# 14. The fundamental theorem of the calculus

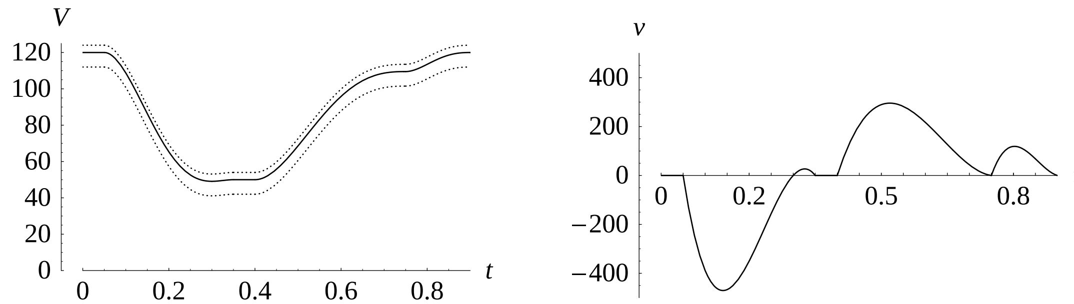


Figure 1: (a) Ventricular volume for subjects with capacities $C = 124$ ml, $C = 120$ ml, $C = 112$ ml and (b) the corresponding inflow. The solid curves are identical to those in Figure 3 of Lecture 1.

From Lecture 13 we already know that ventricular volume at any time $t$ equals ventricular volume at any earlier time $a$ plus subsequent net recharge:

$$V(t) \;=\; V(a) + \int_a^t v(x)\,dx. \tag{1}$$

But from Lecture 6 we also know that inflow equals rate of change of volume:

$$V'(t) \;=\; v(t). \tag{2}$$

So volume is the integral of inflow, in the sense of (1); and inflow is the derivative of volume, in the sense of (2). But this relationship between integration and differentiation is asymmetric, in the sense that although knowledge of $V$ implies knowledge of $v$, knowledge of $v$ does not imply full knowledge of $V$. Why? In essence, because two different subjects with precisely the same inflow trace may have different ventricular capacities. Figure 1a shows (a cartoon of) the variation with time $t$ of ventricular volume

$$V(t) \;=\; \begin{cases} C & \text{if } 0 \le t < 0.05 \\ \frac{43895}{432} + \frac{2450}{3}t - \frac{96250}{9}t^2 + \frac{980000}{27}t^3 - \frac{350000}{9}t^4 & \text{if } 0.05 \le t < 0.35 \\ 50 & \text{if } 0.35 \le t < 0.4 \\ \frac{1620894}{1127} - \frac{489600}{49}t + \frac{28927200}{1127}t^2 - \frac{1360000}{49}t^3 + \frac{12240000}{1127}t^4 & \text{if } 0.4 \le t < 0.75 \\ \frac{293892}{11} - 126000t + \frac{2450000}{11}t^2 - \frac{1568000}{9}t^3 + \frac{560000}{11}t^4 & \text{if } 0.75 \le t \le 0.9 \end{cases} \tag{3}$$

for a human subject with ventricular capacity $V(0) = C$ ml for three different values of $C$. Regardless of this value, the unique variation with time of ventricular inflow is

$$v(t) \;=\; V'(t) \;=\; \begin{cases} 0 & \text{if } 0 \le t < 0.05 \\ \frac{2450}{3} - \frac{192500}{9}t + \frac{980000}{9}t^2 - \frac{1400000}{9}t^3 & \text{if } 0.05 \le t < 0.35 \\ 0 & \text{if } 0.35 \le t < 0.4 \\ -\frac{489600}{49} + \frac{57854400}{1127}t - \frac{4080000}{49}t^2 + \frac{48960000}{1127}t^3 & \text{if } 0.4 \le t < 0.75 \\ -126000 + \frac{4900000}{11}t - \frac{1568000}{3}t^2 + \frac{2240000}{11}t^3 & \text{if } 0.75 \le t \le 0.9 \end{cases} \tag{4}$$

