1. Ordinary functions: a graphical perspective

The most fundamental concept in calculus is that of a function. We first define the concept in general terms, and then proceed immediately to examples.

A *function* is a rule that unambiguously labels things belonging to a given set. That is, each thing has a unique label (although a label can be assigned to more than one thing). The set of all possible things is the function's *domain*, and the set of all possible labels is the function's *range*. Usually, both things and labels are real numbers, in which case we will call the function an *ordinary* function; an ordinary function is most readily defined in terms of its graph, a plot of all possible (THING, LABEL) pairs with THING measured along a horizontal axis and LABEL along a vertical one.
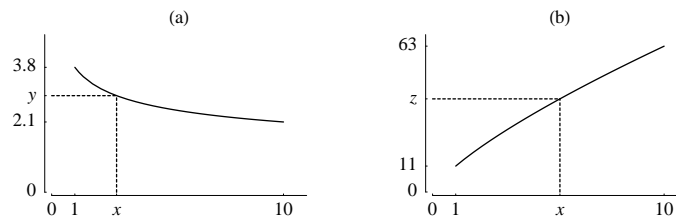

Figure 1: (a) Heart rate $y = h(x)$ beats/sec and (b) oxygen consumption rate $z = q(x)$ ml/min versus body mass x kg.

For example, larger mammals have slower heart (or pulse) rates than smaller mammals. For mammals between 1 kg and 10 kg in body mass, this relationship between size and heart rate can be *modelled* (i.e., approximated) by the curve in Figure 1a.[*] This curve is a graph, a plot of all possible (MASS, RATE) pairs, with body mass measured in kg along the horizontal axis and resting heart rate in beats/sec along the vertical one. The graph defines a function, because each possible body mass is unambiguously labelled by a heart rate. More precisely, each possible x between 1 and 10 is labelled by a y between 2.1 and 3.8 such that (x, y) lies on the graph, as indicated by the dashes running vertically from $(x, 0)$ to (x, y) and horizontally from (x, y) to $(0, y)$.

We give this function a name: we call it h (for heart rate). The domain of h contains all possible numbers between 1 and 10 (on the horizontal axis), whereas the range of h contains all possible numbers between 2.1 and 3.8 (on the vertical axis). More concisely, using $[a, b]$ to denote the set of all numbers between a and b inclusive, the domain and range of h are [1, 10] and [2.1, 3.8], respectively.

Different functions can be defined on the same domain. For example, larger mammals have higher rates of oxygen consumption than smaller mammals. For mammals between 1 and 10 kg, this relationship between size and oxygen consumption rate (OCR) is modelled by the curve in Figure 1b.[†] Again, the curve is a graph, a plot of all possible (MASS, RATE) pairs; and again it defines a function, because each possible body mass x between 1 and 10 kg is uniquely labelled by an OCR z between 11 and 63 ml/min. We give this function a name as well: we call it q (for quaffing air, if you insist on a reason). The domain is again [1, 10], but the range is now [11, 63].

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[*]See, e.g., Dawson, 1991, p. 4.
[†]See, e.g., Dawson, 1991, p. 5.

MAMMAL	REPRESENTATIVE SIZE	SOURCE
zorilla	1 kg	Estes, 1991 p. 429
rock hyrax	3 kg	op. cit., p. 254
female Kirk's dik-dik	5.5 kg	op. cit., p. 47
jackal	10 kg	op. cit., pp. 398-408

Table 1: Some representative African mammal sizes.

