

3. Sequences

The second most fundamental concept in calculus (after that of a function) is that of a *sequence*, which is the special case of a function for which the domain is a set of integers. Often the range is a (discrete) set of real numbers, in which case, we call the sequence an ordinary sequence. In all other respects, a sequence is just like an ordinary function. In particular, it can still be defined in terms of a graph, which is still the plot of all possible (THING, LABEL) pairs with THING measured along a horizontal axis and LABEL along a vertical one; but the graph is now a discrete set of points—unlike the graph of an ordinary function, which is a (usually continuous) curve. To distinguish between the label assigned to, say, k by an ordinary function called f and the label assigned to k by a sequence called f , we use $f(k)$ for the function but f_k for the sequence. Moreover, although a sequence is in principle a function, in practice it is often more convenient to think of it as a list of terms—hence the name. Then f_k is the k -th term of the list $\{f_1, f_2, f_3, \dots\}$, where the dots denote all the extra terms we don't have time to write down, and we use the shorthand $\{f_k\}$ to denote the entire list.

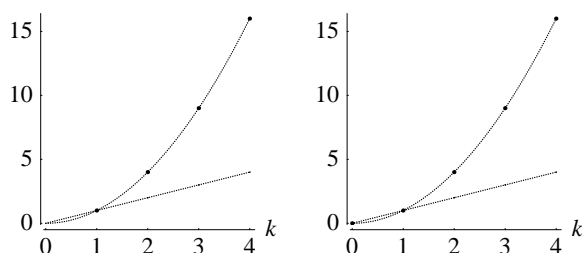


Figure 1: The discrete graphs of the sequences i and s , defined by (1)-(2), together with their continuous analogues (shown dotted).

For example, we can define an identity sequence i and a square sequence s by

$$i_k = k \tag{1}$$

and

$$s_k = k^2, \tag{2}$$

respectively. The graphs of these sequences are shown in Figure 1; the dotted curves are the corresponding graphs of the functions i and s defined by $i(x) = x$ and $s(x) = x^2$. But what is the domain of these sequences? Is it the positive integers, as in the left-hand panel, or the nonnegative integers, as in the right-hand panel? That would depend upon the circumstances; but clearly, we need a short-hand to distinguish between the sequences plotted in the left- and right-hand panels of Figure 1. Accordingly, we use $\{f_k | k \geq 1\}$ to denote $\{f_1, f_2, f_3, \dots\}$ and $\{f_k | k \geq 0\}$ to denote $\{f_0, f_1, f_2, f_3, \dots\}$; however, if the first term is obvious from context, then we simply use $\{f_k\}$ to denote either.*

*In general, we denote the set of all things of type \bullet with property P by $\{\bullet\}$ if P is obvious from context, by $\{P\}$ if \bullet is obvious from context, and by $\{\bullet|P\}$ if neither \bullet nor P is obvious from context.

Sequences may be defined either explicitly, as in the case of i and s above; or implicitly, in terms of a recursion. For an example of the latter, we consider one of the earliest known examples of biological mathematics, dating all the way back to the beginning of the 13th century, when Leonardo Fibonacci of Pisa—reputedly the most distinguished mathematician of the Middle Ages—considered the growth of a highly idealized rabbit population with zero mortality in which every rabbit was paired from birth (until eternity) with a member of the opposite sex. He supposed this population to sprout from a single pair of newborns introduced at time zero. He further assumed that rabbits reached maturity at age one month, and that every adult pair reproduced itself precisely once every month. How many pairs of rabbits would there be in this population on December 31 if the initial pair were introduced on New Year’s Day?

At the time corresponding to either the end of month k or the beginning of month $k + 1$, let a_k denote the number of adult pairs, let y_k denote the number of young pairs, and let u_k denote the grand total. Then

$$u_k = a_k + y_k \tag{3}$$

for any value of k . Because the population starts with a pair of newborns, on January 1 (which we regard not only as the beginning of month 1, but also as the end of month 0) there are no adults, just a pair of juveniles. Hence

$$a_0 = 0, \quad y_0 = 1. \tag{4}$$

At midnight on January 31 (or, if you prefer, zero hours on February 1), Adam & Eve Rabbit reach maturity. There are now no juveniles (because they have just become adults), but we do have a single pair of adults, namely, A & E Rabbit. That is,