(Exercise 1). Clearly, therefore, knowledge of $V$ implies knowledge of $v$. But knowledge of $v$ does not imply full knowledge of $V$, because although the method of Lecture 13 can be used to recover (3) from (4), the value of $C = V(0)$ is undetermined. On the other hand, knowledge of $v$ is almost enough to yield knowledge of $V$: the only additional information required is the magnitude of the volume at a single instant, e.g., $t = 0$. Thus $v$ implies $V$ except for the addition of an arbitrary constant, here $C = V(0)$. In other words, the dotted curves in Figure 1a are identical to the solid curve, except for translation in the vertical direction (either up by 4 ml or down by 8 ml).

The above relationship between $v$ and $V$—namely, inflow equals rate of change of volume, and current volume equals initial volume plus subsequent net recharge—can be reduced to a single line of mathematics by writing

$$v(t) \;=\; V'(t) \quad\Longleftrightarrow\quad V(t) \;=\; C + \int_a^t v(x)\,dx \tag{5a}$$

where

$$C \;=\; V(a). \tag{5b}$$

Furthermore, this relationship is perfectly general: any two functions $f$ and $F$ satisfy

$$f(t) \;=\; F'(t) \quad\Longleftrightarrow\quad F(t) \;=\; C + \int_a^t f(x)\,dx \tag{6a}$$

for arbitrary

$$C \;=\; F(a) \tag{6b}$$

(provided that $F'$ exists, which we assume). The equivalence (6) is called the *fundamental theorem of calculus*; and any function that assigns to $t$ the label

$$C + \int_a^t f(x)\,dx, \tag{7}$$

where $C$ is undetermined, is called an *anti-derivative* of $f$.

The fundamental theorem of calculus more or less says that integration and differentiation are inverse operations. There is also an asymmetry, however, because derivatives are always unique, whereas anti-derivatives are not unique—although all of a function's anti-derivatives differ only by a constant; in other words, $f$ is *the* derivative of $F$ if and only if $F$ is *an* anti-derivative of $f$. A proof of the theorem is sketched in the appendix.

In practice, before applying the fundamental theorem, we usually substitute from one side of (6a) into the other. Clearly, there are two ways in which to do so; and before considering the first, it is convenient to recall a couple of general points about differentiation. First, although results concerning derivatives are in principle statements about a relationship between functions $F$ and $F'$, in practice we often find it most convenient to proceed without explicitly naming $F$; so, for example, instead of writing $F(t) = \ln(t) \Longrightarrow F'(t) = 1/t$ (for $t > 0$), we prefer to write $\frac{d}{dt}\{\ln(t)\} = 1/t$. Second, although a derivative is in principle the limit of a quotient, it is rare in practice that we calculate a limit when we want to find a derivative: instead we employ general results like the chain, product and quotient rules in conjunction with special rules like the one above for the derivative of the logarithm.

Likewise for anti-differentiation. First, although results concerning anti-derivatives are in principle statements about a relationship between functions $f$ and $F$ defined by $F(t) = C + \int_a^t f(x)\,dx$, in practice we often find it most convenient to proceed without explicitly naming $F$ or $f$; so, for example, instead of writing $f(t) = 1/t \Longrightarrow C + \int_a^t f(x)\,dx = \ln(t)$, we could simply write $C + \int_a^t \frac{1}{x}\,dx = \ln(t)$. We could—but it would go against tradition. If $c$ is an arbitrary constant, then the traditional way of stating that the function $F$ defined by $F(t) = \ln(t) + c$ is an anti-derivative of the function $f$ defined by $f(t) = 1/t$ is to write

$$\int \frac{1}{t}\,dt \;=\; \ln(t) + c. \tag{8}$$

Now, it has to be said immediately that this is a *lousy* notation, because if there is one thing that the left-hand side of (8) should be independent of, then it is $t$—contradicting that the right-hand side depends on $t$! It would be very much better to write (8) as

