5. Infinitesimals and differential coefficients

The fourth most fundamental concept in calculus (after those of function, sequence and limit) is that of *infinitesimals*, which are changes of arbitrarily small magnitude—so, small, in fact, that their squares can usually be neglected. Note that we deliberately speak of the concept of infinitesimals (plural), as opposed to the concept of an infinitesimal, because these arbitrarily small quantities are of little use by themselves: they are useful only in pairs. Moreover, an infinitesimal always belongs to a variable, because you can't record a change (however small) unless there is already something there to be changed; and correspondingly, a (useful) pair of infinitesimals always belongs to a pair of related variables. At least for now, we will denote the independent variable by x, the dependent variable by y, the infinitesimal that belongs to x by δx, and the infinitesimal that belongs to y by δy. Note the important point that δx stands for "very small change in x" and is a number all by itself: it does *not* mean $\delta \times x$. Similarly for δy. Another important point about infinitesimals is that although they are all very small, they can still be very different sizes of small (in much the same way that the mass of a proton exceeds the mass of an electron by three orders of magnitude despite the exceedingly small size of both particles).

An example will serve to make these matters clearer. Consider, therefore, a square of side x and area y, for which x and y are related by

$$y = x^2, \tag{1}$$

and suppose that x is changed by a very small (positive or negative) amount δx to become $x + \delta x$. Then, correspondingly, y is changed by a very small (positive or negative) amount δy to become $y + \delta y$. The side is now $x + \delta x$; the area is now $y + \delta y$; and so*

$$y + \delta y = (x + \delta x)^2 = x^2 + 2x \cdot \delta x + (\delta x)^2, \tag{2}$$

implying

$$\delta y = \{y + \delta y\} - y = \{x^2 + 2x \cdot \delta x + (\delta x)^2\} - x^2 = 2x \cdot \delta x + (\delta x)^2 \tag{3}$$

as illustrated by Figure 1.

Suppose, for example, that the side of a square of area 4 square units is increased by 0.01 units from 2 units to 2.01 units to yield a square of area $(2.01)^2 = 4.0401$ square units. Then $x = 2$, $y = 4$, $\delta x = 0.01$, $2x\delta x = 0.04$, $(\delta x)^2 = 0.0001$ and $\delta y = 0.0401$. This example illustrates two points. First, even though δx and δy are both small, δy is four times as large as δx. Second, $(\delta x)^2 = 0.0001$ makes only a tiny contribution to δy, which is almost entirely accounted for by the first term in (3), namely, $2x\delta x$. Thus, for all practical purposes, it suffices to know that $\delta y \approx 2x\delta x$; to two significant figures, we obtain 0.04 regardless of whether we use the exact expression $2x\delta x + (\delta x)^2$ or the approximation $2x\delta x$. The smaller the value of δx, the greater the extent to which $\delta y = \delta(x^2)$ is accounted for by $2x\delta x$ alone; for example, with $\delta x = 10^{-6}$, $2x\delta x$ would be correct to six significant figures. Moreover, the above remarks apply regardless of whether δx is positive or negative. For example, if the side of the square of area 4 is instead decreased by 0.01 from 2 to 1.99 to yield a

*On using the identity $(A + B)^2 = A^2 + 2AB + B^2$ with $A = x$ and $B = \delta x$.

square of area $(1.99)^2 = 3.9601$ square units, we instead have $\delta x = -0.01$, $2x\delta x = -0.04$, $(\delta x)^2 = 0.0001$ and $\delta y = -0.0399 \approx -0.040$, so that $2x\delta x$ is still correct to two significant figures.

