13. Index versus ordinary functions

The definite integral is an example of an index. Although it is largely the only index we shall ever use, it will help in the long run if we first understand not only what an index means in more general terms, but also the relationship between index functions and ordinary functions. That is our goal in this lecture.

A function, as you know, is a rule that labels things unambiguously. The labels are always numbers (at least in this course); and for ordinary functions or sequences, the things are numbers, too. But in general things need not be numbers. In particular, things may be pairs consisting of an ordinary function and a possible subdomain. The function is then an *index* function, and the label it assigns is an index.* So an index function's job, in principle, is to assign labels to function-subdomain pairs. In practice, however, if f is a function and $[a, b]$ is a subdomain, then it is simpler to think of the index function's domain not as a set of all possible $(f, [a, b])$ pairs but rather as a set of all possible (f, a, b) triples—if only because it saves the hassle of repeatedly writing "[" and "]".†

As far as we are concerned, by far the most important example of an index is the definite integral introduced in Lecture 11; however, a couple of simpler examples will serve to clarify the general idea. Consider, therefore, the index functions $\min$ and $\max$, which label function-subdomain triples with the function's greatest and least values, respectively, on the subdomain. So $\min$ is defined by

$$\min(f, a, b) \;=\; f(t_{\min}) \tag{1}$$

where $t_{\min}$ is *any* global minimizer of f on $[a, b]$; and $\max$ is defined by

$$\max(f, a, b) \;=\; f(t_{\max}) \tag{2}$$

where $t_{\max}$ is any global maximizer. For example, Figure 1 shows inflow $y = v(t)$ in Lecture 1's cardiac cycle, from which $\min(v, 0, 0.9) = v(t_{\min}) \approx v(0.14) \approx -470$ml/s; and $\max(v, 0, 0.9) = v(t_{\max}) \approx v(0.52) \approx 296$, whereas $\max(v, 0, 0.35) \approx v(0.33) \approx 27$. Of course, this is hardly new. We discovered long ago, in Lecture 1, that global extrema are properties of both a function and its domain; however, $\max$ and $\min$ formalize this idea by making the dependence on domain explicit.

Moreover, Figure 1 highlights the essential difference between an ordinary function and an index function, which is that an ordinary function, say f, yields *local* properties whereas an index function, such as $\min$ or $\max$, yields *global* properties. For $t \in [a, b]$, f can ignore absolutely every number in $[a, b]$ except t, yet still supply the label $f(t)$. By contrast, $\min$ and $\max$ describe overall properties of f on $[a, b]$, and must pay attention to the entire interval in order to be able to do their job. Despite this difference, there is an important relationship between index functions and ordinary functions, because ind(f, a, b)

*Thus index function and index bear the same relation to one another as derivative and differential coefficient. Index functions are also called functionals, and ordinary functions are also called point functions.

†We say "a set of all possible (f, a, b) triples" rather than "the set of all possible (f, a, b) triples" because in principle the set may be different for different index functions. For example, the domain of $\max$ consists of functions with maxima paired with possible subdomains, the domain of $\min$ consists of functions with minima paired with possible subdomains, and these two domains are not the same; in particular, for K defined in Lecture 2, $(K, 0, 1)$ belongs to the domain of $\min$, but not to that of $\max$.

generates an ordinary function if we either hold both f and a fixed while varying b or (less commonly) hold both f and b fixed while varying a. For example, if $v(t)$ denotes ventricular inflow at time t in our cardiac cycle, and if $v_{\min}(t)$ and $v_{\max}(t)$ denote the least and greatest such inflow achieved since the cycle began, then ordinary functions $v_{\min}$ and $v_{\max}$ are generated by ind = $\min$ and ind = $\max$ according to

$$v_{\min}(t) \;=\; \min(v, 0, t), \qquad v_{\max}(t) \;=\; \max(v, 0, t). \tag{3}$$

The graphs of $v_{\min}$ and $v_{\max}$ are sketched in Figure 1.