$$\int^t \frac{1}{x}\,dx \;=\; \ln(t) + c \tag{9}$$

and interpret the left-hand side as a way of writing $\int_a^t \frac{1}{x}\,dx$ when $a$ is completely arbitrary, because $\int_a^t \frac{1}{x}\,dx = \ln(t) + c$ is just another way of writing $C + \int_a^t \frac{1}{x}\,dx = \ln(t)$.* But it is hard to buck tradition, especially one as deeply ingrained as this one. So instead we have to learn to live with the lousy notation, and regard (8) as simply an inverse way of saying

$$\frac{1}{t} \;=\; \frac{d}{dt}\{\ln(t) + c\}. \tag{10}$$

With the new notation, every result we have ever had concerning derivatives can immediately be rewritten as a corresponding result concerning anti-derivatives. Note, however, that it is traditional to rescale such equivalences so that the function on the left-hand side always, in effect, has coefficient 1. For example, although $\frac{d}{dt}\{t^2\} = 2t \Longleftrightarrow \int 2t\,dt = t^2 + C$ is perfectly correct, you are much more likely to see $\frac{d}{dt}\{t^2\} = 2t \Longleftrightarrow \int t\,dt = \frac{1}{2}t^2 + c$ (with the arbitrary constants $C$ and $c$ related by $C = 2c$). A sample of such equivalences appears in Table 1. Note that $\int f(x)\,dx$ is called an *indefinite integral*, and that an indefinite integral is basically the same thing as an anti-derivative—in precisely the same sense that a differential coefficient is basically the same thing as a derivative.

Second, although a definite integral is in principle the limit of a sum, it is rare in practice that we calculate a limit when we want to find a definite integral; instead we use the fundamental theorem in the form

$$\int_a^t F'(x)\,dx \;=\; F(t) - F(a), \tag{11}$$

which we obtain from (6) by substituting from (6b) and the left-hand side of (6a) into the right-hand side, i.e., by substituting $f(x) = F'(x)$ and $C = F(a)$. Because (11) holds for

*With the arbitrary constants $C$ and $c$ related by $c = -C = -\ln(a)$

| | | | |
|---|---|---|---|
| $\dfrac{d}{dt}\{t^r\} = rt^{r-1}$ | $\iff$ | $\int t^r\,dt = \dfrac{t^{r+1}}{r+1} + c \quad (r \neq -1)$ |
| $\dfrac{d}{dt}\{\ln(t+\alpha)\} = \frac{1}{t+\alpha}$ | $\iff$ | $\int \frac{1}{t+\alpha}\,dt = \ln(t+\alpha) + c \quad (t > -\alpha)$ |
| $\dfrac{d}{dt}\{e^{\alpha t}\} = \alpha e^{\alpha t}$ | $\iff$ | $\int e^{\alpha t}\,dt = \frac{1}{\alpha}e^{\alpha t} + c$ |
| $\dfrac{d}{dt}\{\sin(\alpha t)\} = \alpha\cos(\alpha t)$ | $\iff$ | $\int \cos(\alpha t)\,dt = \frac{1}{\alpha}\sin(\alpha t) + c$ |
| $\dfrac{d}{dt}\{\cos(\alpha t)\} = -\alpha\sin(\alpha t)$ | $\iff$ | $\int \sin(\alpha t)\,dt = -\frac{1}{\alpha}\cos(\alpha t) + c$ |

Table 1: Some equivalences between derivative and anti-derivative. In each case $c$ is an arbitrary "constant of integration."

any $t$ in the domain of $F$, including the right-hand endpoint, we have in particular that

$$\int_a^b F'(x)\,dx = F(b) - F(a). \tag{12}$$

What this equation means in practice is that derivatives are very easy to integrate. For example, we know that $F(t) = \ln(t) \Longrightarrow F'(t) = 1/t$; or, which of course is exactly the same thing, $F(x) = \ln(x) \Longrightarrow F'(x) = 1/x$. So it follows at once from (12) that

$$\int_a^b \frac{1}{x}\,dx = \int_a^b F'(x)\,dx = F(b) - F(a) = \ln(b) - \ln(a) = \ln\Big(\frac{b}{a}\Big). \tag{13}$$