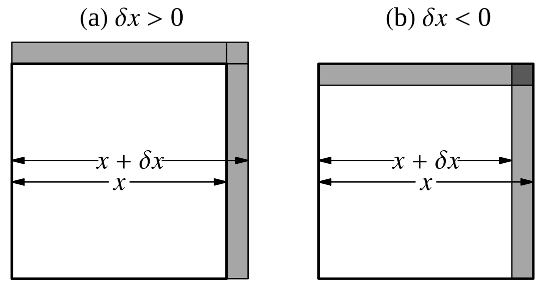

Figure 1: The effect on the area x^2 of a square of infinitesimally changing (= increasing or decreasing) its side x by δx, which may be positive or negative; from (3), we have $\delta(x^2) = 2x \cdot \delta x + (\delta x)^2$ in either case. (a) For $\delta x > 0$, $2x \cdot \delta x$ corresponds to the sum of the two equal areas of the shaded rectangles; $(\delta x)^2$ corresponds to the shaded square; and $\delta(x^2)$ corresponds to the total shaded area. (b) For $\delta x < 0$, $2x \cdot \delta x$ corresponds to the sum of the *negatives* of the areas of the same two rectangles, which now overlap; $(\delta x)^2$ corrects for the double subtraction of their overlapping area; and $\delta(x^2)$ corresponds to the *negative* of the total shaded area.

Similar considerations apply to a cube of side x and volume y, for which x and y are related by

$$y = x^3. \tag{4}$$

Again, let x be changed by a very small (positive or negative) amount δx to become $x + \delta x$. Then, correspondingly, y is changed by δy to become $y + \delta y$. The side is now $x + \delta x$; the volume is now $y + \delta y$; and so†

$$y + \delta y = (x + \delta x)^3 = x^3 + 3x^2 \cdot \delta x + 3x \cdot (\delta x)^2 + (\delta x)^3, \tag{5}$$

implying

$$\begin{aligned} \delta y = \{y + \delta y\} - y &= \{x^3 + 3x^2 \cdot \delta x + 3x \cdot (\delta x)^2 + (\delta x)^3\} - x^3 \\ &= 3x^2 \cdot \delta x + 3x \cdot (\delta x)^2 + (\delta x)^3. \end{aligned} \tag{6}$$

Again, the approximation $\delta y = 3x^2 \delta x$ is accurate enough for all practical purposes when δx is small; e.g., with $x = 2$ and $\delta x = 0.001$ we have $3x^2\delta x = 0.012$, which yields δy correct to three significant figures. In effect, the last two terms of (6) are junk.

Nevertheless, there is a practical difference between the junk $(\delta x)^2$ at the end of the equation $\delta(x^2) = 2x\delta x + (\delta x)^2$ and the junk $3x(\delta x)^2 + (\delta x)^3$ at the end of the equation $\delta(x^3) = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$: the junk in $\delta(x^2)$ contains but a single term, whereas the junk in $\delta(x^3)$ contains two terms. Moreover, if you apply the above method to $\delta(x^4)$ or $\delta(x^5)$, then you will find that the number of junk terms increases to three or four, respectively; and in general, the number of junk terms in $\delta(x^n)$ is $n - 1$. It would therefore be a

† On using the identity $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ with $A = x$ and $B = \delta x$.

good idea to have a notation that keeps all that junk in a trash can—without in any way compromising the exactness of our equation.

Now, what the junk terms $(\delta x)^2$ and $3x(\delta x)^2 + (\delta x)^3$ have in common is that they not only are infinitesimal, but also are so small that you can divide either one by δx and still obtain an infinitesimal quantity; specifically, $(\delta x)^2 \div \delta x$ yields the infinitesimal δx, and $\{3x(\delta x)^2 + (\delta x)^3\} \div \delta x$ yields the infinitesimal $3x\delta x + (\delta x)^2$. This is the essence of infinitesimal junk: it remains infinitesimally small even after division by δx. Another way to say exactly the same thing is that if you first divide by δx and then let δx tend to zero, then the limit you obtain will always be zero. We therefore define "$o(\delta x)$"—called "little oh of delta eks"—to mean terms so small that

$$\lim_{\delta x \to 0} \frac{o(\delta x)}{\delta x} \;=\; 0. \tag{7}$$

Thus $(\delta x)^2 = o(\delta x)$ because $(\delta x)^2 \div \delta x = \delta x \to 0$ as $\delta x \to 0$, and $3x(\delta x)^2 + (\delta x)^3 = o(\delta x)$ because $\{3x(\delta x)^2 + (\delta x)^3\} \div \delta x = 3x\delta x + (\delta x)^2 \to 0$ also as $\delta x \to 0$. Note an important point: because $o(\delta x)$ is not unique—countless different expressions can all be $o(\delta x)$—we can *never* write an equation of the form "$o(\delta x)$ = TERMS." We can use $o(\delta x)$ only in equations of the form "TERMS = $o(\delta x)$." In particular,

$$o(\delta x) + o(\delta x) \;=\; o(\delta x). \tag{8}$$

What happens if you add a pile of junk to a pile of junk? You still have a pile of junk!