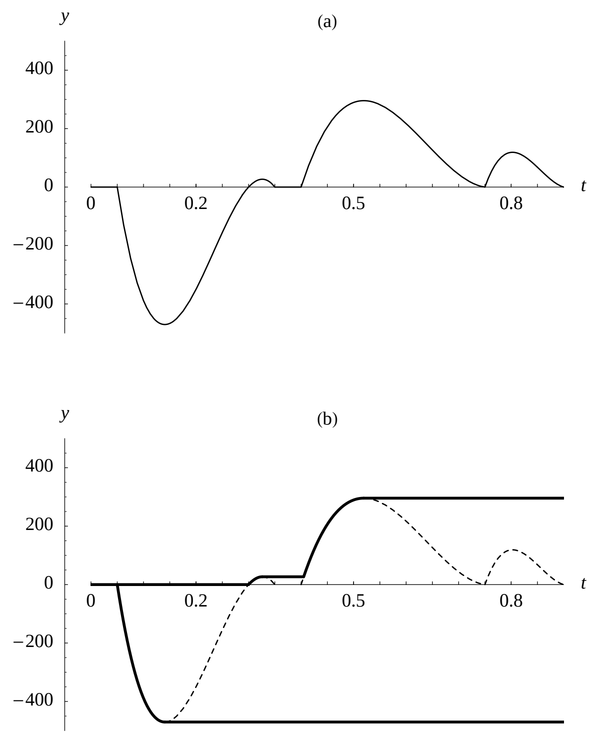


Figure 1: (a) Ventricular inflow $y = v(t)$ in our cardiac cycle. (b) The lower solid curve represents $y = v_{\min}(t)$, the least inflow so far. The upper solid curve represents $y = v_{\max}(t)$, the greatest inflow so far. The dashed curve is $y = v(t)$, the same curve as is shown solid in (a).

In exactly the same way, we can generate an ordinary function from the index function $\int$ that yields the definite integral $\int_a^b f(x)\,dx$ by holding both f and a fixed in $\int(f, a, b)$ while varying b. That is, for any fixed a and f we can define an ordinary function F by

$$F(t) \;=\; \left\{ \begin{array}{c} \text{total signed area between the graph of } f \text{ and} \\ \text{segment } [a, t] \text{ of the horizontal axis, counted} \\ \text{positively above the axis, negatively below it.} \end{array} \right. \;=\; \int_a^t f(x)\,dx. \tag{4}$$

Note an *extremely* important point: here x denotes a generic THING in the domain of f, whereas t denotes a generic THING in the domain of F—and we *must* use different letters, because the right-hand boundary of the defined signed area is always at $x = t$.

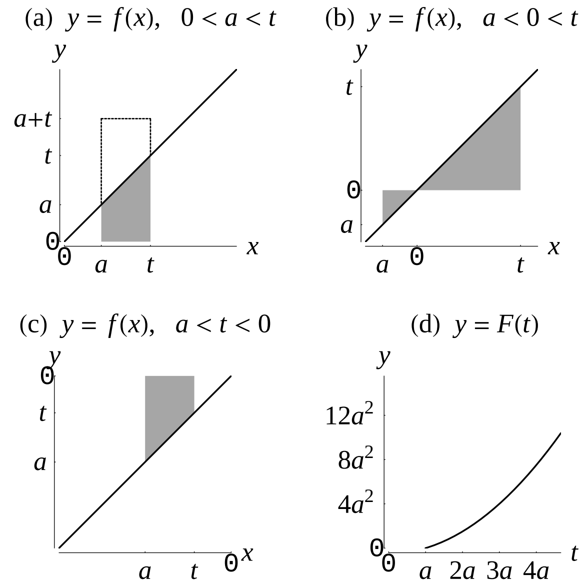


Figure 2: Calculation of $F(t)$ defined by (4).

For the sake of definiteness, suppose that f is defined on $[a, \infty)$ by

$$f(x) \;=\; x. \tag{5}$$

There are two methods by which we can recover an explicit expression for (4) from (5). The first method begins with the observation that for $0 < a < t$, $\int_a^t f(x)\,dx$ is the shaded trapezoidal area in Figure 2a, which is half that of the indicated rectangle, all counted positively. So $F(t) = \frac{1}{2}(t-a)(t+a) = \frac{1}{2}(t^2-a^2)$. For $a < 0 < t$, on the other hand, $\int_a^t f(x)\,dx$ is the area of the larger shaded triangle in Figure 2b *minus* the area of the smaller shaded triangle, or $F(t) = \frac{1}{2}t^2 - \frac{1}{2}a^2$. Finally, for $a < t < 0$, $\int_a^t f(x)\,dx$ is the *negative* of the area of the shaded trapezium in Figure 2c, or $-\frac{1}{2}|t-a|(|t|+|a|) = -\frac{1}{2}(t-a)(\{-t\}+\{-a\}) = \frac{1}{2}(t^2-a^2)$, as before. In other words, in every case,