Let $h(x)$ denote the label that h assigns to x. Then the graph in Figure 1a is the set of all possible $(x, h(x))$ pairs with $1 \leq x \leq 10$. In principle, only two letters are needed to represent a function, one for its name (e.g., h or q) and one for the generic THING (e.g., x). In practice, however, it is often useful to reserve a third letter (e.g., y or z) for a generic LABEL. Then the graph of h can also be described as the set of all possible (x, y) pairs satisfying $y = h(x), 1 \leq x \leq 10$. Likewise, the graph of q is the set of all possible (x, z) pairs such that $z = q(x), 1 \leq x \leq 10$. These two graphs can be used to estimate both heart rate and OCR for any mammal between 1 and 10 kg in size. For example, with the help of Table 1, the model predicts that $h(1)$ = 2.1 beats/sec and $h(10)$ = 3.8 beats/sec are representative heart rates for a zorilla and a jackal, respectively; similarly, $q(3)$ = 25 ml/min and $q(5.5)$ = 40 ml/min are (again, according to our model) representative OCRs for a hyrax and a pygmy antelope .

Note that heart rate keeps dropping as one moves to the right along the curve in Figure 1a: h is a *decreasing* function. Similarly, because OCR keeps rising as one moves to the right along the curve in Figure 1b, q is an *increasing* function. The generic term for increasing or decreasing is *monotonic*. Monotonic functions have an important property: they are *invertible*,[‡] i.e., each possible LABEL belongs to precisely one THING. For example, h is invertible because each possible heart rate between 2.1 and 3.8 is associated with precisely one body mass (according to the model), and q is invertible because each possible OCR between 11 and 63 is associated with precisely one body mass.

If a function is invertible, then we define its *inverse* function by interchanging the roles of domain and range. For example, let g denote the inverse of h in Figure 1a. Then we obtain the graph of g by flipping over the graph of h, in such a way that we interchange axes while holding holding $(0, 0)$—the origin of coordinates—fixed. This maneuver transports the range of h from the vertical axis to the horizontal axis to become the domain of g, and the domain of h from the horizontal axis to the vertical axis to become the range of g. The resulting graph of g is the set of all possible $(y, g(y))$ pairs with $2.1 \leq y \leq 3.8$ (or, if you prefer, the set of all possible (y, x) pairs such that $x = g(y), 2.1 \leq y \leq 3.8$): see Figure 2b. The inverse of q, which we denote by r, is similarly obtained: see Figure 2d. We can use g and r to estimate the body size associated with a given heart rate or OCR. For example, the models associate a heart rate of 3 beats per second with a body size of $g(3)$ = 2.6 kg and an OCR of 48 ml/min with a body size of $r(48)$ = 7 kg. Incidentally, the standard way to represent a function's inverse is to use the superscript $^{-1}$; thus $g = h^{-1}$, and $r = q^{-1}$. Note the important point that $h^{-1}(y)$ does *not* mean $1/h(y)$; on the contrary,

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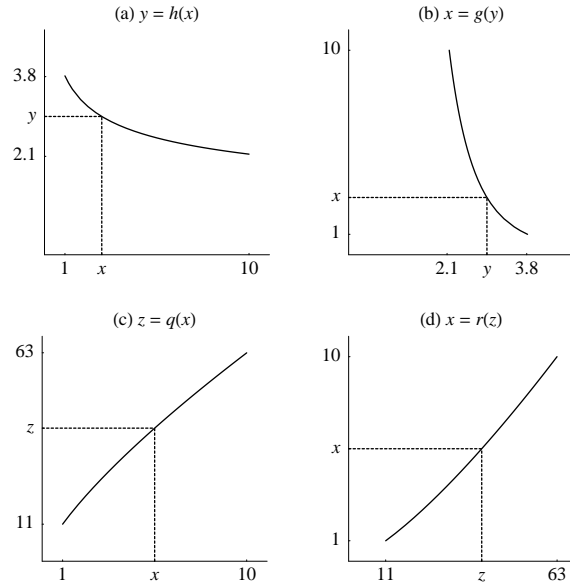[‡]Invertible functions are also known as *one-one* functions.

Figure 2: $g = h^{-1}$ (upper panels) and $r = q^{-1}$ (lower panels)

because h is monotonic, $x = h^{-1}(y)$ is just another way of saying that $y = h(x)$.