The upshot of all the above is that in place of (3) or (6) we can now write

$$\delta y \;=\; 2x\,\delta x + o(\delta x) \tag{9}$$

or

$$\delta y \;=\; 3x^2\,\delta x + o(\delta x), \tag{10}$$

respectively, *with absolute precision*: (9)-(10) are not approximate, they are exact! Although they suppress some of the information contained in (3) or (6), this information is of no use for our purposes; therefore, (9)-(10), despite being less informative, are *superior* to (3) or (6) because they are more compact and therefore more elegant—without being imprecise.

Every relationship between variables like (1) or (4) implies a relationship between infinitesimals like (3) or (6) that records the variation with x of the effect on y of a small change in x, and the key to finding this relationship between infinitesimals is always to observe that $x + \delta x$ and $y + \delta y$ satisfy the same relationship between variables as x and y. Furthermore, the relationship between infinitesimals can always be rewritten in the form of (9) or (10). The coefficient of δx on the right-hand side is called the *differential coefficient*; that is,

$$\delta y \;=\; \{\text{DIFFERENTIAL COEFFICIENT}\} \cdot \delta x + o(\delta x). \tag{11}$$

But "differential coefficient" is a lot to have to write: wouldn't it be better to have a simple notation instead? Dividing (11) by δx yields

$$\frac{\delta y}{\delta x} \;=\; \text{DIFFERENTIAL COEFFICIENT} + \frac{o(\delta x)}{\delta x}. \tag{12}$$

Now taking the limit as $\delta x \to 0$ and using (7) yields

$$\begin{aligned}\lim_{\delta x \to 0} \frac{\delta y}{\delta x} &= \text{DIFFERENTIAL COEFFICIENT} + \lim_{\delta x \to 0} \frac{o(\delta x)}{\delta x} \\ &= \text{DIFFERENTIAL COEFFICIENT} + 0\end{aligned}$$

or

$$\text{DIFFERENTIAL COEFFICIENT} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}. \tag{13}$$

We see that the differential coefficient is always the limit as $\delta x \to 0$ (and hence also $\delta y \to 0$) of the ratio $\delta y/\delta x$. So we need a notation that is evocative of this limit, and the traditional choice is dy/dx. That is, we define

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}, \tag{14}$$

‡ It is important to note, however, that whereas δx and δy are both numbers and $\delta y/\delta x$ is their ratio, neither dx nor dy is a number, and therefore dy/dx is not a ratio; rather, it is simply an evocative shorthand for the right-hand side of (14), and hence for the differential coefficient.

We can use (14) to find differential coefficients that are hard to find by purely algebraic manipulation (i.e., without invoking the concept of a limit). Suppose, e.g., that

$$y = \frac{1}{x} \tag{15}$$

or $xy = 1$. Then $(x + \delta x)(y + \delta y) = 1$ as well, implying $xy + \delta x \cdot y + x \cdot \delta y + \delta x \cdot \delta y = 1$. But $xy = 1$, so that $\delta x \cdot y + x \cdot \delta y + \delta x \cdot \delta y = 0$, from which $(x + \delta x)\delta y = -y\delta x$ or

$$\delta y = -\frac{y\,\delta x}{x + \delta x} = -\frac{\delta x}{x(x + \delta x)}. \tag{16}$$

So, from (14) and Lecture 4's limit combination rule, the differential coefficient is

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{-1}{x(x + \delta x)} = \frac{-1}{x(x + 0)} = -\frac{1}{x^2}; \tag{17}$$

and from (11), the corresponding relationship between infinitesimals is

$$\delta y = -\frac{1}{x^2} \cdot \delta x + o(\delta x). \tag{18}$$

Our next example requires the trigonometric identity

$$\sin(A)\cos(B) + \cos(A)\sin(B) = \sin(A + B), \tag{19}$$

both in its general form and in the special case

$$\sin^2(A) + \cos^2(A) = 1 \tag{20}$$

‡So that (11) is more succinctly written as $\delta y = \frac{dy}{dx} \cdot \delta x + o(\delta x)$.