$$F(t) \;=\; \tfrac{1}{2}(t^2 - a^2). \tag{6}$$

The graph of this function is plotted in Figure 2d. Note that F *must* be strictly increasing, because the shaded area always gets bigger as you move to the right in Figures 2a-c. The second method for recovering (6) is simply to set $b = t$ in the second of the four special results we obtained in Lectures 11-12, namely,

$$\int_a^b 1\,dx = b-a, \quad \int_a^b x\,dx = \tfrac{b^2-a^2}{2}, \quad \int_a^b x^2\,dx = \tfrac{b^3-a^3}{3}, \quad \int_a^b x^3\,dx = \tfrac{b^4-a^4}{4}. \tag{7}$$

Then (6) is an immediate consequence of (5). Needless to say, the second method is quicker—but the first is more instructive, which is why we have included it.

An important example of an ordinary function generated by the index function $\int$ is ventricular volume in our cardiac cycle. Recall from Lecture 11 that the true net recharge between $t = 0.28$ and $t = 0.35$ equalled the total signed area between the horizontal axis and the graph of ventricular inflow v on [0.28, 0.35]. But the true net recharge, i.e., the net volume of blood transported into the ventricle between $t = 0.28$ and $t = 0.35$ is also the amount by which the volume in the ventricle increases. So if $V(t)$ denotes the volume of fluid in the ventricle at time t, then we have

$$\int_{0.28}^{0.35} v(t)\,dt \;=\; L \;=\; V(0.35) - V(0.28). \tag{8}$$

A few moments' thought now reveals, however, that the truth of the above relationship between volume and inflow does not depend in any way on having chosen [0.28,0.35] as the interval of time on which to calculate true net recharge: for any other interval $[a, b]$, and by precisely the same argument as we used above, the true net recharge is the total signed area and increases the volume of blood in the ventricle from $V(a)$ to $V(b)$. That is,

$$\int_{a}^{b} v(t)\,dt \;=\; V(b) - V(a) \tag{9}$$

or, equivalently,

$$V(b) \;=\; V(a) + \int_{a}^{b} v(t)\,dt \tag{10}$$

or—which is exactly the same thing—

$$V(b) \;=\; V(a) + \int_{a}^{b} v(x)\,dx. \tag{11}$$

Allowing $b = t$ to vary, we obtain the ordinary function V defined by

$$V(t) \;=\; V(a) + \int_{a}^{t} v(x)\,dx. \tag{12}$$

Physiologically speaking, ventricular volume at any later time t equals ventricular volume at any earlier time a plus subsequent net recharge.

To obtain an explicit expression for $V(t)$ in the case where $a = 0$, we first recall from Lecture 2 that the initial ventricular volume is

$$V(0) \;=\; 120 \tag{13}$$

millileters and that inflow is defined by[‡]

$$v(x) \;=\; \begin{cases} 0 & \text{if } 0 \le x < 0.05 \\ \frac{2450}{3} - \frac{192500}{9}x + \frac{980000}{9}x^2 - \frac{1400000}{9}x^3 & \text{if } 0.05 \le x < 0.35 \\ 0 & \text{if } 0.35 \le x < 0.4 \\ -\frac{489600}{49} + \frac{57854400}{1127}x - \frac{4080000}{49}x^2 + \frac{48960000}{1127}x^3 & \text{if } 0.4 \le x < 0.75 \\ -126000 + \frac{4900000}{11}x - \frac{1568000}{3}x^2 + \frac{2240000}{11}x^3 & \text{if } 0.75 \le x \le 0.9. \end{cases} \tag{14}$$

[‡]On rewriting (11) from Lecture 2 as a cubic polynomial join.

Then we use our general results from Lecture 11, namely,

$$\int_a^b f(x)\,dx \;=\; \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx \tag{15}$$

for any c such that $a \le c \le b$ and

$$\int_a^b \{q\,f(t) + k\,g(t)\}\,dt \;=\; q\int_a^b f(t)\,dt \;+\; k\int_a^b g(t)\,dt \tag{16}$$

for any (real) q and k, together with (7) and the obvious result that no area is enclosed between the axis and itself or

$$\int_a^b 0\,dx \;=\; 0. \tag{17}$$

For $0 \le t \le 0.05$ we obtain

$$V(t) \;=\; V(0) + \int_0^t v(x)\,dx \;=\; V(0) + \int_0^t 0\,dx \;=\; 120 + 0 = 120. \tag{18}$$

Because this result holds for all $t \in [0, 0.05]$, it implies in particular that $V(0.05) = 120$.

