16. Integration by substitution

Let's begin by re-stating the essence of the fundamental theorem of calculus: differentiation is the opposite of integration in the sense that

$$F'(u) \;=\; f(u) \qquad \Longleftrightarrow \qquad \int f(u)\,du \;=\; F(u) + C \tag{1}$$

for some constant C or, equivalently,

$$F'(u) \;=\; f(u) \qquad \Longleftrightarrow \qquad \int\limits_a^b f(u)\,du \;=\; F(b) - F(a) \tag{2}$$

for arbitrary a and b, where $\int_a^b f(u)\,du$ denotes the definite integral of f over the subdomain $[a, b]$, i.e., the signed area enclosed by the graph $y = f(u)$, the horizontal coordinate axis $y = 0$ and vertical line segments at $u = a$ and $u = b$, with area counted positively above the axis and negatively below it.

Because the fundamental theorem is about a relationship between f and F on an entire domain, it does not matter what symbol we use to denote an arbitrary element of that domain. Furthermore, the information content of the fundamental theorem is in no way altered by re-labelling G and g as F and f, respectively (provided, of course, we are 100% consistent about it). Thus an identical statement of the fundamental theorem is that

$$g(x) \;=\; G'(x) \qquad \Longleftrightarrow \qquad G(x) + C = \int g(x)\,dx \tag{3}$$

for some constant C or, equivalently,

$$g(x) \;=\; G'(x) \qquad \Longleftrightarrow \qquad G(\beta) - G(\alpha) \;=\; \int\limits_\alpha^\beta g(x)\,dx \tag{4}$$

for arbitrary α and β, where $\int_\alpha^\beta g(x)\,dx$ denotes the definite integral of g over the subdomain $[\alpha, \beta]$, i.e., the signed area enclosed by the graph $y = g(x)$, the horizontal coordinate axis $y = 0$ and vertical line segments at $x = \alpha$ and $x = \beta$; again, of course, area is counted positively above the axis and negatively below it.

Now suppose that u and x are related by $u = \phi(x)$, where ϕ is an *invertible* (either increasing or decreasing) function on a subdomain of interest. Then there exists an inverse function, say ζ, such that

$$u \;=\; \phi(x) \qquad \Longleftrightarrow \qquad x \;=\; \zeta(u) \tag{5}$$

(Figure 1). Furthermore, define a composition G by

$$G(x) \;=\; F(\phi(x)), \tag{6}$$

which implies, of course, that

$$F(u) \;=\; G(\zeta(u)). \tag{7}$$

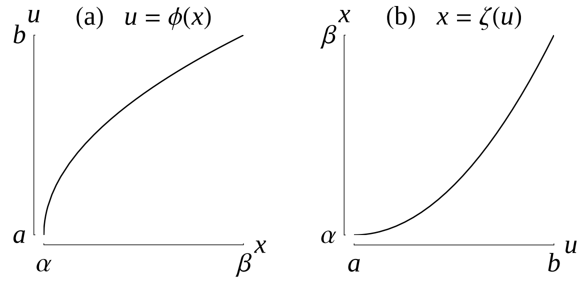

Figure 1: (a) A typical substitution and (b) its inverse; typically both functions are increasing (as, for example, in all of the exercises at the end of this lecture). Integration with respect to x from α to β corresponds to integration with respect to u from a to b, and vice versa.

Applying the chain rule to each of these equations in turn, we have *both* that

$$G'(x) \;=\; F'(\phi(x))\,\phi'(x) \tag{8}$$

and

$$F'(u) \;=\; G'(\zeta(u))\zeta'(u). \tag{9}$$

But $F'(u) = f(u) \implies F'(\phi(x)) = f(\phi(x))$, and $G'(x) = g(x) \implies G'(\zeta(u)) = g(\zeta(u))$. So, from above,

$$g(x) \;=\; f(\phi(x))\,\phi'(x) \tag{10}$$

and

$$f(u) \;=\; g(\zeta(u))\zeta'(u). \tag{11}$$

Now we put it all together to get *both*

$$\begin{aligned}\int f(u)\,du \;&=\; F(u)+C \;=\; F(\phi(x))+C \\ &=\; G(x)+C \;=\; \int g(x)\,dx \\ &=\; \int f(\phi(x))\,\phi'(x)\,dx \;=\; \int \left\{ f(u)\,\frac{du}{dx} \right\} dx\end{aligned} \tag{12}$$

and

$$\begin{aligned}\int g(x)\,dx \;&=\; G(x)+C \;=\; G(\zeta(u))+C \\ &=\; F(u)+C \;=\; \int f(u)\,du \\ &=\; \int g(\zeta(u))\,\zeta'(u)\,du \;=\; \int \left\{ g(x)\,\frac{dx}{du} \right\} du.\end{aligned} \tag{13}$$

So the substitution $u = \phi(x)$ can be used to convert an integral with respect to u into an integral with respect to x; and, correspondingly, the inverse substitution $x = \zeta(u)$ can be used to convert an integral with respect to x into an integral with respect to u. This process is known as *integration by substitution*.