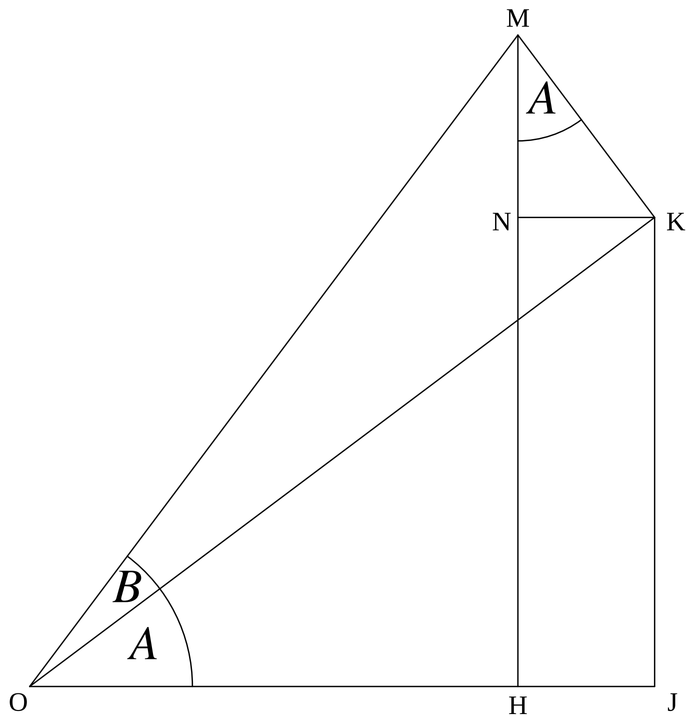

Figure 2: Proof of (19). From the diagram, it is clear that ∠HOK = ∠KMN = A; and so $\sin(A) = \frac{JK}{OK} = \frac{NK}{MK}$, $\cos(A) = \frac{OJ}{OK} = \frac{MN}{MK}$, $\sin(B) = \frac{KM}{OM}$ and $\cos(B) = \frac{OK}{OM}$. Hence $\sin(A+B) = \frac{HM}{OM} = \frac{HN+NM}{OM} = \frac{JK+NM}{OM} = \frac{JK}{OM} + \frac{NM}{OM} = \frac{JK}{OK}\frac{OK}{OM} + \frac{NM}{MK}\frac{MK}{OM} = \sin(A)\cos(B) + \cos(A)\sin(B)$. Similarly, $\cos(A+B) = \frac{OH}{OM} = \frac{OJ-HJ}{OM} = \frac{OJ}{OM} - \frac{NK}{OM} = \frac{OJ}{OK}\frac{OK}{OM} - \frac{NK}{MK}\frac{MK}{OM} = \cos(A)\cos(B) - \sin(A)\sin(B)$.

where $B = \frac{1}{2}\pi - A$.[§] Accordingly, suppose that

$$y \;=\; \sin(x) \tag{21}$$

and hence

$$y + \delta y \;=\; \sin(x + \delta x). \tag{22}$$

Subtracting (21) from (22), then using (19) with $A = x$ and $B = \delta x$, we obtain

$$\begin{aligned} \delta y \;=\; \sin(x+\delta x) - \sin(x) \;&=\; \sin(x)\cos(\delta x) + \cos(x)\sin(\delta x) - \sin(x) \\ &=\; \{\cos(\delta x) - 1\}\sin(x) \;+\; \cos(x)\;\sin(\delta x), \end{aligned} \tag{23}$$

[§]A proof of (19) is sketched in Figure 2. For an alternative proof, see Exercise 7 of Lecture 9.

