2. Ordinary functions: an algebraic perspective

There were (virtually) no equations in Lecture 1, because we defined functions only in terms of their graphs. But functions can also be defined in terms of formulae.
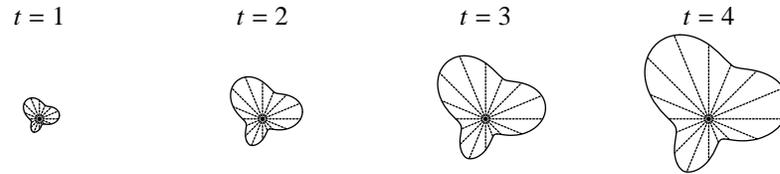

Figure 1: A growing patch of weeds.

Suppose, for example, that a patch of weeds is growing vegetatively on flat terrain by expanding uniformly in all directions. We can define an ordinary function A by labelling each possible age t in months of this clone with the area $A(t)$ that it covers. In Figure 1, the spokes all emanate from the point where the clone began its life, at time $t = 0$, as a single plant covering an area so small that we can set $A(0) = 0$ with negligible loss of accuracy. Initially, the patch assumes a shape at random; subsequently, its shape does not change because it expands uniformly outwards from the initial point. That is, in Figure 1, which shows the patch at ages 1, 2, 3 and 4, every spoke at $t = 2, 3$ or 4 is precisely t times as long as the corresponding spoke at $t = 1$. Therefore, $A(2)$ is 4 times as large as $A(1)$, $A(3)$ is 9 times as large $A(1)$ and $A(4)$ is 16 times as large as $A(1)$. More generally, $A(t)$ is t^2 times as large as $A(1)$. So the function A is defined algebraically by

$$A(t) = \alpha t^2 \tag{1}$$

where $\alpha = A(1)$. A value for α can sometimes be estimated from field data. For example, Cousens and Mortimer (1995, p. 76) suggest that, in the absence of interfering vegetation, a clone of the herbaceous perennial *Cyperus esculentis* is approximately circular. Their Figure 3.12 suggests that, during a clone's first six months, its radius increases at a rate of about two ninths of a meter per month. Thus a representative value for area at age 1 month is $\alpha = \pi(2/9)^2 \approx 0.155$, yielding

$$A(t) = \tfrac{4\pi}{81}t^2 \approx 0.155t^2 \tag{2}$$

square meters.

Defining a function algebraically has no effect on its graph. The graph of A is still the set of all possible $(t, A(t))$ pairs or, if y is an area, the set of all possible (t, y) pairs such that $y = A(t)$. For example, the graph with equation $y = \tfrac{4\pi}{81}t^2$ is sketched in Figure 2.

Note that three symbols appear in (1), namely, A, α, and t. The first, A, is the name of the function. The other two, α and t, represent quantities on which area depends: if you change either, then you change the area. But there is an important difference between α and t: For any given patch, the value of α is fixed. Thus a given patch can have different areas only by virtue of t being different, whereas different patches can have different

areas either because they are older or younger (different t) or because they grow faster or slower (different α, even if t is the same). Accordingly, we need to distinguish between a quantity that can vary for the subject of interest (here, a patch of weeds) and a quantity that is fixed for the subject of interest, but which in principle could take different values. We call the first a *variable*, and the second a *parameter*. Thus α is a parameter and t is a variable. Of course, changing t changes $A(t)$, so $A(t)$ is also a variable. We distinguish between THING-variables and LABEL-variables by calling t the *independent* variable and $y = A(t)$ the *dependent* variable.
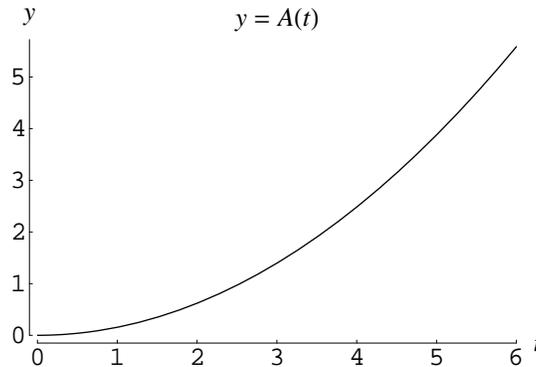

Figure 2: Weed patch size as a function of time.