For $0.05 \le t \le 0.35$ we use (15) with $f = v$, $a = 0$, $c = 0.05$, $b = t$, (7) with $a = 0.05$, $b = t$ and (12) with $t = 0.05$:

$$\begin{aligned}
V(t) \;&=\; V(0) \;+ \int_0^t v(x)\,dx \;=\; V(0) \;+ \int_0^{0.05} v(x)\,dx \;+ \int_{0.05}^t v(x)\,dx \\
&=\; V(0.05) + \int_{0.05}^t v(x)\,dx \\
&=\; V(0.05) + \int_{0.05}^t \left\{\tfrac{2450}{3} - \tfrac{192500}{9}x + \tfrac{980000}{9}x^2 - \tfrac{1400000}{9}x^3\right\}dx \\
&=\; 120 + \tfrac{2450}{3}\int_{0.05}^t 1\,dx - \tfrac{192500}{9}\int_{0.05}^t x\,dx + \tfrac{980000}{9}\int_{0.05}^t x^2\,dx - \tfrac{1400000}{9}\int_{0.05}^t x^3\,dx \\
&=\; 120 + \tfrac{2450}{3}(t - 0.05) - \tfrac{192500}{9}\cdot\tfrac{t^2 - 0.05^2}{2} + \tfrac{980000}{9}\cdot\tfrac{t^3 - 0.05^3}{3} - \tfrac{1400000}{9}\cdot\tfrac{t^4 - 0.05^4}{4} \\
&=\; \tfrac{43895}{432} + \tfrac{2450}{3}t - \tfrac{96250}{9}t^2 + \tfrac{980000}{27}t^3 - \tfrac{350000}{9}t^4
\end{aligned} \tag{19}$$

after simplification. Because this result holds for all $t \in [0.05, 0.35]$, it implies in particular that $V(0.35) = \frac{43895}{432} + \frac{2450}{3} \times 0.05 - \frac{96250}{9} \times 0.05^2 + \frac{980000}{27} \times 0.05^3 - \frac{350000}{9} \times 0.05^4 = 50$.

For $0.35 \le t \le 0.4$ we similarly obtain

$$\begin{aligned}
V(t) \;&=\; V(0) \;+ \int_0^t v(x)\,dx \;=\; V(0) \;+ \int_0^{0.35} v(x)\,dx \;+ \int_{0.35}^t v(x)\,dx \\
&=\; V(0.35) + \int_{0.35}^t v(x)\,dx \\
&=\; 50 + \int_{0.35}^t 0\,dx \;=\; 50 + 0 = 50.
\end{aligned} \tag{20}$$

In particular, $V(0.4) = 50$. Similarly, for $0.4 \leq t \leq 0.75$ we obtain $V(t) \;=$

$$
\begin{aligned}
V(0) \;+ \int_0^t v(x)\,dx \;&=\; V(0) \;+ \int_0^{0.4} v(x)\,dx \;+ \int_{0.4}^t v(x)\,dx \\
&= V(0.4) + \int_{0.4}^t v(x)\,dx \\
&= V(0.4) + \int_{0.4}^t \left\{-\tfrac{489600}{49} + \tfrac{57854400}{1127}x - \tfrac{4080000}{49}x^2 + \tfrac{48960000}{1127}x^3\right\} dx \\
&= 50 - \tfrac{489600}{49}\int_{0.4}^t 1\,dx + \tfrac{57854400}{1127}\int_{0.4}^t x\,dx - \tfrac{4080000}{49}\int_{0.4}^t x^2\,dx + \tfrac{48960000}{1127}\int_{0.4}^t x^3\,dx \\
&= 50 - \tfrac{489600}{49}(t - 0.4) + \tfrac{57854400}{1127}\cdot\tfrac{t^2-0.4^2}{2} - \tfrac{4080000}{49}\cdot\tfrac{t^3-0.4^3}{3} + \tfrac{48960000}{1127}\cdot\tfrac{t^4-0.4^4}{4} \\
&= \tfrac{1620894}{1127} - \tfrac{489600}{49}t + \tfrac{28927200}{1127}t^2 - \tfrac{1360000}{49}t^3 + \tfrac{12240000}{1127}t^4
\end{aligned}
\tag{21}
$$