Not every function is invertible, however, because not every function is increasing or decreasing. For example, in each cardiac cycle the volume of blood in a human left ventricle decreases from about 120 ml at the end of diastole to about 50 ml at the end of systole and then increases to 120 ml again for the start of the next cycle. In a (resting) heart that beats 67 times a minute, a cycle lasts for 0.9 seconds. If we regard a cycle as beginning when the mitral valve closes to block off the venous return, then the graph in Figure 3a is a typical set of (TIME, VOLUME) pairs. This graph defines a function, say V, because each possible time t in the domain [0, 0.9] is unambiguously labelled by a ventricular volume $y = V(t)$. But V is not an invertible function, because each possible volume between 50 ml and 120 ml is reached once during systole and once during diastole. Hence every y satisfying $50 < y < 120$ is associated with two different values of t; for example, $V(t) = 100$ both when $t = 0.12$ and when $t = 0.62$. Furthermore, $y = 120$ is associated with every t such that $0 \leq t \leq 0.05$; this is the isovolumetric contraction phase, when the mitral valve has closed and ventricular blood pressure rapidly rises to open the aortic valve. Similarly, $y = 50$ is associated with every t such that $0.35 \leq t \leq 0.4$; this is the isovolumetric relaxation phase, when the aortic valve has closed and ventricular blood pressure rapidly falls again to open the mitral valve. Because the graph $y = V(t)$ is flat for $0 \leq t \leq 0.05$ and again for $0.35 \leq t \leq 0.4$, we say that V is constant both on [0, 0.05] and on [0.35, 0.4].

Now, if you look very carefully at Figure 3a, you will see that the systolic end volume of 50 ml at $t = 0.35$ is not the lowest volume achieved during a cycle; rather, the lowest volume is 49.1 ml, which is achieved at $t = 0.3$. Although the mitral valve is still closed when $0.3 \leq t \leq 0.35$, ventricular volume increases due to a brief arterial backflow (which
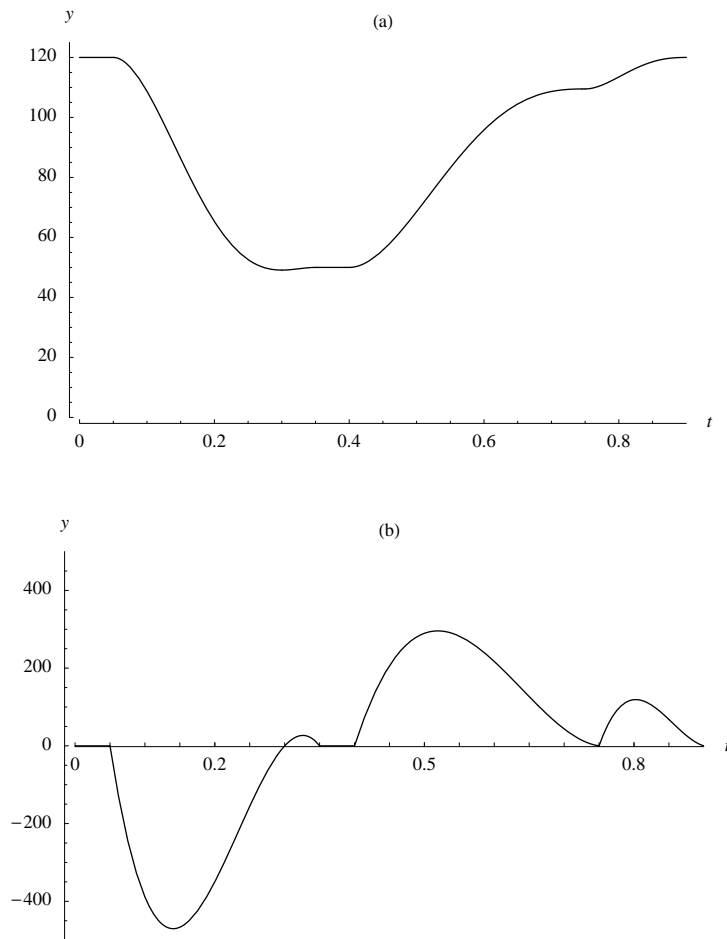
Figure 3: Relationship between (a) volume of blood $y = V(t)$ ml in left ventricle and (b) ventricular inflow $y = v(t)$ ml/s at time t. Note that the above traces of blood volume and flow are merely cartoons of data I abstracted from Folkow and Neil (1971, p. 157) and Levick (1995, pp. 16-20). These cartoons capture all essential aspects of the relationship between ventricular volume and blood flow for pedagogical purposes. For example, the diagrams include (in order) an isovolumetric contraction phase, when both input and output valves are closed; an ejection phase, when the arterial flow rises to a maximum and then decreases to zero as the ventricle empties; a backflow phase, during which the ventricle refills slightly; an isovolumetric relaxation phase, during which both valves are closed again; a primary phase of ventricular filling, when the chamber fills rapidly, at first by suction; and an atrial contraction phase, which forces additional blood into the ventricle. Nevertheless, by virtue of being cartoons, Figure 3a-b are not intended to be accurate in every detail. For example, the relaxation phase at the beginning of diastole is only 0.05 seconds (Levick suggests 0.08); backflow is only about 1% of stroke volume (Levick suggests that it may be closer to 5%); and so on.