from which a clever multiplication by 1 and use of (20) with $A = \delta x$ yields

$$\begin{aligned}
\frac{\delta y}{\delta x} &= \frac{\cos(\delta x) - 1}{\delta x}\sin(x) \;+\; \cos(x)\,\frac{\sin(\delta x)}{\delta x} \\
&= \frac{\{\cos(\delta x) + 1\}\{\cos(\delta x) - 1\}}{\{\cos(\delta x) + 1\}\delta x}\sin(x) \;+\; \cos(x)\,\frac{\sin(\delta x)}{\delta x} \\
&= \frac{\{\cos^2(\delta x) - 1\}}{\{\cos(\delta x) + 1\}\delta x}\sin(x) \;+\; \cos(x)\,\frac{\sin(\delta x)}{\delta x} \\
&= \frac{-\sin^2(\delta x)}{\{\cos(\delta x) + 1\}\delta x}\sin(x) \;+\; \cos(x)\,\frac{\sin(\delta x)}{\delta x} \\
&= -\sin(\delta x)\frac{\sin(x)}{\cos(\delta x) + 1}\frac{\sin(\delta x)}{\delta x} \;+\; \cos(x)\,\frac{\sin(\delta x)}{\delta x}.
\end{aligned} \tag{24}$$

On using the limit combination rule, we now readily find the differential coefficient:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\delta x \to 0}\frac{\delta y}{\delta x} \\
&= -\lim_{\delta x \to 0}\sin(\delta x)\lim_{\delta x \to 0}\frac{\sin(x)}{\cos(\delta x) + 1}\lim_{\delta x \to 0}\frac{\sin(\delta x)}{\delta x} \;+\; \cos(x)\lim_{\delta x \to 0}\frac{\sin(\delta x)}{\delta x} \\
&= -\sin(0)\frac{\sin(x)}{\cos(0) + 1}\cdot 1 \;+\; \cos(x)\cdot 1 \\
&= 0\cdot\tfrac{1}{2}\sin(x) + \cos(x) \;=\; \cos(x).
\end{aligned} \tag{25}$$

The corresponding relationship between infinitesimals is

$$\delta y \;=\; \cos(x)\cdot\delta x + o(\delta x). \tag{26}$$

We conclude by discussing a geometrical interpretation of the differential coefficient in terms of the graph of y versus x. For the sake of definiteness, suppose that the graph is hill-shaped, like that in Figure 3a. Then (x, y) and $(x + \delta x, y + \delta y)$ are neighboring points on the graph; δx is the infinitesimal horizontal displacement from (x, y) to $(x + \delta x, y)$; and δy is the infinitesimal change in altitude from $(x + \delta x, y)$ to $(x + \delta x, y + \delta y)$. So the average slope, or average gradient, between these two points is

$$\frac{\text{ALTITUDE CHANGE}}{\text{HORIZONTAL DISPLACEMENT}} \;=\; \frac{\delta y}{\delta x}. \tag{27}$$

This average gradient is the actual gradient of the dashed line in Figure 3a. The closer together the two large dots, the more nearly the dashed line coincides with the dotted line, which is called the *tangent* line at the focal point, because in this neighborhood it meets the curve precisely once, whereas the dashed line—or *chord*—meets it twice. As a consequence, the closer together the two large dots, the more nearly the average chord gradient $\delta y/\delta x$ coincides with the gradient of the tangent. If we allow $x + \delta x$ to become arbitrarily close to x, i.e., if we allow $\delta x \to 0$, then—because both dots lie on the curve—we automatically ensure that $y + \delta y$ becomes arbitrarily close to y, or $\delta y \to 0$, and hence that $\delta y/\delta x$ becomes arbitrarily close to the gradient of the tangent. So

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} \;=\; \text{GRADIENT OF TANGENT LINE AT POINT WITH COORDINATES } (x, y). \tag{28}$$

In other words, (14), i.e.,

$$\frac{dy}{dx} \;=\; \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \tag{29}$$

defines the gradient of the tangent at the point (x, y), which in turn measures the steepness of the hill at this point.

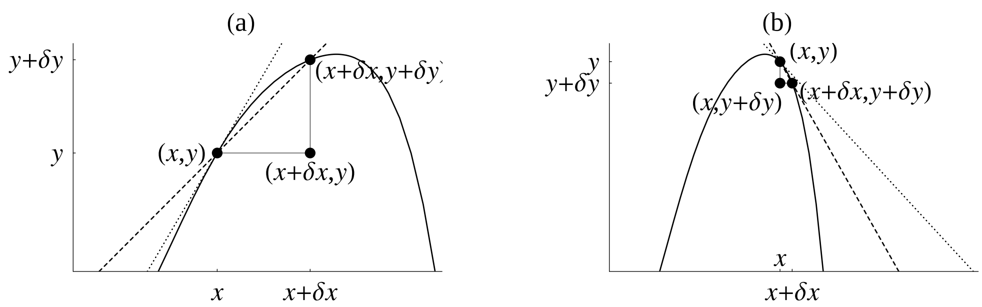

Figure 3: The geometrical interpretation of the differential coefficient as the gradient or slope of the tangent line.