after simplification. Because this result holds for all $t \in [0.4, 0.75]$, it implies in particular that $V(0.75) = \frac{1620894}{1127} - \frac{489600}{49} \times 0.75 + \frac{28927200}{1127} \times 0.75^2 - \frac{1360000}{49} \times 0.75i^3 + \frac{12240000}{1127} \times 0.75^4 = \frac{219}{2}$. Finally, for $0.75 \leq t \leq 0.9$ we obtain $V(t) \;=$

$$
\begin{aligned}
V(0) \;+ \int_0^t v(x)\,dx \;&=\; V(0) \;+ \int_0^{0.75} v(x)\,dx \;+ \int_{0.75}^t v(x)\,dx \\
&= V(0.75) + \int_{0.75}^t v(x)\,dx \\
&= V(0.75) + \int_{0.75}^t \left\{-126000 + \tfrac{4900000}{11}x - \tfrac{1568000}{3}x^2 + \tfrac{2240000}{11}x^3\right\} dx \\
&= \tfrac{219}{2} - 126000\int_{0.75}^t 1\,dx + \tfrac{4900000}{11}\int_{0.75}^t x\,dx - \tfrac{1568000}{3}\int_{0.75}^t x^2\,dx + \tfrac{2240000}{11}\int_{0.75}^t x^3\,dx \\
&= \tfrac{219}{2} - 126000\,(t - 0.75) + \tfrac{4900000}{11}\cdot\tfrac{t^2-0.75^2}{2} - \tfrac{1568000}{3}\cdot\tfrac{t^3-0.75^3}{3} + \tfrac{2240000}{11}\cdot\tfrac{t^4-0.75^4}{4} \\
&= \tfrac{293892}{11} - 126000t + \tfrac{2450000}{11}t^2 - \tfrac{1568000}{9}t^3 + \tfrac{560000}{11}t^4
\end{aligned}
\tag{22}
$$

after simplification; note in particular that $V(0.9) = \frac{293892}{11} - 126000 \times 0.75 + \frac{2450000}{11} \times 0.75^2 - \frac{1568000}{9} \times 0.75^3 + \frac{560000}{11} \times 0.75^4 = 120$. In sum,

$$
V(t) \;=\; \begin{cases}
120 & \text{if } \; 0 \leq t < 0.05 \\
\frac{43895}{432} + \frac{2450}{3}t - \frac{96250}{9}t^2 + \frac{980000}{27}t^3 - \frac{350000}{9}t^4 & \text{if } \; 0.05 \leq t < 0.35 \\
50 & \text{if } \; 0.35 \leq t < 0.4 \\
\frac{1620894}{1127} - \frac{489600}{49}t + \frac{28927200}{1127}t^2 - \frac{1360000}{49}t^3 + \frac{12240000}{1127}t^4 & \text{if } \; 0.4 \leq t < 0.75 \\
\frac{293892}{11} - 126000t + \frac{2450000}{11}t^2 - \frac{1568000}{9}t^3 + \frac{560000}{11}t^4 & \text{if } \; 0.75 \leq t \leq 0.9,
\end{cases}
\tag{23}
$$

the result we claimed in Lecture 2.

Exercises

1. Let $f(t)$ denote the rate at which blood is discharged into the aorta at time t during the systolic phase of a human cardiac cycle. If the cycle begins at $t = 0$, then the discharge at time t—i.e., the volume of blood discharged by time t—is $F(t) = \int_0^t f(x)\,dx$. If the outflow is (very crudely) modelled as the piecewise-linear join defined by

$$f(t) \;=\; \begin{cases} 0 & \text{if} \quad 0 \le t < \frac{1}{20} \\ 465(20t-1) & \text{if} \quad \frac{1}{20} \le t < \frac{1}{10} \\ 465 & \text{if} \quad \frac{1}{10} \le t < \frac{3}{20} \\ 310(3-10t) & \text{if} \quad \frac{3}{20} \le t < \frac{3}{10}, \end{cases}$$

calculate $F(t)$ for all $t \in [0, \frac{3}{10}]$.