(for $b > a > 0$). But we have had to name $F$ explicitly to obtain this result, and in practice we might be happier if that weren't necessary. So we introduce another notation: we use $F(x)\,|_a^b$ to denote the *jump* in $F(x)$ between $x = a$ and $x = b$; i.e., we define

$$F(x)\,|_a^b = F(b) - F(a). \tag{14}$$

Now we can re-state (12) as

$$\int_a^b F'(x)\,dx = F(x)\,|_a^b \tag{15}$$

with the advantage that we can evaluate definite integrals without ever having to name $F$ explicitly. For example, in terms of (15), (13) reduces to

$$\int_a^b \frac{1}{x}\,dx = \int_a^b \frac{d}{dx}\{\ln(x)\}\,dx = \ln(x)\,|_a^b = \ln(b) - \ln(a) = \ln\Big(\frac{b}{a}\Big). \tag{16}$$

We can similarly recover the special results we obtained at the end of Lecture 11 (by evaluating the limit that defines the definite integral); for example,

$$\int_a^b x^2\,dx = \int_a^b \frac{d}{dx}\{\tfrac{1}{3}x^3\}\,dx = \tfrac{1}{3}x^3\Big|_a^b = \tfrac{1}{3}b^3 - \tfrac{1}{3}a^3 = \tfrac{1}{3}(b^3 - a^3). \tag{17}$$

Not only does using the fundamental theorem involve less work, but also we can obtain a much more general result, for (if necessary, with the help of Table 1) we have

$$\int_a^b x^n\,dx \;=\; \int_a^b \frac{d}{dx}\left\{\frac{x^{n+1}}{n+1}\right\}dx \;=\; \left.\frac{x^{n+1}}{n+1}\right|_a^b \;=\; \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \;=\; \frac{b^{n+1}-a^{n+1}}{n+1}, \tag{18}$$

provided that $n \neq -1$; in that case, we have (16).

The other way to substitute from one side of (6a) into the other is to substitute from the right-hand side into the left-hand side. Then we obtain

$$f(t) \;=\; \frac{d}{dt}\left\{C + \int_a^t f(x)\,dx\right\} \;=\; \frac{dC}{dt} + \frac{d}{dt}\left\{\int_a^t f(x)\,dx\right\}$$

or

$$\frac{d}{dt}\left\{\int_a^t f(x)\,dx\right\} \;=\; f(t) \tag{19}$$

because $\frac{dC}{dt} = 0$; intuitively, the derivative of an integral is the original function. This form of the fundamental theorem is often most useful when used in conjunction with the chain rule. Suppose, for example, that

$$y \;=\; \int_{\sin(u)}^1 \sqrt{1+x^2}\,dx \tag{20}$$

and we want to know $\frac{dy}{du}$. Then, on using (19) with $a = 0$ and $f(x) = \sqrt{1+x^2}$, the substitution $t = \sin(u)$ yields

$$\begin{aligned}
\frac{dy}{du} \;=\; \frac{dy}{dt}\frac{dt}{du} \;&=\; \frac{d}{dt}\left\{\int_t^1 \sqrt{1+x^2}\,dx\right\}\frac{dt}{du} \\
&=\; \frac{d}{dt}\left\{-\int_1^t \sqrt{1+x^2}\,dx\right\}\frac{d}{du}\{\sin(u)\} \\
&=\; -\frac{d}{dt}\left\{\int_1^t \sqrt{1+x^2}\,dx\right\}\cos(u) \\
&=\; -\sqrt{1+t^2}\,\cos(u) \;=\; -\cos(u)\sqrt{1+\sin^2(u)}.
\end{aligned} \tag{21}$$