Here several remarks are in order. First, everything we have said about Figure 3a applies with equal force to Figure 3b, where the graph is the same but we have selected a different focal point (x, y). The only difference is that δy is now negative. Because δx is still positive, $\delta y/\delta x$ is now negative; and so, from (29), dy/dx is also negative—unless the focal point is precisely at the summit (and therefore not as drawn in the diagram). What happens in this special case is that, although $\delta y/\delta x$ is strictly negative, it gets arbitrarily close to zero as the two dots on the curve coalesce; and so, from (29), $dy/dx = 0$ (and the tangent line is horizontal).¶ Thus $dy/dx > 0$ going up the hill (from left to right, i.e., in the direction of increasing x), $dy/dx = 0$ at the top, and $dy/dx < 0$ going down the other side. Similar considerations apply to traversing a valley (again from left to right): $dy/dx < 0$ going down, $dy/dx = 0$ at the bottom, and $dy/dx > 0$ going up the other side.

Second, dy/dx is a notation for the gradient or slope of the tangent line to a curve at an arbitrary, variable point with coordinates (x, y)—but we may happen to be especially interested in a particular, fixed point, say the point at which $x = a$. Then we need a new notation to denote the gradient at that particular point. The one we use is

$$\left.\frac{dy}{dx}\right|_{x=a}. \tag{30}$$

Third, we now know everything we need to know to find the equation of the tangent line to a given curve at any particular point. An example will serve best to illustrate the method. Accordingly, suppose that we would like to know the equation of the tangent

¶Thus, $\delta y/\delta x < 0$ *usually* means $dy/dx < 0$, but because it is possible for $\delta y/\delta x < 0$ to mean only that $dy/dx = 0$ (e.g., at a summit), all that we can deduce in general from $\delta y/\delta x < 0$ in the limit as $\delta x \to 0$ is that $dy/dx \leq 0$. Similarly, with regard to the function K in Lectures 2 and 3, all we can deduce from $K(u) < 1$ in the limit as $u \to \infty$ is that $K(\infty) \leq 1$. These results illustrate a more general result, namely, that limiting processes weaken strong inequalities.

line to the curve with equation $y = \sin(x)$ at the point where $x = \frac{1}{6}\pi$, and hence $y = \sin\left(\frac{1}{6}\pi\right) = \frac{1}{2}$. From (25), we find that the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{x=\frac{1}{6}\pi} = \cos(x)\big|_{x=\frac{1}{6}\pi} = \cos\left(\frac{1}{6}\pi\right) = \frac{1}{2}\sqrt{3}. \tag{31}$$

So the equation of the tangent line is that of the line through $\left(\frac{1}{6}\pi, \frac{1}{2}\right)$ with slope m or

$$y - \frac{1}{2} = m\left(x - \frac{1}{6}\pi\right) \implies y = \frac{\sqrt{3}}{2}x - \frac{\pi\sqrt{3}}{12} + \frac{1}{2}, \tag{32}$$

which intersects the axis $y = 0$ at the point where $x = \frac{\pi}{6} - \frac{1}{\sqrt{3}} = -0.054$; see Figure 4.

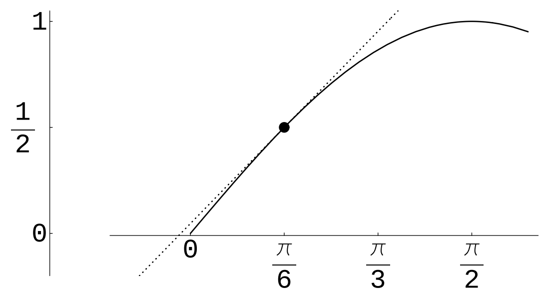

Figure 4: The tangent line to $y = \sin(x)$ at the point with coordinates $\left(\frac{1}{6}\pi, \frac{1}{2}\right)$.