2. Functions g and G are defined on $[0,3]$ by

$$g(t) \;=\; \begin{cases} 4-3t & \text{if} \quad 0 \le t < 1 \\ 2-t^3 & \text{if} \quad 1 \le t \le 3 \end{cases} \qquad \text{and} \qquad G(t) \;=\; \int_0^t g(x)\,dx.$$

Verify that g is continuous, and find an explicit formula for $G(t)$ for all $t \in [0,3]$.

3. Functions g and G are defined on $[0,4]$ by

$$g(t) \;=\; \begin{cases} 4-t^2 & \text{if} \quad 0 \le t < 1 \\ t^3+2 & \text{if} \quad 1 \le t < 3 \\ 10t-1 & \text{if} \quad 3 \le t \le 4 \end{cases} \qquad \text{and} \qquad G(t) \;=\; \int_0^t g(x)\,dx.$$

Verify that g is continuous, and find an explicit formula for $G(t)$ for all $t \in [0,4]$.

4. Functions ξ and ϕ are defined on $[0,3]$ by

$$\xi(t) \;=\; \begin{cases} -2 & \text{if} \quad 0 \le t < 1 \\ 7t-9 & \text{if} \quad 1 \le t < 2 \\ 3t-1 & \text{if} \quad 2 \le t \le 3 \end{cases} \qquad \text{and} \qquad \phi(t) \;=\; \int_0^t \xi(x)\,dx.$$

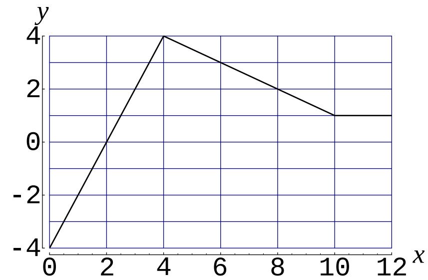
Verify that ξ is continuous, find an explicit formula for $\phi(t)$ for all $t \in [0,3]$ and plot the graphs of ξ and ϕ.

5. Functions ξ and ϕ are defined on $[0,\infty)$ by

$$\xi(t) \;=\; \begin{cases} 4-2t & \text{if} \quad 0 \le t < 3 \\ t-5 & \text{if} \quad 3 \le t < 6 \\ 1 & \text{if} \quad 6 \le t < \infty \end{cases} \qquad \text{and} \qquad \phi(t) \;=\; \int_0^t \xi(x)\,dx.$$

Verify that ξ is continuous, find an explicit formula for $\phi(t)$ for all $t \in [0,3]$ and plot the graphs of ξ and ϕ.

6. A piecewise-linear function g is defined by the graph $y = g(x)$ shown below.



A function G is defined on $[0, 12]$ by $G(t) = \int_0^t g(x)\, dx$. Obtain an explicit formula for $G(t)$.

7. Functions g and G are defined on $[0, 3]$ by

$$g(t) = \begin{cases} 5 - 8t & \text{if} \quad 0 \le t < 1 \\ t^3 - 4 & \text{if} \quad 1 \le t < 2 \\ 2t^2 + 3t - 10 & \text{if} \quad 2 \le t \le 3 \end{cases} \qquad \text{and} \qquad G(t) = \int_0^t g(x)\, dx.$$

Verify that g is continuous, and find an explicit formula for $G(t)$ for all $t \in [0, 3]$.

8. A piecewise-linear function f is defined on $[0, 6]$ by

$$f(x) = \begin{cases} 2 & \text{if} \quad 0 \le x < 1 \\ 2x & \text{if} \quad 1 \le x < 2 \\ 8 - 2x & \text{if} \quad 2 \le x < 4 \\ 0 & \text{if} \quad 4 \le x \le 6. \end{cases}$$

The functions F, L and U are defined on the same domain by $F(t) = \int_0^t f(x)\, dx$, $L(t) = \min(f, 0, t)$ and $U(t) = \max(f, 0, t)$.

(a) Find explicit expressions for $f(t)$, $L(t)$ and $U(t)$, for all $t \in [0, 6]$.

(b) Sketch the graphs of f, L and U, distinguishing them clearly.

(c) Show that $\int_0^3 f(x)\, dx = 8$ and calculate both $\int_0^6 L(t)\, dt$ and $\int_0^6 U(t)\, dt$.