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In sum, the fundamental theorem of calculus says that the derivative of a function's integral is precisely that function, in the sense that

$$\frac{d}{dx}\left\{\int_a^x f(t)\,dt\right\} \;=\; f(x) \tag{22}$$

and that the integral of a function's derivative is essentially that function, in the sense that

$$\int F'(x)\,dx \;=\; F(x) + c \tag{23}$$

where $c$ is an arbitrary constant. Note that (19) and(22) are just two different ways of saying exactly the same thing (with $x$ and $t$ merely juxtaposed); and don't forget that what (24) really means is

$$\int_a^x F'(t)\,dt \;=\; F(x) - F(a) \;=\; F(t)\Big|_a^x \tag{24}$$

for any $x$, where $a$ is an arbitrary constant.

## Appendix: A sketch of a proof of the fundamental theorem of calculus

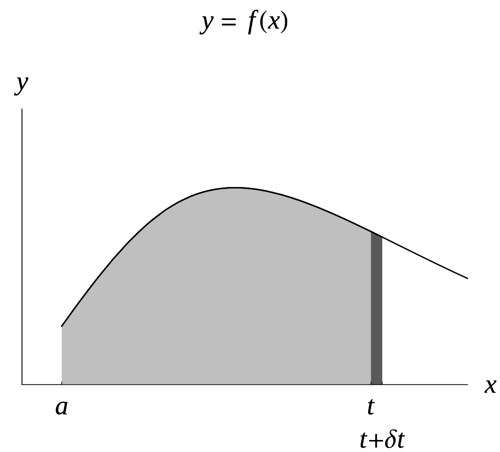


Figure 2: For $\Phi$ defined by (25), the lighter shaded area represents $\Phi(t)$ and the total shaded area represents $\Phi(t+\delta t)$; thus the darker shaded area represents $\Phi(t+\delta t) - F(t)$

The purpose of this appendix is to show why (6) must hold in general (without actually proving it rigorously). There are two results to be established, namely, that the right-hand side of (6a) implies the left-hand side, and that the left-hand side of (6a) implies the right-hand side. We begin with the former, that is, we first establish that $F(t) = C + \int_a^t f(x)\,dx \Longrightarrow f(t) = F'(t)$, which means assuming $F(t) = C + \int_a^t f(x)\,dx$ and deducing $f(t) = F'(t)$.

Accordingly, assume that $F(t) = C + \int_a^t f(x)\,dx$. It will help to define

$$\Phi(t) \;=\; \int_a^t f(x)\,dx, \tag{25}$$

which is represented by the lighter shaded area in Figure 2, i.e., the signed area between the graph of $f$ and segment $[a, t]$ of the horizontal $x$-axis. Note once more the important point that, because $t$ is an endpoint of the interval in question, we *cannot* use $t$ to label the axis—but we can use any symbol other than $t$ (or $y$, obviously), and we have chosen to use $x$. Let us now increase $t$ infinitesimally, to $t + \delta t$. Then $\Phi(t)$ changes infinitesimally to $\Phi(t+\delta t)$, i.e., the signed area between the graph of $f$ and the segment $[a, t+\delta t]$ of the $x$-axis, as represented by the total shaded area in Figure 2. Hence the difference, i.e.,

$\Phi(t + \delta t) - \Phi(t)$, is represented by the darker shaded area. But for sufficiently small $\delta t$, this area must fall between that of a rectangle of width $\delta t$ and height $f(t)$ and that of a rectangle of width $\delta t$ and height $f(t+\delta t)$—because, if $\delta t$ is sufficiently small, then we can assume that $f$ is either increasing, decreasing or constant on $[t, t+\delta t]$. In other words, for the darker shaded (signed) area, we have

$$\int_t^{t+\delta t} f(x)\,dx \;=\; f(t)\delta t + o(\delta t). \tag{26}$$

As we saw a moment ago, however, this quantity must also equal $\Phi(t+\delta t) - \Phi(t)$. So

$$\Phi(t+\delta t) - \Phi(t) \;=\; f(t)\delta t + o(\delta t), \tag{27}$$

implying

$$\frac{\Phi(t+\delta t) - \Phi(t)}{\delta t} \;=\; f(t) + \frac{o(\delta t)}{\delta t}. \tag{28}$$

