10. Finite sums and infinite series

You already know what a sequence is, and from any sequence you can induce another by adding the first n terms. Usually, we use a lower-case letter for the original sequence and an upper-case letter for the induced sequence, i.e., the sequence of finite sums; thus, if the original sequence is $\{s_k\}$, then the sequence of finite sums will be $\{S_n\}$ defined by

$$S_n \;=\; s_1 + s_2 + \ldots + s_{n-1} + s_n \;=\; \sum_{k=1}^{n} s_k \tag{1}$$

(using the summation notation introduced at the end of Lecture 3). Often there's a trick for obtaining a simple expression for S_n, in other words, a formula for the finite sum. For example, if $s_k = k$ then we can write the sum twice as follows, first forwards, then backwards:

$$\begin{aligned} S_n &= 1 + \quad 2 \quad + \quad 3 \quad + \ldots + (n-2) + (n-1) + n \;=\; \sum_{k=1}^{n} k \\ S_n &= n + (n-1) + (n-2) + \ldots + \quad 3 \quad + \quad 2 \quad + 1 \;=\; \sum_{k=1}^{n} \{n - (k-1)\}. \end{aligned} \tag{2}$$

Now, if we add each term to the one directly above, then we find that the result is $n+1$ in all n cases. So twice S_n must sum to n times $n+1$. That is, $2S_n = n(n+1)$ or

$$S_n \;=\; 1 + 2 + 3 + \ldots + n \;=\; \tfrac{1}{2}n(n+1). \tag{3}$$

For example, $1+2+3+4+\ldots+99+100 = \frac{1}{2} \times 100 \times 101 = 5050$. But that particular trick in essence works only for the example on which we have used it.

A trick that works much more often is to rewrite s_k as the difference between the k-th and the $(k-1)$-th term of a judiciously chosen different sequence, say $\{p_k\}$. For if

$$s_k \;=\; p_k - p_{k-1} \tag{4}$$

for all $k = 1, \ldots, n$ then*

$$\begin{aligned} S_n &= s_n + s_{n-1} + s_{n-2} + s_{n-3} + \ldots + s_4 + s_3 + s_2 + s_1 \\ &= (p_n - p_{n-1}) + (p_{n-1} - p_{n-2}) + (p_{n-2} - p_{n-3}) + (p_{n-2} - p_{n-3}) + \ldots \\ &\qquad + \; (p_4 - p_3) + (p_3 - p_2) + (p_2 - p_1) + (p_1 - p_0) \\ &= p_n + (-p_{n-1} + p_{n-1}) + (-p_{n-2} + p_{n-2}) + (-p_{n-3} + p_{n-3}) + \ldots \\ &\qquad + \; (-p_3 + p_3) + (-p_2 + p_2) + (-p_1 + p_1) - p_0 \\ &= p_n - p_0 \end{aligned} \tag{5}$$

because all terms cancel in pairs, except for the very first and last.

*Note that (4) implies $\{p_k\} = \{p_k \mid k \geq 0\}$, whereas $\{s_k\} = \{s_k \mid k \geq 1\}$

Suppose, for example, that we wish to calculate the finite sum

$$\tfrac{1}{3} + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^3 + \ldots + \left(\tfrac{1}{3}\right)^n \;=\; \sum_{k=1}^{n} \left(\tfrac{1}{3}\right)^k, \tag{6}$$

for which $s_k = \left(\frac{1}{3}\right)^k$. Here a judicious choice is

$$p_k \;=\; -\tfrac{3}{2} \cdot \left(\tfrac{1}{3}\right)^{k+1}, \tag{7}$$

because now

$$\begin{aligned} p_k - p_{k-1} \;&=\; -\tfrac{3}{2}\left(\tfrac{1}{3}\right)^{k+1} + \tfrac{3}{2}\left(\tfrac{1}{3}\right)^k \;=\; \tfrac{3}{2} \cdot \left\{-\tfrac{1}{3} + 1\right\}\left(\tfrac{1}{3}\right)^k \\ &=\; \tfrac{3}{2} \cdot \tfrac{2}{3} \cdot \left(\tfrac{1}{3}\right)^k \;=\; \left(\tfrac{1}{3}\right)^k \;=\; s_k \end{aligned} \tag{8}$$

and (5) implies

$$S_n \;=\; p_n - p_0 \;=\; -\tfrac{3}{2} \cdot \left(\tfrac{1}{3}\right)^{n+1} + \tfrac{3}{2} \cdot \left(\tfrac{1}{3}\right)^{0+1} \;=\; \tfrac{1}{2}\left\{1 - \left(\tfrac{1}{3}\right)^n\right\}. \tag{9}$$

In other words,

$$\sum_{k=1}^{n} \left(\tfrac{1}{3}\right)^k \;=\; \tfrac{1}{2}\left\{1 - \left(\tfrac{1}{3}\right)^n\right\}. \tag{10}$$

For example, $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5 + \left(\frac{1}{3}\right)^6 + \left(\frac{1}{3}\right)^7 = \frac{1}{2}\left\{1 - \left(\frac{1}{3}\right)^7\right\} = \frac{1093}{2187}$.