$$a_1 = 1, \quad y_1 = 0. \tag{5}$$

During the month of February, A & E Rabbit reproduce themselves. So, on February 29 at midnight (it’s leap year, what else?), a first pair of young is counted; and A & E Rabbit are still around, so

$$a_2 = 1, \quad y_2 = 1. \tag{6}$$

During March, A & E Rabbit reproduce again, so a fresh pair of young is counted in the midnight census of March 31. Only a single pair of young is counted at that time, however, because A & E’s first son and daughter have just become adults. On the other hand, we now have two pairs of adults (A & E plus kids). So

$$a_3 = 2, \quad y_3 = 1. \tag{7}$$

Continuing in this manner, we find that the number of young at the end of month k is identical to the number of adults at the beginning of month k , or—which is exactly the same thing—at the end of month $k - 1$. That is,

$$y_k = a_{k-1}. \tag{8}$$

This result is illustrated by Figure 2, where time increases downwards, unfilled circles correspond to juveniles, filled circles correspond to adults, and all circles on the same

vertical line correspond to the same pair of rabbits; for example, $y_4 = a_3$ because there are two unfilled circles on the level corresponding to time $k = 4$ and two filled circles on the level above. Similarly, because number of filled circles at any level equals total number of circles (both unfilled and filled) at the level above, number of adult pairs at the end of month k equals number of young pairs at the end of month $k - 1$ plus number of adult pairs at the end of month $k - 1$, i.e.,

$$a_k = y_{k-1} + a_{k-1}. \quad (9)$$

Because (8)-(9) are true for any k , we can replace k by $k + 1$ to obtain

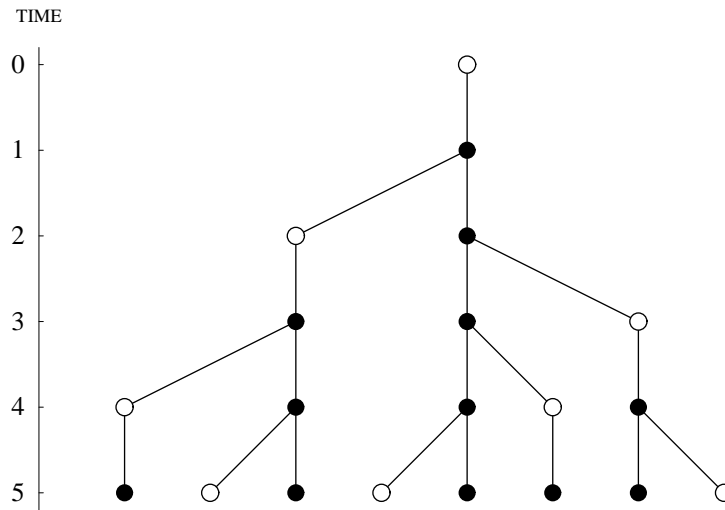


Figure 2: Growth of Fibonacci population: young are white, adults are black.

$$y_{k+1} = a_k \quad (10a)$$

and

$$a_{k+1} = y_k + a_k. \quad (10b)$$

Now, from (3) with $k + 1$ in place of k , from (10), (3) and (9), and from (3) with $k - 1$ in place of k , we obtain

$$\begin{aligned} u_{k+1} &= a_{k+1} + y_{k+1} \\ &= (y_k + a_k) + a_k \\ &= u_k + a_k \\ &= u_k + (y_{k-1} + a_{k-1}) \\ &= u_k + u_{k-1}. \end{aligned} \quad (11)$$

In words: number of pairs next month equals this month's total plus last month's total. Why? Because this month's total increases by the number of newborn pairs, which equals this month's number of adult pairs, which equals last month's total number of pairs. So,

from (3)-(5) and (11), the number of rabbit pairs at time k is defined recursively by

$$u_0 = 1 \quad (12a)$$

$$u_1 = 1 \quad (12b)$$

$$u_{k+1} = u_k + u_{k-1} \quad \text{if } k \geq 1 \quad (12c)$$

yielding $u_2 = u_1 + u_0 = 1 + 1 = 2$, $u_3 = u_2 + u_1 = 2 + 1 = 3$, $u_4 = 3 + 2 = 5$, $u_5 = 5 + 3 = 8$, and so on, as you can readily verify by counting all circles at a given level in Figure 2. We call (12c) a *recurrence relation* (because recurrent use of it yields the sequence). The graph of $\{u_k\}$ is plotted in Figure 3a. Note that u_k gets larger and larger as k gets larger and larger. Indeed there is no number, however large, that u_k cannot exceed for large enough k . We identify this state of affairs by saying that the sequence $\{u_k\}$ *diverges*, or that u_k tends to infinity as k tends to infinity, written

$$u_k \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (13)$$

Because there is no mortality to hold rabbits in check, the behavior of the sequence is neither surprising nor realistic. Nevertheless, zero mortality may not be unreasonable for a year or so, in which case, we can answer the question we began with: the prediction for midnight on December 31 is $u_{12} = 233$ rabbit pairs.

