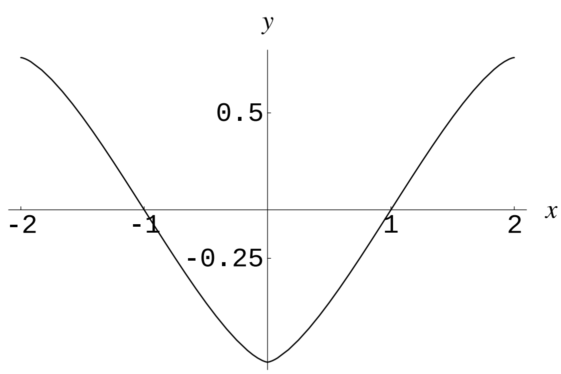
$$f(x) \;=\; \begin{cases} -\sqrt{-x(2+x)} & \text{if} \quad -2 \le x < 0 \\ \sqrt{x(2-x)} & \text{if} \quad 0 \le x \le 2 \end{cases}$$

we have $I(f, -1, 2) = \frac{1}{4}\pi$; and it can be shown (by using the chain and product rules) that F defined by

$$F(x) \;=\; \begin{cases} -\frac{1}{2}\{\arcsin(x+1) + (x+1)\sqrt{-x(2+x)}\} & \text{if} \quad -2 \le x < 0 \\ \frac{1}{2}\{\arcsin(x-1) + (x-1)\sqrt{x(2-x)}\} & \text{if} \quad 0 \le x \le 2 \end{cases}$$

(whose graph is sketched below) is an anti-derivative of f. So $F(2) - F(-1) = \frac{1}{2}\arcsin(1) + \sqrt{0} + \frac{1}{2}\{\arcsin(0) + 0\} = \frac{1}{2} \cdot \frac{1}{2}\pi + 0 + 0 = I(f, -1, 2)$.

$y = F(x)$



In all three cases, therefore, it appears that "F is an anti-derivative of f" $\Longrightarrow I(f, a, b) = F(b) - F(a)$ or—which is exactly the same thing—

$$F'(x) \;=\; f(x) \;\Longrightarrow\; I(f, a, b) \;=\; F(b) - F(a).$$

And appearances are not deceiving: what appears to be, is actually so. Moreover, not only is the above result always true,[§] but also its converse is always true in the sense that

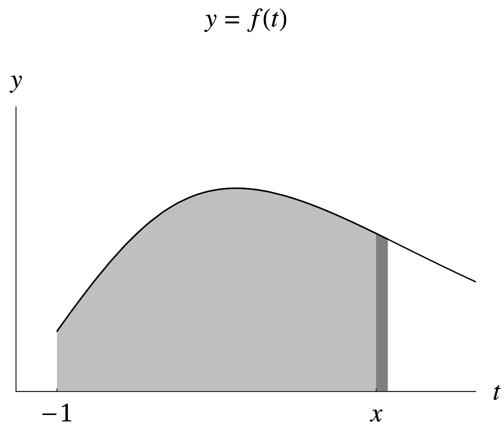
$$I(f, a, b) \;=\; F(b) - F(a) \text{ for ALL } a \text{ and } b \;\Longrightarrow\; F'(x) \;=\; f(x).$$

Combining results:

$$I(f, a, b) \;=\; F(b) - F(a) \text{ for all } a \text{ and } b \;\Longleftrightarrow\; F'(x) \;=\; f(x).$$

This two-way implication is known as the Fundamental Theorem of calculus.

[§]With appropriate qualifications—e.g., F must be piecewise differentiable.