The function A defined by (2) is a special case of a *power function*. A more general power function f is defined on $[0, \infty)$ by the formula

$$f(t) \;=\; \alpha t^\beta, \tag{3a}$$

where the two parameters α and β are called the *coefficient* and *exponent*, respectively. Power functions are often used to model relationships among biological variables. For example, the functions h, q, g and r defined graphically by Figures 1.1-1.2 are all power functions: they can instead be defined algebraically, by the formulae in Table 1. Note that, because a function is neither a thing nor a label, but rather a rule for assigning labels to things, it does not matter in the least what symbol we use for the generic THING in a formula: the rule will be the same for every such symbol. Thus the function f defined by $f(t) = \alpha t^\beta$ is not just similar but *identical* to the function f defined by

$$f(x) \;=\; \alpha x^\beta \tag{3b}$$

or $f(y) = \alpha y^\beta$, because all three formulae are merely different ways of saying that, for any THING in the domain of f,

$$f(\text{THING}) \;=\; \alpha \text{THING}^\beta. \tag{3c}$$

Another function often used to model relationships among biological variables is defined by the formula

$$f(x) \;=\; \alpha x + \beta \tag{4}$$

2

FUNCTION	COEFFICIENT	EXPONENT	EQUIVALENT DEFINITIONS
h (Figure 1.1a)	$\frac{229}{60}$	$-\frac{1}{4}$	$h(x) = \frac{229}{60x^{1/4}} \approx \frac{3.817}{\sqrt{\sqrt{x}}}$
q (Figure 1.1b)	$\frac{56}{5}$	$\frac{3}{4}$	$q(x) = \frac{56}{5}\left(x^{1/4}\right)^3 \approx 11.2\left(\sqrt{\sqrt{x}}\right)^3$
g (Figure 1.2b)	$\left(\frac{229}{60}\right)^4$	-4	$g(y) = \left(\frac{229}{60y}\right)^4 \approx 212.2y^{-4}$
r (Figure 1.2d)	$\left(\frac{5}{56}\right)^{\frac{4}{3}}$	$\frac{4}{3}$	$r(y) = \left(\frac{5y}{56}\right)^{\frac{4}{3}} \approx 0.04y^{4/3}$

Table 1: Algebraic definitions of power functions introduced in Lecture 1. Note that power-function relationships are often called *allometric laws* when body size is the independent variable.

where α and β are parameters. In this case f is called a *linear* function because its graph is always a straight line. For example, Thompson (1942, p. 209) used a linear function with $\alpha = 0.5829$ and $\beta = -8.085$ to model how mandible length y increases with total body length x in the reindeer beetle; see Figure 3. This model predicts, e.g., mandible length $f(60) = 0.583 \times 608.06 = 26.9$ mm for reindeer beetles with body length 60 mm.
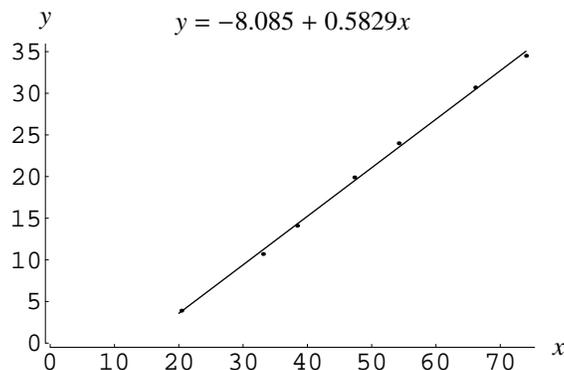

Figure 3: Mandible versus total body length in *Cyclommatus tarandus*.