A more interesting sequence compares the number of rabbit pairs at the end of each month with the number at the end of the previous month. We define the *Fibonacci ratio*, ϕ_k , to be the ratio between the number of rabbit pairs at the end of month k and the number at the end of month $k - 1$, i.e., we define

$$\phi_k = \frac{u_k}{u_{k-1}} \quad (14)$$

for $k \geq 1$: note that (14) is meaningless for $k = 0$, because u_{-1} is undefined. This ratio is a measure of how rapidly the population has grown during month k . Replacing k by $k + 1$ in (14) and using (12), we obtain

$$\phi_{k+1} = \frac{u_{k+1}}{u_{k+1-1}} = \frac{u_{k+1}}{u_k} = \frac{u_k + u_{k-1}}{u_k} = 1 + \frac{u_{k-1}}{u_k} = 1 + \frac{1}{\frac{u_k}{u_{k-1}}} = 1 + \frac{1}{\phi_k}. \quad (15)$$

Also, again from (14) and (12), we have $\phi_1 = u_1/u_0 = 1/1 = 1$. So the Fibonacci ratio is defined recursively by

$$\phi_1 = 1 \quad (16a)$$

$$\phi_{k+1} = 1 + \frac{1}{\phi_k} \quad \text{if } k \geq 1 \quad (16b)$$

yielding $\phi_2 = 1 + 1/\phi_1 = 2$, $\phi_3 = 1 + 1/\phi_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $\phi_4 = 1 + 1/\phi_3 = 1 + \frac{2}{3} = \frac{5}{3}$, etc.

The behavior of $\{\phi_k\}$ as k increases indefinitely is very different from that of $\{u_k\}$, as the graph in Figure 3b makes clear. The ratio ϕ_k alternately increases and decreases, by smaller amounts each time, until eventually it settles down to a number near 1.6—let's call it L . We say that L is the *limit* to which $\{\phi_k\}$ *converges* as k tends to infinity, written

$$L = \lim_{k \rightarrow \infty} \phi_k. \quad (17)$$

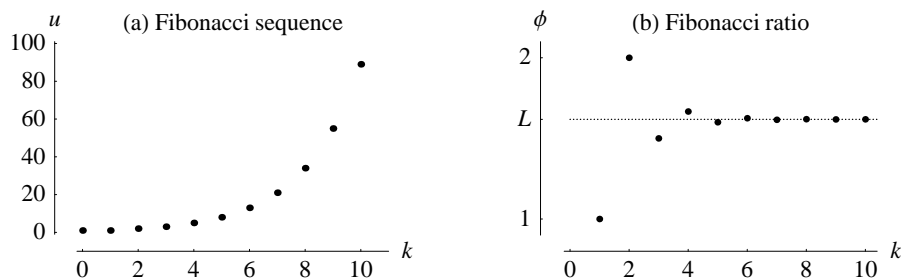


Figure 3: The Fibonacci sequence (left) and ratio (right). Convergence of the latter is said to be *oscillatory*, because consecutive terms of the sequence lie on opposite sides of the limit.

Equivalently, we say that ϕ_k tends to L as k tends to infinity, written

$$\phi_k \rightarrow L \quad \text{as} \quad k \rightarrow \infty. \quad (18)$$

But what, precisely, is this limit to which the sequence settles down? That is, what is the precise value of the number indicated by the dotted line in Figure 3b? We find it by observing that, as k becomes increasingly large (i.e., as $k \rightarrow \infty$) and therefore ϕ_k tends to L , the left-hand side of

$$\phi_{k+1} = 1 + \frac{1}{\phi_k}$$

must also tend to L (because ϕ_{k+1} is always even closer to the L than ϕ_k is), while the right-hand side tends to $1 + \frac{1}{L}$; and so

$$L = 1 + \frac{1}{L}, \quad (19)$$

from which it follows readily that

$$L = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618 \quad (20)$$

(see Exercise 1). This number is called the golden ratio, and plays an important role in, e.g., studies of phyllotaxis (Jean, 1994). It also has aesthetic properties: see Exercise 2.