An even more judicious choice, namely,

$$p_k \;=\; \frac{ar^k}{r-1} \tag{11}$$

yields the standard formula for the sum of a finite geometric sum of n terms with first term a and ratio r, where $r \neq 1$. For (11) implies

$$p_k - p_{k-1} \;=\; \frac{ar^k}{r-1} - \frac{ar^{k-1}}{r-1} \;=\; \frac{ar^k - ar^{k-1}}{r-1} \;=\; \frac{ar^{k-1}(r-1)}{r-1} \;=\; ar^{k-1}. \tag{12}$$

So, on setting

$$s_k \;=\; ar^{k-1} \tag{13}$$

in (5) we obtain

$$S_n \;=\; p_n - p_0 \;=\; \frac{ar^n}{r-1} - \frac{ar^0}{r-1} \;=\; \frac{a(r^n-1)}{r-1}. \tag{14}$$

In other words,

$$\sum_{k=1}^{n} ar^{k-1} \;=\; a + ar + ar^2 + ar^3 \ldots + ar^{n-2} + ar^{n-1} \;=\; \frac{a(r^n-1)}{r-1}, \quad r \neq 1. \tag{15}$$

For $r = 1$ we need no tricks:

$$\sum_{k=1}^{n} a\,1^{k-1} \;=\; a + a + a + a \ldots + a + a(n \text{ times}) \;=\; na. \tag{16}$$

$\sum_{k=1}^{n} k$	$=$	$1+2+3+\ldots+n$	$=$	$\frac{1}{2}n(n+1)$
$\sum_{k=1}^{n} k^2$	$=$	$1^2+2^2+3^2+\ldots+n^2$	$=$	$\frac{1}{6}n(n+1)(2n+1)$
$\sum_{k=1}^{n} k^3$	$=$	$1^3+2^3+3^3+\ldots+n^3$	$=$	$\frac{1}{4}n^2(n+1)^2$
$\sum_{k=1}^{n} k^4$	$=$	$1^4+2^4+3^4+\ldots+n^4$	$=$	$\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$
$\sum_{k=1}^{n} k^5$	$=$	$1^5+2^5+3^5+\ldots+n^5$	$=$	$\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$
$\sum_{k=1}^{n} k^6$	$=$	$1^6+2^6+3^6+\ldots+n^6$	$=$	$\frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$

Table 1: The sums of the powers of the first n integers for the first six integer exponents.

In sum:

$$\sum_{k=1}^{n} ar^{k-1} = \begin{cases} \frac{a(r^n-1)}{r-1} & \text{if} \quad r \neq 1 \\ na & \text{if} \quad r = 1. \end{cases} \tag{17}$$

Of course, (10) is the special case for which $n = 7$ and $a = r = \frac{1}{3}$.

The only thing about the trick that is the slightest bit, well, tricky is judiciously guessing p_k. But for all of the results in Table 1—some of which are needed in Lecture 12—the judicious choice of p_k always turns out to be just S_k itself ... which you know, because you know S_n from the table. Now you know everything you need to know to complete Exercise 1 by yourself.

In many of the above cases, s_k increases with k, and so $\{s_k\}$ and $\{S_n\}$ both diverge. If $s_k \to 0$ as $k \to \infty$, however, it is possible for the sequence $\{S_n\}$ to converge to a limit, which we denote by S_∞. That is,

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} s_k. \tag{18}$$

Then the right-hand side of (18) is called an *infinite series* and is usually written as

$$\text{either} \quad \sum_{k=1}^{\infty} s_k \quad \text{or} \quad s_1 + s_2 + s_3 + s_4 + \ldots \tag{19}$$

The left-hand side of (18) is the *sum* to which this infinite series converges. For example, because†

$$\lim_{n\to\infty} r^n = 0 \quad \text{whenever } |r| < 1, \tag{20}$$

†See Equation (22) of Lecture 3.

(15) implies that

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \dots$$
$$= \lim_{n\to\infty} \frac{a(r^n - 1)}{r - 1} = \frac{a(0-1)}{r-1} = \frac{a}{1-r} \quad \text{whenever } |r| < 1 \tag{21}$$

(on using the limit combination rule). Thus the infinite geometric series with first term a and ratio r converges to the sum $\frac{a}{1-r}$ whenever $|r| < 1$ (but diverges whenever $|r| \geq 1$). In particular, for $a = r = \frac{1}{3}$ we obtain

$$\tfrac{1}{3} + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^3 + \left(\tfrac{1}{3}\right)^4 + \dots = \tfrac{1}{2}. \tag{22}$$

As a further example, consider the infinite series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}. \tag{23}$$

Here $s_k = \frac{1}{k(k+1)} = \frac{k}{k+1} - \frac{k-1}{k}$. So with $p_k = \frac{k}{k+1}$ in (4)-(5) we obtain

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = p_n - p_0 = \frac{n}{n+1} - 0 = \frac{n}{n+1} \tag{24}$$

and thus deduce from (18)-(19) that

$$S_\infty = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1. \tag{25}$$

In other words, $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \dots$ (forever) $= 1$.

Exercises

1. Verify Table 1.

2. Show that

$$\frac{1}{1.2.4} + \frac{1}{2.3.5} + \frac{1}{3.4.6} + \frac{1}{4.5.7} + \dots = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+3)} = \tfrac{7}{36}.$$

Hint: A judicious choice is $p_k = \frac{k(7k^2+42k+59)}{36(k+1)(k+2)(k+3)}$.

Suitable problems from standard calculus texts

Stewart (2003): p. 720, ## 14, 15, 17-20 and 26 (for which use $p_n = \frac{n(5n+13)}{6(n+2)(n+3)}$).

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.