Although, as we have just demonstrated, linear functions or power functions can sometimes be useful in their own right as simple two-parameter models (of, e.g., relationships between biological variables), their real importance in mathematics is that more complex models can be constructed from them by addition (or subtraction), by multiplication, by division, by using several formulae on contiguous subdomains or by constructing "functions of functions." The corresponding new functions are known, respectively, as *sums*, *products*, *quotients*, *joins* or *compositions*. We give one example of each.

An example of a product is systolic outflow in our cardiac cycle, defined algebraically as a product of linear functions:

$$f(t) = \frac{350}{9}(20t-1)(3-10t)(7-20t), \quad 0.05 \le t \le 0.35. \tag{5a}$$

The very same systolic outflow can also be defined as a sum of power functions (illustrat-

ing that functions may be expressed in more than one way):

$$f(t) \;=\; -\tfrac{2450}{3} + \tfrac{192500}{9}t - \tfrac{980000}{9}t^2 + \tfrac{1400000}{9}t^3, \quad 0.05 \le t \le 0.35. \tag{5b}$$

Note that a sum of power functions with integer exponents is called a *polynomial*, whose *degree* or *order* is the highest exponent: here $f(t)$ is a polynomial of order three.

An example of a frequently used quotient (of two linear functions) is Q defined by

$$Q(x) \;=\; \frac{\alpha x}{x + \beta}, \quad 0 \le x < \infty; \tag{6}$$

note that (provided $\beta > 0$) the denominator can never equal zero. For example, plant scientists have used the relationship $y = Q(x)$ to model both the effect of potassium concentration on rate of ion absorption in roots (see, e.g., Fitter & Hay, 1987, pp. 89-90) and the effect of carbon dioxide concentration on the rate of photosynthesis in leaves (see, e.g., Rabinowitch, 1951, p. 920). In the case of photosynthesis, the independent variable x denotes interstitial CO_2 concentration (which may differ from that in the surrounding air) in micromoles per mole; the dependent variable $y = Q(x)$ denotes the rate in micromoles per square meter per second at which CO_2 is assimilated by a leaf; the parameter α denotes the "saturation" rate of photosynthesis (in micromol $m^{-2}s^{-1}$), which cannot be exceeded even at extremely high CO_2 concentration; and the parameter β denotes the "half-saturation concentration," i.e., the value of x at which $y = \tfrac{1}{2}\alpha$. Although values for these parameters are readily available, it is more convenient to "scale" them out of the equation by defining

$$u \;=\; \frac{x}{\beta}, \quad z \;=\; \frac{y}{\alpha}. \tag{7}$$

Then, from (6), we obtain

$$z \;=\; \frac{y}{\alpha} \;=\; \frac{Q(x)}{\alpha} \;=\; \frac{x}{x + \beta} \;=\; \frac{\frac{x}{\beta}}{\frac{x}{\beta} + 1} \;=\; \frac{u}{u + 1} \;=\; K(u), \tag{8}$$

say. The graph of K is plotted in Figure 4. Notice that, no matter how large the value of u, $K(u)$ can never exceed 1—but also, $K(u)$ can never attain the value 1.

In practice, however, the photosynthetic reaction appears to saturate at quite moderate concentrations of CO_2*. That is, there appears to exist a concentration C such that $Q(x) = \alpha$ if $x \ge C$, suggesting that photosynthesis might be better described by the function L whose graph is plotted as a solid curve in Figure 4. This function is a join of functions defined on two contiguous subdomains, because it is defined algebraically by

$$L(u) \;=\; \begin{cases} \frac{2-\sqrt{2}}{2+\sqrt{2}}\left(2 + \sqrt{2} - \tfrac{1}{2}u\right)u & \text{if} \quad 0 \le u \le 2 + \sqrt{2} \\ 1 & \text{if} \quad 2 + \sqrt{2} < u < \infty \end{cases} \tag{9a}$$

$$\approx \begin{cases} 0.586(1 - 0.146u)u & \text{if} \quad 0 \le u \le 3.41 \\ 1 & \text{if} \quad 3.41 < u < \infty. \end{cases} \tag{9b}$$

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*For example, at 40-50 micromol/liter under 407 lux in the water plant *Cabomba caroliniana* (Rabinowitch, 1951, p. 897). Representative values of C/β can be inferred from Table 27.I of Rabinowitch (1951, pp. 892-893). Most of these values fall between 2 and 5 (although some are higher), so that $2 + \sqrt{2} \approx 3.41$ is not an unreasonable choice in (9).