Exercises

1. Find the differential coefficient for $y = \cos(x)$.
Hint: Use the second identity whose proof is sketched in Figure 2.

2. Find the differential coefficient for $y = \dfrac{x}{x+1}$.

3. Find the tangent line to $y = \dfrac{x}{x+1}$ at the point with coordinates $\left(1, \frac{1}{2}\right)$.

4. Find the tangent line to $y = \cos(x)$ at the point with coordinates $\left(\frac{1}{3}\pi, \frac{1}{2}\right)$.

5. Find the tangent line to $y = \sqrt{x}$ at the point with coordinates $(4, 2)$.

6. Let x denote the absolute temperature (in degrees Kelvin) of an ideal radiator or *black body* and let y denote the energy it radiates per unit area per second or power per unit area. Then, according to the Stefan-Boltzmann law, $y = \sigma x^4$ where σ ($\approx 5.67 \times 10^{-8} JK^{-4}m^{-2}s^{-1}$) is the Stefan-Boltzmann constant. How is an infinitesimal increase δy in power per unit area related to an infinitesimal increase δx in temperature? What is the differential coefficient of y with respect to x?

Suitable problems from standard calculus texts

Stewart (2003): p. 156 , ## 7-14.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

Solutions or hints for selected exercises

1. We have both

$$y \;=\; \cos(x)$$

and

$$y + \delta y \;=\; \cos(x + \delta x).$$

Subtracting and using $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ with $A = x$ and $B = \delta x$, we obtain

$$\begin{aligned}\delta y \;=\; \cos(x+\delta x) - \cos(x) \;&=\; \cos(x)\cos(\delta x) - \sin(x)\sin(\delta x) - \cos(x)\\ &=\; \{\cos(\delta x) - 1\}\cos(x) \;-\; \sin(x)\,\sin(\delta x),\end{aligned}$$

from which a clever multiplication by 1 and use of (20) with $A = \delta x$ yields

$$\begin{aligned}\frac{\delta y}{\delta x} \;&=\; \frac{\cos(\delta x) - 1}{\delta x}\cos(x) \;-\; \sin(x)\,\frac{\sin(\delta x)}{\delta x}\\ &=\; \frac{\{\cos(\delta x)+1\}\{\cos(\delta x) - 1\}}{\{\cos(\delta x)+1\}\delta x}\cos(x) \;-\; \sin(x)\,\frac{\sin(\delta x)}{\delta x}\\ &=\; \frac{-\sin^2(\delta x)}{\{\cos(\delta x)+1\}\delta x}\cos(x) \;-\; \sin(x)\,\frac{\sin(\delta x)}{\delta x}\\ &=\; -\sin(\delta x)\frac{\cos(x)}{\cos(\delta x)+1}\frac{\sin(\delta x)}{\delta x} \;-\; \sin(x)\,\frac{\sin(\delta x)}{\delta x}.\end{aligned}$$

From the limit combination rule, we readily find that the differential coefficient is

$$\begin{aligned}\frac{dy}{dx} \;&=\; \lim_{\delta x\to 0}\frac{\delta y}{\delta x}\\ &=\; -\lim_{\delta x\to 0}\sin(\delta x)\,\lim_{\delta x\to 0}\frac{\cos(x)}{\cos(\delta x)+1}\,\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x} \;-\; \sin(x)\,\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x}\\ &=\; -\sin(0)\frac{\cos(x)}{\cos(0)+1}\cdot 1 \;-\; \sin(x)\cdot 1\\ &=\; 0\cdot\tfrac{1}{2}\cos(x) - \sin(x) \;=\; -\sin(x),\end{aligned}$$

and that the corresponding relationship between infinitesimals is

$$\delta y \;=\; -\sin(x)\cdot\delta x \;+\; o(\delta x).$$

2. We have both

$$y = \frac{x}{x+1}$$

and

$$y + \delta y = \frac{x + \delta x}{(x + \delta x) + 1} = \frac{x + \delta x}{x + 1 + \delta x}.$$