It isn't hard to understand why the fundamental theorem must be true.[¶] Let's define a differentiable function F by

$$F(x) \;=\; I(f, a, x).$$

In the diagram above, $F(x)$ is represented by the lighter shaded area, the area between the graph of f and the segment $[a, x]$ of the horizontal axis. Note the important point that, because x is an endpoint of the interval in question, we *cannot* use x to label the axis—but we can use any symbol other than x (or y, obviously), and we have chosen to use t. Thus $I(f, a, x)$ means signed area between the horizontal axis and the graph $y = f(t)$ to the right of $t = a$ and to the left of $t = x$. It now follows at once that

$$F(x+h) \;=\; I(f, a, x+h),$$

the signed area between the horizontal axis and the graph $y = f(t)$ to the right of $t = a$ and to the left of $t = x+h$, is represented by the total shaded area (light + dark). Hence the difference, $F(x+h) - F(x)$, is represented by the dark shaded area. But for sufficiently small h, this area must fall between that of a rectangle of width h and height $f(x)$ and that of a rectangle of width h and height $f(x+h)$—because, if h is sufficiently small, then we can assume that f is either increasing, decreasing or constant on $[x, x+h]$. Thus, for sufficiently small h, the dark shaded area must lie between $hf(x)$ and $hf(x+h)$, and hence must have the form $hf(x+\theta h)$ where $0 \leq \theta < 1$). That is

$$F(x+h) - F(x) \;=\; hf(x+\theta h) \quad\Longrightarrow\quad \frac{F(x+h) - F(x)}{h} \;=\; f(x+\theta h).$$

Now take the limit as $h \to 0$. We obtain

$$f(x) \;=\; \lim_{h\to 0} f(x+\theta h) \;=\; \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \;=\; F'(x).$$

Thus $F(x) = I(f, a, x) \Longrightarrow F'(x) = f(x)$. But $F(a) = I(f, a, a) = 0$ by the first general property. So $F(x) = I(f, a, x)$ is just another way of saying that F is the *particular* anti-derivative of f with the property that $F = 0$ when $x = 0$. Because definite integration in effect means selecting a particular anti-derivative:

[¶]To be sure, this is not the same as proving it rigorously, but for Calculus I it will more than suffice.

1. selecting a generic anti-derivative is more commonly known as indefinite integration *and*

2. the standard notation for the definite integral of f between a and x is not $I(f, a, x)$ but rather

$$\int_a^x f(t)\, dt.$$

Note, however, the extremely important point that the above quantity depends only on f, a and x: it does not in any way depend on t, for reasons that should be apparent from the diagram on the previous page.[‖] We can now rewrite the fundamental theorem as

$$\int_a^b f(t)\, dt \;=\; F(b) - F(a) \text{ for all } \; a \text{ and } b \iff F'(x) \;=\; f(x)$$

(and it doesn't matter in the least which anti-derivative of f we pick because if F is one anti-derivative and Φ is another then of necessity $\Phi(x) = F(x) + C$ for some constant C, so that $\Phi(b) - \Phi(a) = F(b) + C - \{F(a) + C\}$ is identical to $F(b) - F(a)$). Furthermore, it follows at once that

$$\int_a^b F'(t)\, dt \;=\; F(b) - F(a),$$

and note—that same extremely important point—that the above statement is *identical* (i.e., not just similar, but identical) to the statement that

$$\int_a^b F'(x)\, dx \;=\; F(b) - F(a)$$

for the simple reason that F, a and b are the only things in either equation that anything depends on (for which reason, it is frequently said that t is just a dummy variable in the first equation, and that x is just a dummy variable in the second). Note also that it is often convenient to write $F(b) - F(a)$ as $F(x)\Big|_a^b$.

Finally, do you remember how much trouble we went to in calculating the integral of f defined by $f(x) = e^x - 1$ over the interval $[-1, 1]$ by directly using the legal definition? Well, now that we know the fundamental theorem, all of that work can be skipped—all we need say is

$$\int_{-1}^1 e^x - 1\, dx \;=\; \int_{-1}^1 \frac{d}{dx}\{e^x - x\}\, dx \;=\; \{e^x - x\}\Big|_{-1}^1 \;=\; e^1 - 1 - \{e^{-1} - (-1)\} \;=\; e - \frac{1}{e} - 2$$

. . . and Bob's your uncle!

[‖] So $I(f, a, x)$ is a far superior notation; regardless, I'm not going to use it much—if at all—beyond today.