Two very useful results about convergence of sequences, namely, that

$$\lim_{k \rightarrow \infty} \frac{1}{k^r} = 0 \quad (21)$$

for any positive number r and that

$$\lim_{k \rightarrow \infty} r^k = 0 \quad (22)$$

for any number r whose absolute value is less than 1 (i.e., $|r| < 1$ or $-1 < r < 1$) are illustrated by Figures 4 and 5, respectively. From these two results we can deduce the convergence of many another sequence.

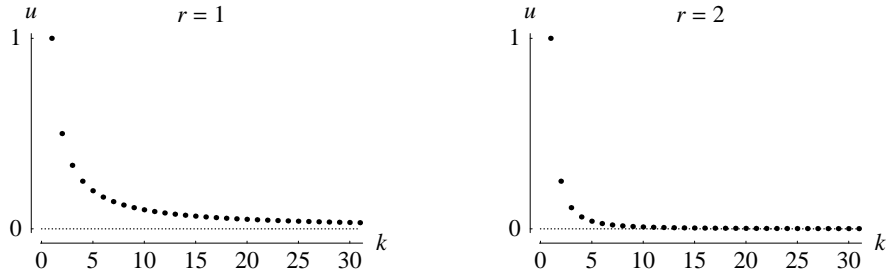


Figure 4: The sequence $\{u_k | k \geq 1\}$ defined by $u_k = r^{-k}$ for $r = 1$ and 2. Convergence in this case is said to be *monotonic*, because the sequence approaches the limit from one side only.

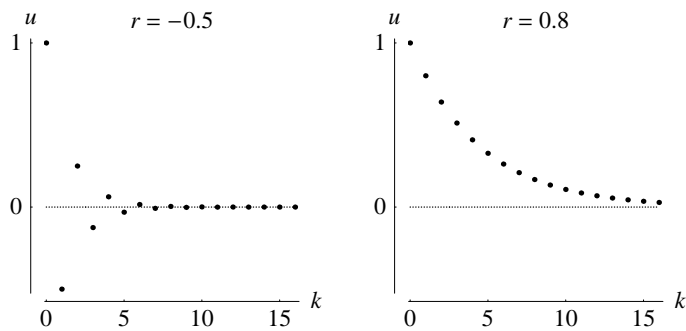


Figure 5: Typical convergence of the sequence $\{u_k | k \geq 0\}$ defined by $u_k = r^k$ with $|r| < 1$, which is oscillatory for negative r but monotonic for positive r .

For example, consider the sequence $\{K_n\}$ obtained by sampling Lecture 2's rate-of-photosynthesis function at integer values. In place of $K(x) = \frac{x}{1+x}$ for $x \geq 0$ we obtain

$$K_n = \frac{n}{1+n} \quad (23)$$

for $n \geq 0$; note that we are using n in place of k for the generic domain element (to avoid any possible confusion with the name of the sequence itself). Because (21) implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0, \quad (24)$$

i.e., that $\frac{1}{n}$ can be made as close to zero as we please by making n sufficiently large, we obtain

$$\lim_{n \rightarrow \infty} K_n = \lim_{n \rightarrow \infty} \frac{n}{1+n} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n} + 1} = \frac{1}{0+1} = 1 \quad (25)$$

as confirmed by Figure 6a. Similarly,

$$\lim_{n \rightarrow \infty} \frac{2 \cdot 3^n}{2^n + 3^{n+1}} = \lim_{n \rightarrow \infty} \frac{2}{\left(\frac{2}{3}\right)^n + 3} = \frac{2}{0+3} = \frac{2}{3} \quad (26)$$

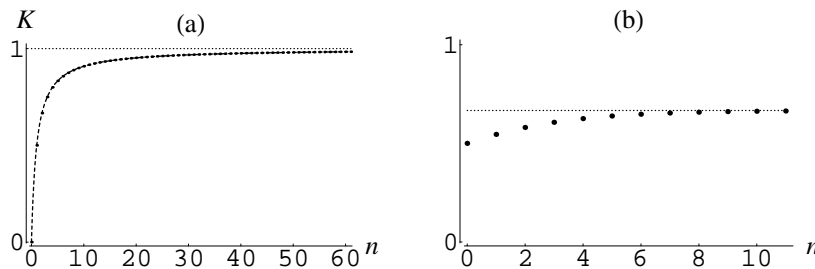


Figure 6: (a) Convergence to 1 of the sequence $\{K_n\}$ defined by $K_n = \frac{n}{1+n}$ and (b) convergence to $\frac{2}{3}$ of the sequence defined by (26).