Now taking the limit as $\delta t \to 0$ yields $\Phi'(t) = f(t) + 0 = f(t)$. We have therefore established that

$$\Phi(t) \;=\; \int_a^t f(x)\,dx \quad\Longrightarrow\quad \Phi'(t) \;=\; f(t) \tag{29}$$

implying

$$F'(t) \;=\; \frac{d}{dt}\{C + \Phi(t)\} \;=\; \frac{dC}{dt} + \Phi'(t) \;=\; 0 + f(t) \;=\; f(t) \tag{30}$$

Now we establish that the left-hand side of (6) implies the right-hand side, or $f(t) = F'(t) \implies F(t) = C + \int_a^t f(x)\,dx$, which means assuming $f(t) = F'(t)$ and deducing $F(t) = C + \int_a^t f(x)\,dx$. Accordingly, assume that $f(t) = F'(t)$, or—which of course is exactly the same thing—$f(x) = F'(x)$. Then, from the definition of differential coefficient (Lectures 5-6), we have

$$F(x+\delta x) - F(x) \;=\; f(x)\,\delta x \;+\; o(\delta x). \tag{31}$$

Recall from Lecture 11 one of the two notations for the definite integral of $f$ between $a$ and $b$ that we did not adopt (because it was too cumbersome):

$$\lim_{\delta t \to 0} \sum_{t \in [a,b]} f(t)\,\delta t \;=\; \left\{ \begin{array}{c} \text{total signed area between the graph of } f \text{ and} \\ \text{segment } [a,b] \text{ of the horizontal axis, counted} \\ \text{positively above the axis, negatively below it.} \end{array} \right.$$

We can replace $b$ by $t$ in this equivalence, but only if we first replace $t$ by $x$ on the left-hand side (for the reason emphasized above):

$$\int_a^t f(x)\,dx \;=\; \lim_{\delta x \to 0} \sum_{x \in [a,t]} f(x)\,\delta x \;=\; \left\{ \begin{array}{c} \text{total signed area between the graph of } f \text{ and} \\ \text{segment } [a,t] \text{ of the horizontal axis, counted} \\ \text{positively above the axis, negatively below it.} \end{array} \right. \tag{32}$$

Now, because $o(\delta x) \to 0$ as $\delta x \to 0$, (31)-(32) imply

$$\begin{aligned} \int_a^t f(x)\,dx \;&=\; \lim_{\delta x \to 0} \sum_{x \in [a,t]} f(x)\,\delta x \;=\; \lim_{\delta x \to 0} \sum_{x \in [a,t]} \{f(x)\,\delta x \;+\; o(\delta x)\} \\ &=\; \lim_{\delta x \to 0} \sum_{x \in [a,t]} \{F(x+\delta x) - F(x)\}. \end{aligned} \tag{33}$$

In the limit as $\delta x \to 0$, this summation behaves like the summed sequence of differences in Lecture 10: because each $F(x)$ is in essence $F(x + \delta x)$ for the next difference down, the terms all cancel in pairs—except for the very first and last. That is,

$$\lim_{\delta x \to 0} \sum_{x \in [a,t]} \{F(x + \delta x) - F(x)\} \;=\; F(t) - F(a). \tag{34}$$

Together, (33)-(34) and (6b) imply

$$\int_a^t f(x)\,dx \;=\; F(t) - C \tag{35}$$

or $F(t) = C + \int_a^t f(x)\,dx$, as required.