closes the aortic valve). Thus the graph of V falls all the way from (0.05, 120) to (0.3, 49.1) but rises thereafter from (0.3, 49.1) to (0.35, 50); it is flat between (0.35, 50) and (0.4, 50), but then rises all the way to (0.9, 120). So V is an uninvertible function only if we insist that its domain is [0, 0.9]. If instead we restrict its domain to [0.05, 0.3], then V is decreasing, and therefore invertible. We call this function the *restriction* of V to [0.05, 0.3], and we refer to [0.05, 0.3] as a *subdomain* of V. Likewise, if we restrict V's domain to [0.4, 0.9], then V is increasing, and again invertible. The moral here is that properties of a function can depend on its domain of definition, and so no function is properly defined until its domain has been specified. In particular, invertibility is a property of both function and domain, because V is invertible on [0.05, 0.3] or [0.4, 0.9] but not on [0, 0.9].

Another property of both function and domain is that of *extremum*, a generic term for *minimum* or *maximum*. A global minimum or maximum is the lowest or highest value a function takes anywhere on its domain, whereas a local minimum or maximum occurs where the graph of the function switches from decreasing to increasing or vice versa. Any element of the domain corresponding to a maximum or minimum is called a *maximizer* or *minimizer*, respectively, the generic term being *extremizer*. Note that extrema always lie in the range of the function, and extremizers always lie in the domain. For example, the graph in Figure 3b defines ventricular inflow as a function of time. Let's call this function v (so that the graph has equation $y = v(t), 0 \leq t \leq 0.9$). Inspection reveals the global minimum and maximum of v on [0, 0.9] to be $y = -470$ and $y = 296$ ml/s, respectively, with global minimizer $t = 0.14$ and global maximizer $t = 0.52$. There are three local maxima, namely, $y = 27$ with maximizer $t = 0.33$, $y = 296$ with maximizer $t = 0.52$ and $y = 119$ with maximizer $t = 0.8$; however, there is only one local minimum, namely, -470. A global extremizer is always either a local extremizer or an endpoint.

Although a global extremum is unique on any given domain, there may be several extremizers; for example, any t such that $0 \leq t \leq 0.05$ and t = 0.9 are all global maximizers for ventricular volume in Figure 3a. Nevertheless, changing domains can change extrema. For example, if we were interested only in systolic blood flow, then we would restrict the domain of v to [0, 0.35] and the global minimum would still be -470, but the global maximum would now be 27. Similarly, if interested only in diastole, we would restrict the domain of v to [0.35, 0.9]: the global maximum would still be 296, but the global minimum would now be zero.