9. A piecewise-linear function f is defined on $[0, 7]$ by

$$f(x) = \begin{cases} 3 + 2x & \text{if} \quad 0 \le x < 2 \\ 13 - 3x & \text{if} \quad 2 \le x < 4 \\ x - 3 & \text{if} \quad 4 \le x < 5 \\ 2 & \text{if} \quad 5 \le x \le 7. \end{cases}$$

The functions F, L and U are defined on the same domain by $F(t) = \int_0^t f(x)\, dx$, $L(t) = \min(f, 0, t)$ and $U(t) = \max(f, 0, t)$.

(a) Find explicit expressions for $f(t)$, $L(t)$ and $U(t)$, for all $t \in [0, 6]$.

(b) Sketch the graphs of f, L and U, distinguishing them clearly.

(c) Show that $\int_0^7 f(x)\, dx = \frac{47}{2}$ and calculate both $\int_0^7 L(t)\, dt$.

Solutions or hints for selected exercises

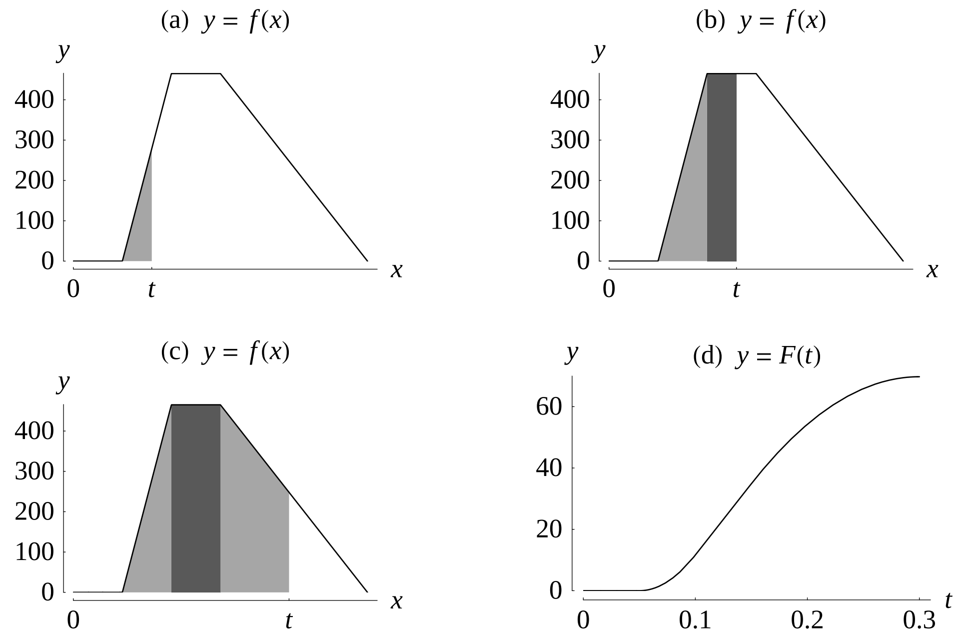


Figure 3: Solution to Exercise 1.

1. There are four cases, according to which subdomain of $[0, \frac{3}{10}]$ contains t. The easiest case is when $t \in [0, \frac{1}{20}]$: because f$(x) = 0$ for $0 \le x \le \frac{1}{20}$ and $t \le \frac{1}{20}$, we have $f(x) = 0$ for $0 \le x \le t$, and so $F(t) = \int_0^t f(x)\,dx = 0$. This result holds for all $t \in [0, \frac{1}{20}]$, including $t = \frac{1}{20}$. So, in particular, $F(\frac{1}{20}) = 0$. In the second case, $t \in [\frac{1}{20}, \frac{1}{10}]$. Then, from (15) with $a = 0$, $c = \frac{1}{20}$ and $b = t$ we have $F(t) = \int_0^t f(x)\,dx = \int_0^{1/20} f(x)\,dx + \int_{1/20}^t f(x)\,dx = F(\frac{1}{20}) + \int_{1/20}^t f(x)\,dx = 0 + \int_{1/20}^t f(x)\,dx = \int_{1/20}^t f(x)\,dx$ from our earlier calculation. From Figure 3a, however, $\int_{1/20}^t f(x)\,dx$ is the area of a triangle with base $t - \frac{1}{20}$ and height $f(t) = 465(20t - 1)$. So for $t \in [\frac{1}{20}, \frac{1}{10}]$ we have

$$F(t) \;=\; \tfrac{1}{2}\left(t - \tfrac{1}{20}\right) 465(20t - 1) \;=\; 4650\left(t - \tfrac{1}{20}\right)^2.$$