# Exercises

**1.** Differentiate (3) to obtain (4) and verify that $v$ is continuous.
**Hint**: Polynomials are always continuous, so you need only verify continuity at $t = 0.05$, $t = 0.35$, $t = 0.4$ and $t = 0.75$.

**2.** For ventricular volume $V$, if $b$ (= 0.9 in Figure 1a) is the period of the cardiac cycle, then $\max(V, 0, b) - \min(V, 0, b)$ is called the *stroke volume*.

(a) What is the stroke volume in Figure 1a?

(b) Why must it be true that $\int_0^b v(t)\,dt = 0$?

(c) Verify (b) for $v$ defined by (4).

**3.** Functions $g$ and $G$ are defined on $[0, 4]$ by

$$g(t) \;=\; \begin{cases} 4 - t^2 & \text{if} \quad 0 \le t < 1 \\ t^3 + 2 & \text{if} \quad 1 \le t < 3 \\ 10t - 1 & \text{if} \quad 3 \le t \le 4 \end{cases} \qquad \text{and} \qquad G(t) \;=\; \int_0^t g(x)\,dx.$$

Using a different method from the one you used for Exercise 3 of Lecture 13, find an explicit formula for $G(t)$ for all $t \in [0, 4]$.

**4.** (a) Calculate $\dfrac{d}{dx}\left\{\dfrac{-1}{2x^2}\right\}$

(b) Show that $\int\limits_1^t x^{-3}\,dx \;=\; \dfrac{1}{2}\left(1 - \dfrac{1}{t}\right)\left(1 + \dfrac{1}{t}\right)$.

Assume that $t > 1$ (hence $1 \le x \le t$ for all relevant $x$).

**Hint**: Use (15) with $a = 1$, $b = t$ and a judicious choice of $F$.

**5.** (a) Calculate $\dfrac{d}{dx}\left\{\dfrac{-1}{3x^3}\right\}$

(b) Show that $\int\limits_1^t x^{-4}\,dx \;=\; \dfrac{(t-1)(t^2+t+1)}{3t^3}$.

Assume that $t > 1$ (hence $1 \le x \le t$ for all relevant $x$).

**Hint**: Use (15) with $a = 1$, $b = t$ and a judicious choice of $F$.

**6.** (a) Calculate $\dfrac{d}{dx}\left\{\dfrac{x^2}{3-x}\right\}$

(b) Show that $\int\limits_1^t \dfrac{x(6-x)}{(3-x)^2}\,dx \;=\; \dfrac{(t-1)(2t+3)}{2(3-t)}$.

Assume that $1 \le t < 3$ (hence $1 \le x < 3$ for all relevant $x$).

**Hint**: Use (15) with $a = 1$, $b = t$ and a judicious choice of $F$.

# Suitable problems from standard calculus texts

Stewart (2003): p. 402, ## 7-24 and 28-52; pp. 411-412, ## 1-58. Where necessary, use (19) and the chain rule.

# Reference


Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.


# Solutions or hints for selected exercises

**2.** (b) $V_{max}(t) - V_{min}(t) = 120 - 50 = 70$ ml. (b) Because volume is conserved.

**3.** $G$ defined on $[0, 4]$ by

$$G(t) \;=\; \begin{cases} 4t - \frac{1}{3}t^3 + c_1 & \text{if} \quad 0 \le t < 1 \\ \frac{1}{4}t^4 + 2t + c_2 & \text{if} \quad 1 \le t < 3 \\ 5t^2 - t + c_3 & \text{if} \quad 3 \le t \le 4 \end{cases}$$

is an anti-derivative of $g$, where the integration constants $c_1$, $c_2$ and $c_3$ are different for each subdomain of the join. These constants are determined by $G(0) = 0$ and the continuity of $G$, i.e., $G(1^-) = G(1^+)$, $G(3^-) = G(3^+)$. So $c_1 = 0$, $c_2 = \frac{17}{12}$ and $c_3 = -\frac{43}{3}$. Note that $G$ must be continuous even if $g$ is not (although in this case, it is).