Because inflow is never positive during systole (because the ventricle is draining), however, it may be easier to think in terms of outflow. Outflow is just the negative of inflow: if inflow has equation $y = v(t)$, then outflow has equation $y = -v(t)$, and so if outflow is represented by the function f then $f(t) = -v(t)$ throughout the domain. We write $f = -v$ and call $-v$ the *negative* of v. Figure 4b shows the graph of $-v$, with that of v above it for comparison. We see that any minimum of v (regardless of whether it is local or global) is a maximum of $-v$, that any maximum of v is a minimum of $-v$, and vice versa. This result is general: it applies to any function.

Finally, note that, in this lecture, we have been careful to distinguish between functions and labels by always using different letters. Initially, this is an excellent habit. Once you understand the difference between function and label, however, it rarely causes confusion to use the same letter for both. Henceforward, therefore, we allow ourselves the luxury of saying things like, "Let t denote time, and let $V = V(t)$ be ventricular volume
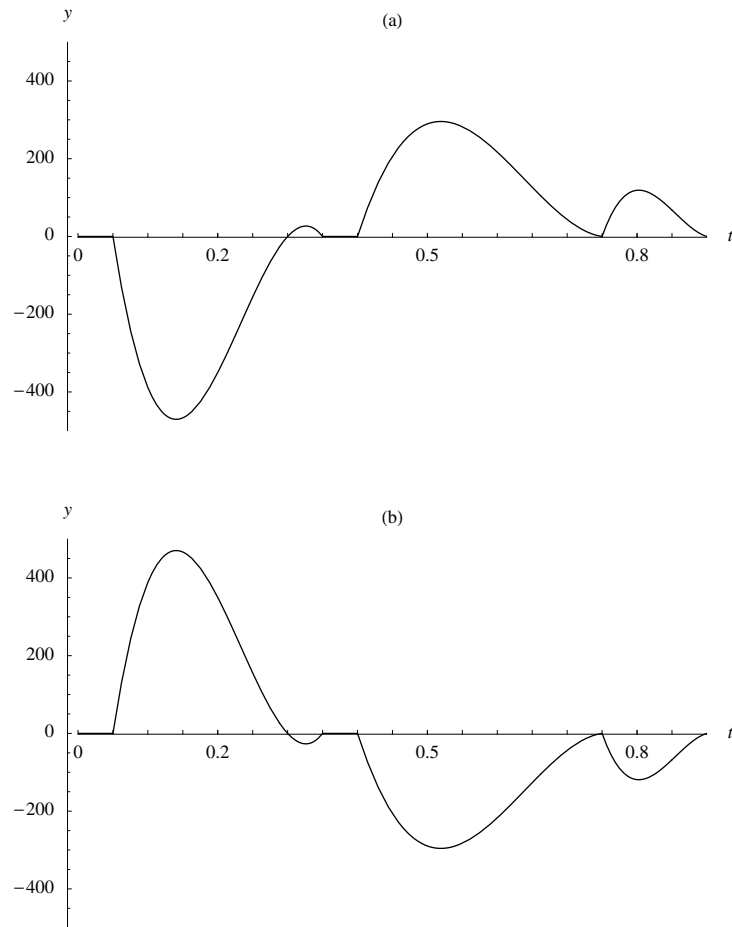
Figure 4: Relationship between (a) ventricular inflow $y = v(t)$ ml/s and (b) ventricular outflow $y = f(t) = -v(t)$ ml/s at time t.

at time t." The first V is technically a label, and the second V is technically a function, but it will be obvious from context which meaning for V is intended. It's rather like using "Darwin" to refer either to a well known Victorian scientist or to one of his many books (usually that which appeared in 1859): Darwin the scientist is not the same as Darwin the book, but it is always obvious from context which meaning is intended.

Exercises

1. The global minimum of 49.1 ml for ventricle volume in Figure 3a occurs at $t = 0.3$, precisely where v changes sign from negative to positive in Figure 3b. Why must this be so?

Suitable calculus textbook problems

Stewart (2003): p. 286, ## 3-9 (absolute means global); p. 75, ## 5-8 and 20.

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