Note that far more than two such contiguous subdomains can be used to create a join; for example, volume

$$
V(t) = \begin{cases}
120 & \text{if} \quad 0 \le t < 0.05 \\
\frac{43895}{432} + \frac{2450}{3}t - \frac{96250}{9}t^2 + \frac{980000}{27}t^3 - \frac{350000}{9}t^4 & \text{if} \quad 0.05 \le t < 0.35 \\
50 & \text{if} \quad 0.35 \le t < 0.4 \\
\frac{1620894}{1127} - \frac{489600}{49}t + \frac{28927200}{1127}t^2 - \frac{1360000}{49}t^3 + \frac{12240000}{1127}t^4 & \text{if} \quad 0.4 \le t < 0.75 \\
\frac{293892}{11} - 126000t + \frac{2450000}{11}t^2 - \frac{1568000}{9}t^3 + \frac{560000}{11}t^4 & \text{if} \quad 0.75 \le t \le 0.9
\end{cases}
\tag{10}
$$

and ventricular outflow

$$
f(t) = \begin{cases}
0 & \text{if} \quad 0 \le t < 0.05 \\
\frac{350}{9}(20t-1)(3-10t)(7-20t) & \text{if} \quad 0.05 \le t < 0.35 \\
0 & \text{if} \quad 0.35 \le t < 0.4 \\
\frac{81600}{1127}(30t-23)(5t-2)(3-4t) & \text{if} \quad 0.4 \le t < 0.75 \\
\frac{14000}{33}(12t-11)(4t-3)(9-10t) & \text{if} \quad 0.75 \le t \le 0.9
\end{cases}
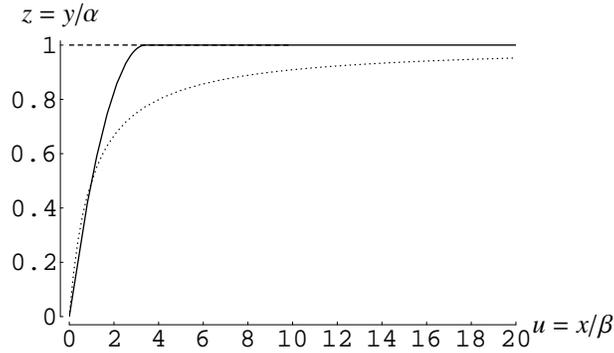\tag{11}
$$

in our cardiac cycle both make use of five.

Figure 4: Graphs of K (dashed curve) and L (solid curve) defined by (8)-(9). Note that $L(1) = \frac{1}{2}$ and $L(2 + \sqrt{2}) = 1$. Note also that even if L really is a better model for CO_2 assimilation rate versus CO_2 concentration, K may still be a better model for CO_2 assimilation rate versus other variables, e.g., light intensity.

We define compositions as follows. Let U be a function with domain $[a, b]$, and let P be another function whose domain is the range of U. Suppose $a \le x \le b$. Then $U(x)$ lies in the range of x, which means that $U(x)$ lies in the domain of P, which means that $P(U(x))$ is well defined. So

$$
R(x) = P(U(x)), \quad a \le x \le b
\tag{12}
$$

defines a function R whose domain and range coincide with those of U and P, respectively, and this is the function we call the composition of U and P.