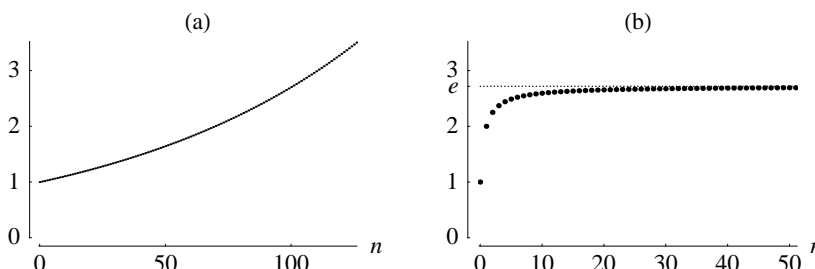


Figure 7: (a) Divergence to infinity of the sequence $\{(1.01)^n\}$ and (b) convergence to e defined by (28) of the sequence $\{u_n\}$ defined by (27).

as confirmed by Figure 6b.

Convergence to a limit is often a balance between two opposing tendencies. Consider, for example, the sequence $\{u_n\}$ defined by

$$u_n = \left(1 + \frac{1}{n}\right)^n. \quad (27)$$

We already know from (24) that $1 + \frac{1}{n} \rightarrow 1$ as $n \rightarrow \infty$. But $r^n \rightarrow \infty$ for $r > 1$, no matter how close to 1 r is; for example, $\{(1.01)^n\}$ diverges, as illustrated by Figure 7a. Thus the opposing tendencies inherent in (27) are that dividing by n tends to make numbers smaller as n grows larger, and raising to the power of n tends to make numbers larger as n grows larger. As Figure 7b illustrates, the result of the opponents' contest is that $\{u_n\}$ settles down to a number between 2 and 3. This number, i.e., the limit of the sequence, is so important in mathematics that we give it a special name: we call it e . That is, we define

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n. \quad (28)$$

It can be shown that $e \approx 2.71828$. Or consider the sequence $\{u_n\}$ defined by

$$u_n = n \sin\left(\frac{1}{n}\right) \quad (29)$$

(where the argument of \sin is measured in radians). Here u_n is a product of two terms. The first term is n , which grows indefinitely large as $n \rightarrow \infty$ (because that is what $n \rightarrow \infty$

means). The second term is $\sin\left(\frac{1}{n}\right)$, which moves closer and closer to $\sin(0) = 0$ as $n \rightarrow \infty$ because $\frac{1}{n} \rightarrow 0$. Again there are two opposing tendencies: $n \rightarrow \infty$ while $\sin\left(\frac{1}{n}\right) \rightarrow 0$. The result of this contest is illustrated by Figure 8:

$$\lim_{n \rightarrow \infty} n \sin\left(\frac{1}{n}\right) = 1. \quad (30)$$

For another example of a contest between two opposing tendencies, see Exercise 6.

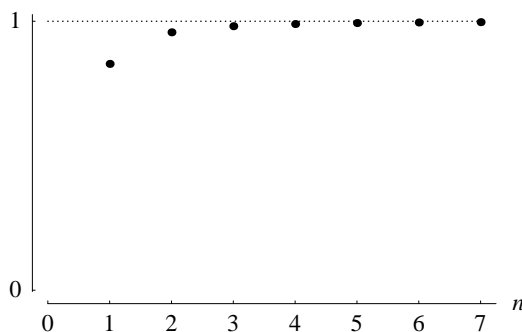


Figure 8: Convergence to 1 of the sequence $\{u_n\}$ defined by (29).

Note, finally, that the limit of a convergent sequence need not lie in its range, and usually doesn't. (Don't forget that a sequence, despite being often most conveniently regarded as a list of numbers, is at heart is a function, and therefore has both a domain and a range.) For example, there is no value of n for which $\frac{1}{n} = 0$, yet $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$ by (24); there is no value of n for which $\frac{n}{n+1} = 1$, yet $\frac{n}{n+1} \rightarrow 1$ as $n \rightarrow \infty$ by (25); there is no value of n for which $n \sin\left(\frac{1}{n}\right) = 1$, yet $n \sin\left(\frac{1}{n}\right) \rightarrow 1$ as $n \rightarrow \infty$ by (30); and so on.