Subtracting, we obtain

$$\delta y = \frac{x + \delta x}{x + 1 + \delta x} - \frac{x}{x+1} = \frac{(x + \delta x)(x+1) - (x + 1 + \delta x)x}{(x + 1 + \delta x)(x+1)}$$

which simplifies to

$$\delta y = \frac{\delta x}{(x + 1 + \delta x)(x+1)}$$

so that

$$\frac{\delta y}{\delta x} = \frac{1}{(x + 1 + \delta x)(x+1)}.$$

Now, from the limit combination rule, the differential coefficient is

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{1}{(x + 1 + 0)(x+1)} = \frac{1}{(x+1)^2}$$

and the corresponding relationship between infinitesimals is

$$\delta y = \frac{\delta x}{(x+1)^2} + o(\delta x).$$

3. From the previous exercise we have $\frac{dy}{dx} = \frac{1}{(x+1)^2}$, so the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{1}{(x+1)^2}\right|_{x=1} = \tfrac{1}{4}.$$

Its equation is therefore $y - \frac{1}{2} = \frac{1}{4}(x-1)$ or $y = \frac{1}{4}(x+1)$.

4. From Exercise 1 we have $\frac{dy}{dx} = -\sin(x)$, so the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{1}{(x+1)^2}\right|_{x=1} = \tfrac{1}{4}.$$

Its equation is therefore $y - \frac{1}{2} = \frac{1}{4}(x-1)$ or $y = \frac{1}{4}(x+1)$.

5. For positive x we have both

$$y = \sqrt{x}$$

and

$$y + \delta y = \sqrt{x + \delta x}.$$

Subtracting, we obtain

$$\delta y \;=\; \sqrt{x+\delta x} \;-\; \sqrt{x} \;=\; (\sqrt{x+\delta x} \;-\; \sqrt{x})\cdot 1 \;=\; (\sqrt{x+\delta x} \;-\; \sqrt{x})\frac{\sqrt{x+\delta x} \;+\; \sqrt{x}}{\sqrt{x+\delta x} \;+\; \sqrt{x}}$$

which simplifies to

$$\begin{aligned} \delta y \;&=\; \frac{(\sqrt{x+\delta x} \;-\; \sqrt{x})(\sqrt{x+\delta x} \;+\; \sqrt{x})}{\sqrt{x+\delta x} \;-\; \sqrt{x}} \;=\; \frac{(\sqrt{x+\delta x})^2 \;-\; (\sqrt{x})^2}{\sqrt{x+\delta x} \;+\; \sqrt{x}} \\ &=\; \frac{x+\delta x \;-\; x}{\sqrt{x+\delta x} \;+\; \sqrt{x}} \;=\; \frac{\delta x}{\sqrt{x+\delta x} \;+\; \sqrt{x}} \end{aligned}$$

so that

$$\frac{\delta y}{\delta x} \;=\; \frac{1}{\sqrt{x+\delta x} \;+\; \sqrt{x}}.$$

Now, from the limit combination rule, the differential coefficient is

$$\frac{dy}{dx} \;=\; \lim_{\delta x \to 0}\frac{\delta y}{\delta x} \;=\; \frac{1}{\sqrt{x+0} \;+\; \sqrt{x}} \;=\; \frac{1}{\sqrt{x}+\sqrt{x}} \;=\; \frac{1}{2\sqrt{x}}$$

(which exists as long as $x > 0$) and the corresponding relationship between infinitesimals is

$$\delta y \;=\; \frac{1}{2}\frac{\delta x}{\sqrt{x}} \;+\; o(\delta x).$$

So the slope of the tangent line at (4,2) is

$$m \;=\; \left.\frac{dy}{dx}\right|_{x=4} \;=\; \left.\frac{1}{2\sqrt{x}}\right|_{x=4} \;=\; \tfrac{1}{2\sqrt{4}} \;=\; \tfrac{1}{4}$$

and its equation is $y-2 = \frac{1}{4}(x-4)$ or $y = \frac{1}{4}x+1$.

6. $$\delta y \;=\; 4\sigma x^3\,\delta x + 6\sigma x^2\,\delta x^2 + 4\sigma x\,\delta x^3 + \sigma\delta x^4 \;=\; 4\sigma x^3\,\delta x + o(\delta x) \Longrightarrow \frac{dy}{dx} \;=\; 4\sigma x^3.$$