More than two functions can be combined in this way; in particular, if the domain of S is the range of R, then we can define T by $T(x) = S(R(x)) = S(P(U(x)))$, which is a composition of three functions, etc. For example, let U with domain $[0, 1]$ and range $[1, 5]$ be defined by

$$U(x) = 1 + 4x, \quad 0 \leq x \leq 1 \tag{13}$$

(a linear function), let P with domain $[1, 5]$ and range $[1, \sqrt{5}]$ be defined by

$$P(y) = \sqrt{y}, \quad 1 \leq y \leq 5 \tag{14}$$

(a power function), and let S with domain $[1, \sqrt{5}]$ and range $[1, \frac{1}{2}(1 + \sqrt{5})]$ be defined by

$$S(z) = \tfrac{1}{2} + \tfrac{1}{2}z, \quad 1 \leq z \leq \sqrt{5} \tag{15}$$

(another linear function); here we use y and z to denote typical elements of the domains of, respectively, P and S (or equivalently, of the ranges of U and P). Then

$$T(x) = S(P(U(x))) = \tfrac{1}{2}(1 + \sqrt{1 + 4x}), \tag{16}$$

i.e., the composition of U, P and S, defines a brand new function T from $[0, 1]$ (the domain of U) to $[1, \frac{1}{2}(1 + \sqrt{5})]$ (the range of S).

Finally, do you have any idea what T has to do with rabbits? No? Then you may just have to wait until Lecture 8—although the answer is also implicit in Lecture 3.

Exercises

1. A tumor grows by expanding uniformly in all directions from a single cell of negligible volume. If ζ is its volume at age 1, what is its volume at age t?

2. Find A^{-1} algebraically for A defined by (2).

3. Verify algebraically that $g = h^{-1}$ and $r = q^{-1}$ in Table 1.

4. In a photosynthetic reaction, rate of assimilation (as a proportion of the saturation rate) at concentration x (measured in units of half-saturation concentration) is $K(x)$, where K is the join defined on $[0, \infty)$ by

$$K(x) = \begin{cases} \frac{8}{9} - \frac{1}{2}\left(x - \frac{4}{3}\right)^2 & \text{if} \quad 0 \leq x \leq 1 \\ 1 - \frac{1}{6x^2} & \text{if} \quad 1 < x < \infty \end{cases}$$

   (a) Plot the graph of K on subdomain $[0, 10]$.
   (b) Why must K be invertible?
   (c) Find K^{-1} both algebraically and graphically.

5. Individuals of the eusocial wasp *Polistes dominulus Christ* vary in strength, which can be measured by an index between 0 (weakest) and 1 (strongest). According to Theraulaz, Bonabeau and Deneubourg (1995, p. 315), as wasps move around the nest, encounters between pairs of individuals occur more or less randomly. The probability y that such an individual is willing to engage in a fight depends upon its strength x according to $y = f(x)$, where f is defined on $[0, 1]$ by

$$f(x) = \tfrac{1}{2}(1 - x + 2x^5).$$

Find the range of f (as accurately as you can).

Suitable calculus textbook problems

Stewart (2003): pp. 23-24, ## 47-53; p. 75, # 24.

References

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Fitter, A. H. & Hay, R. K. M. 1987 *Environmental Physiology of Plants*. New York: Academic Press.

Rabinowitch, E. L. 1951 *Photosynthesis and Related Processes*, vol. 2. New York: Interscience Publishers.

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Theraulaz, G., Bonabeau, E. & Deneubourg, J.-L. 1995 Self-organization of hierarchies in animal societies: the case of the primitively eusocial wasp, *Polistes dominulus* christ. *Journal of Theoretical Biology* **174**, pp. 313–323.

Thompson, D. W. 1942 *On Growth and Form*. Cambridge: Cambridge University Press.

Solutions or hints for selected exercises

1. ζt^3.

2. $\sqrt{\frac{y}{\alpha}}$.

5. By plotting f you can determine that its range is approximately $[\frac{11}{40}, 1]$. By the end of the course you'll be able to show that it is precisely $[a, 1]$, where $a = \frac{1}{2}\left(1 - \frac{2}{5}\sqrt[4]{\frac{8}{5}}\right)$.