The concept of a limit is the third most fundamental concept in calculus (after those of function and sequence). It doesn't apply only to sequences: it applies to ordinary functions as well. But that will be the topic of the following lecture.

Appendix: Summation notation

This is the most convenient point at which to introduce summation notation, which will come in very handy later. Observe from Figure 2 that the total number of (white or black) dots at any level is always equal to the number of white dots at that level or higher in the diagram. Why? Because the only way that a rabbit pair can ever enter this population is by being born, and so the total number of rabbit pairs at any time must always equal the total number of pairs that have ever been young. In other words, for any time k ,

$$u_k = y_0 + y_1 + \dots + y_k, \quad (31)$$

where the dots stand for all the terms we don't have time to write down, namely, the numbers of pairs of rabbits that were counted as young between the end of the second month and the end of month $k - 1$, inclusive. But the right-hand side of (31) is arguably

rather cumbersome, so instead we write

$$u_k = \sum_{n=0}^k y_n, \quad (32)$$

whose right-hand side is an elegant shorthand for the sum of all terms of the form y_n for values of n between $n = 0$ and $n = k$. In a similar vein, because every polynomial is a sum of terms of the form $c_n t^n$ (where n is a nonnegative integer and c_0, c_1, c_2, \dots , are given parameters), we can use a shorthand for “the sum of all terms of the form $c_n t^n$, for values of n between zero and m ” to write this polynomial more succinctly as

$$\sum_{n=0}^m c_n t^n, \quad (33)$$

where m is the order of the polynomial. For example, with $m = 3$, systolic outflow $f(t) = -\frac{2450}{3} + \frac{192500}{9}t - \frac{980000}{9}t^2 + \frac{1400000}{9}t^3$ in our cardiac cycle can be written as

$$\sum_{n=0}^m c_n t^n, \quad (34)$$

where $c_0 = -\frac{2450}{3}$, $c_1 = \frac{192500}{9}$, $c_2 = -\frac{980000}{9}$ and $c_3 = \frac{1400000}{9}$. See also Exercises 8-9.

Exercises

1. Show that (20) is the only positive root of equation (19).
2. It is often said that a two-pane window is most pleasing to the eye if its ratio of height to width equals the upper pane’s ratio of width to height. Verify that the ratio in question must be the golden ratio.
3. A sequence $\{x_k\}$ is defined recursively by $x_0 = 1$ and

$$x_{k+1} = \frac{1}{2} \left(x_k + \frac{9}{x_k} \right)$$

for all $k \geq 0$. Show that the sequence converges. What does it converge to?

4. A sequence $\{x_k\}$ is defined recursively by $x_0 = 1$ and

$$x_{k+1} = \frac{1}{2} \left(x_k + \frac{10}{x_k} \right)$$

for all $k \geq 0$. Show that the sequence converges. What does it converge to?

5. A sequence $\{x_k\}$ is defined recursively by $x_0 = -\frac{2}{5}$ and

$$x_{k+1} = \frac{2x_k^2 + 3}{2(2x_k + 1)}$$

for all $k \geq 0$. Show that the sequence converges. What does it converge to?

6. For $|r| < 1$ and $s > 0$, what is

$$\lim_{n \rightarrow \infty} n^s r^n?$$

Hint: Plot the sequence for different values of r and s .

7. The Fibonacci sequence $\{u_k\}$ defined recursively, and hence implicitly, by (12) can also be explicitly defined for all $k \geq 0$ by

$$u_k = \frac{1}{\sqrt{5}} \left(L^{k+1} + \frac{(-1)^k}{L^{k+1}} \right)$$

where L is defined by (20). Verify that this expression satisfies (12), and also that $u_{k+1}u_{k-1} + (-1)^k = u_k^2$ for all $k \geq 1$.