In particular, $F(\frac{1}{10}) = 4650\left(\frac{1}{10}\right)^2 = \frac{93}{8}$. In the third case, $t \in [\frac{1}{10}, \frac{3}{20}]$. Then, from (15) with $a = 0$, $c = \frac{1}{10}$ and $b = t$ we have $F(t) = \int_0^t f(x)\,dx = \int_0^{1/10} f(x)\,dx + \int_{1/10}^t f(x)\,dx = F(\frac{1}{10}) + \int_{1/10}^t f(x)\,dx = \frac{93}{8} + \int_{1/10}^t f(x)\,dx$ from our earlier calculation. From Figure 3b, however, $\int_{1/10}^t f(x)\,dx$ is the area of a rectangle with base $t - \frac{1}{10}$ and height $f(t) = 465$. So for $t \in [\frac{1}{10}, \frac{3}{20}]$ we have

$$F(t) \;=\; \tfrac{93}{8} + \left(t - \tfrac{1}{10}\right) 465 \;=\; 465t - \tfrac{279}{8}.$$

In particular, $F(\frac{3}{20}) = 465 \cdot \frac{3}{20} - \frac{279}{8} = \frac{279}{8}$. The last case to consider is when $t \in [\frac{3}{20}, \frac{3}{10}]$. Then, from (15) with $a = 0$, $c = \frac{3}{20}$ and $b = t$ we have $F(t) = \int_0^t f(x)\,dx = \int_0^{3/20} f(x)\,dx + \int_{3/20}^t f(x)\,dx = F(\frac{3}{20}) + \int_{3/20}^t f(x)\,dx = \frac{279}{8} + \int_{3/20}^t f(x)\,dx$ from our earlier calculation. But from Figure 3c, $\int_{3/20}^t f(x)\,dx$ is the area of a trapezium of width $t - \frac{3}{20}$, maximum height $f(\frac{3}{20}) = 465$ and minimum height $f(t) = 310(3-10t)$. So for $t \in [\frac{3}{20}, \frac{3}{10}]$ we have

$$F(t) \;=\; \tfrac{179}{8} + \tfrac{1}{2}\left(t - \tfrac{1}{10}\right)\{465 + 310(3-10t)\} \;=\; 930t - 1550t^2 - \tfrac{279}{4}$$

after simplification. Gathering our results together, we find that F is the piecewise-quadratic polynomial join defined by

$$F(t) \;=\; \begin{cases} 0 & \text{if} \quad 0 \le f < \frac{1}{20} \\ 4650t^2 - 465t + \frac{93}{8} & \text{if} \quad \frac{1}{20} \le f < \frac{1}{10} \\ 465t - \frac{279}{8} & \text{if} \quad \frac{1}{10} \le f < \frac{3}{20} \\ 930t - 1550t^2 - \frac{279}{4} & \text{if} \quad \frac{3}{20} \le f < \frac{3}{10}. \end{cases}$$

The graph of this function is plotted in Figure 3d. Note that it is strictly increasing (and therefore invertible) because the area of the shading always gets bigger as you move to the right in Figures 3a-c.

2. G is defined on $[0,3]$ by $G(t) \;=\; \begin{cases} 4t - \frac{3}{2}t^2 & \text{if} \quad 0 \le t < 1 \\ \frac{3}{4} + 2t - \frac{1}{4}t^4 & \text{if} \quad 1 \le t < 3. \end{cases}$

7. G is defined on $[0,3]$ by $G(t) \;=\; \begin{cases} 5t - 4t^2 & \text{if} \quad 0 \le t < 1 \\ \frac{1}{4}t^4 - 4t + \frac{19}{4} & \text{if} \quad 1 \le t < 2 \\ \frac{2}{3}t^3 + \frac{3}{2}t^2 - 10t + \frac{113}{12} & \text{if} \quad 2 \le t \le 3. \end{cases}$

9. (b) $F(t) \;=\; \begin{cases} t^2 + 3t & \text{if} \quad 0 \le t < 2 \\ 13t - \frac{3}{2}t^2 - 10 & \text{if} \quad 2 \le t < 4 \\ \frac{1}{2}t^2 - 3t + 22 & \text{if} \quad 4 \le t < 5 \\ 2t + \frac{19}{2} & \text{if} \quad 5 \le t \le 7 \end{cases}$ (c) $\frac{43}{3}$.