8. Write out the following polynomials in full:

(a) $\sum_{n=1}^3 \{2n - 1\} t^n$

(b) $\sum_{n=0}^4 \{2n^2 - 7n\} t^n$

(c) $\sum_{n=1}^4 \{2n^3 + n^2\} t^{n-1}$

9. Write out the following polynomials in full:

(a) $\sum_{n=0}^4 \{3n + n^2\} t^n$

(b) $\sum_{n=1}^3 \{n^3 - 10n - 2\} t^n$

(c) $\sum_{n=2}^5 \{2n^3 + n^2\} t^{n-2}$

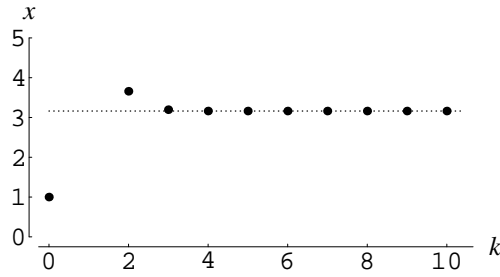
Reference

Jean, R. V. 1994 *Phyllotaxis: A Systemic Study in Plant Morphogenesis*. Cambridge: Cambridge University Press.

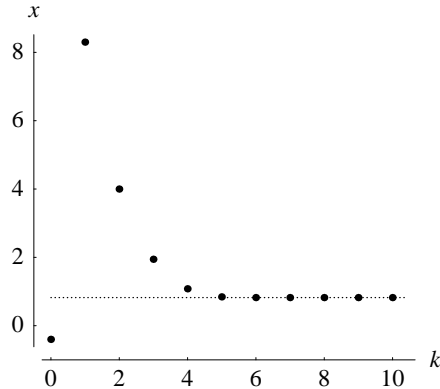
Solutions or hints for selected exercises

1. $L = 1 + \frac{1}{L} \implies L^2 = L + 1 \implies L^2 - L = 1 \implies L^2 - L + \frac{1}{4} = \frac{5}{4} \implies \left(L - \frac{1}{2}\right)^2 = \frac{5}{4}$
 $\implies L - \frac{1}{2} = \pm \frac{\sqrt{5}}{2}$. But L is clearly bigger than $\frac{1}{2}$, so that $L - \frac{1}{2}$ cannot be negative.
Hence $L - \frac{1}{2} = \frac{\sqrt{5}}{2} \implies L = \frac{1}{2}(1 + \sqrt{5})$.

4. $x_1 = \frac{1}{2}(x_0 + 10/x_0) = \frac{1}{2}\left(1 + \frac{10}{1}\right) = \frac{11}{2}$; $x_2 = \frac{1}{2}(x_1 + 10/x_1) = \frac{1}{2}\left(\frac{11}{2} + \frac{20}{11}\right) = \frac{161}{44} \approx 3.65909$; $x_3 = \frac{1}{2}(x_2 + 10/x_2) = \frac{1}{2}\left(\frac{161}{44} + \frac{440}{161}\right) = \frac{45281}{14168} \approx 3.19601$; etc. See the diagram.

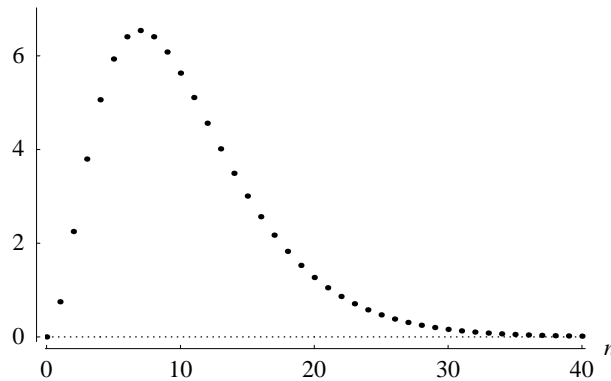


5. $x_1 = 8.3; x_2 \approx 3.99943; x_3 \approx 1.94418; x_4 \approx 1.08009; x_5 \approx 0.84381; \text{etc.}$



The sequence converges to $\frac{1}{2}\{\sqrt{7} - 1\} \approx 0.822876$.

6. As illustrated below for $r = \frac{3}{4}$ and $s = 2$, $\lim_{n \rightarrow \infty} n^s r^n = 0$ because as $n \rightarrow \infty$, $r^n \rightarrow 0$ much faster than $n^s \rightarrow \infty$.



9. (c)

$$\begin{aligned} & \{2 \cdot 2^3 + 2^2\}t^{2-2} + \{2 \cdot 3^3 + 3^2\}t^{3-2} + \{2 \cdot 4^3 + 4^2\}t^{4-2} + \{2 \cdot 5^3 + 5^2\}t^{5-2} \\ & = 20 + 63t + 144t^2 + 275t^3. \end{aligned}$$