The corresponding equations for definite integrals are as follows. First, to convert an integral with respect to u into an integral with respect to x, note that $u = \phi(x) \implies \delta u = \phi'(x)\delta x + o(\delta x)$ from Lecture 6, and that $u \in [a, b] \implies x \in [\zeta(\alpha), \zeta(\beta)]$ from (5). Hence*

$$\begin{aligned}
\int_{u=a}^{u=b} f(u)\,du &= \lim_{\delta u \to 0} \sum_{u\in[a,b]} f(u)\,\delta u \;=\; \lim_{\delta x \to 0} \sum_{x\in[\zeta(\alpha),\zeta(\beta)]} f(\phi(x))\{\phi'(x)\delta x + o(\delta x)\} \\
&= \lim_{\delta x \to 0} \sum_{x\in[\zeta(\alpha),\zeta(\beta)]} f(\phi(x))\phi'(x)\delta x \;+\; \lim_{\delta x \to 0} \sum_{x\in[\zeta(\alpha),\zeta(\beta)]} f(\phi(x))\delta x \lim_{\delta x \to 0} \frac{o(\delta x)}{\delta x} \\
&= \int_{x=\zeta(a)}^{x=\zeta(b)} f(\phi(x))\,\phi'(x)\,dx \;+\; \int_{\zeta(a)}^{\zeta(b)} f(\phi(x))\,dx \cdot 0
\end{aligned}$$

or

$$\int_{u=a}^{u=b} f(u)\,du \;=\; \int_{x=\zeta(a)}^{x=\zeta(b)} f(\phi(x))\,\phi'(x)\,dx. \tag{14}$$

Second, to convert an integral with respect to x into an integral with respect to u, note that $x = \zeta(u) \implies \delta x = \zeta'(u)\delta u + o(\delta u)$ and $x \in [\alpha, \beta] \implies u \in [\phi(a), \phi(b)]$, from (5). Hence

$$\begin{aligned}
\int_{x=\alpha}^{x=\beta} g(x)\,dx &= \lim_{\delta x \to 0} \sum_{x\in[\alpha,\beta]} g(x)\,\delta x \;=\; \lim_{\delta u \to 0} \sum_{u\in[\phi(\alpha),\phi(\beta)]} g(\zeta(u))\{\zeta'(u)\delta u + o(\delta u)\} \\
&= \lim_{\delta u \to 0} \sum_{u\in[\phi(\alpha),\phi(\beta)]} g(\zeta(u))\zeta'(u)\delta u \;+\; \lim_{\delta u \to 0} \sum_{u\in[\phi(\alpha),\phi(\beta)]} g(\zeta(u))\delta u \lim_{\delta u \to 0} \frac{o(\delta u)}{\delta u} \\
&= \int_{u=\phi(\alpha)}^{u=\phi(\beta)} g(\zeta(u))\,\zeta'(u)\,du \;+\; \int_{\phi(\alpha)}^{\phi(\beta)} g(\zeta(u))\,du \cdot 0
\end{aligned}$$

or

$$\int_{x=\alpha}^{x=\beta} g(x)\,dx \;=\; \int_{u=\phi(\alpha)}^{u=\phi(\beta)} g(\zeta(u))\,\zeta'(u)\,du. \tag{15}$$

In either case, note that integration by substitution requires both the original substitution and the inverse substitution: specifically, the original substitution is used to

1. Rewrite the old integrand in terms of the new integration variable
2. Differentiate to find the (nonlinear) scaling factor by which the old integrand must be multiplied to become the new integrand

and the inverse substitution is used to

3. Convert the old integration limits into the new integration limits.

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*Note that $\lim\limits_{\delta x \to 0} \sum\limits_{x\in[\zeta(\alpha),\zeta(\beta)]} f(\phi(x))\delta x = \int\limits_{\zeta(a)}^{\zeta(b)} f(\phi(x))\,dx$, which is *not* the integral whose value we seek: and we don't care about its value, because the only thing that matters in obtaining (14) is that multiplying by zero yields zero. Likewise for $\lim\limits_{\delta u \to 0} \sum\limits_{u\in[\phi(\alpha),\phi(\beta)]} g(\zeta(u))\delta u = \int\limits_{u=\phi(\alpha)}^{u=\phi(\beta)} g(\zeta(u))\,du$ in obtaining (15).

For example, to calculate

$$I = \int_0^{\frac{1}{63}} \frac{64\sqrt{x}}{(1+x)^{\frac{7}{2}}}\,dx \tag{16}$$

we can use the substitution

$$x = \zeta(u) = \frac{u}{16-u}. \tag{17}$$

Because $x = \frac{u}{16-u} \Longrightarrow (16-u)x = u \Longrightarrow 16x - ux = u \Longrightarrow 16x = ux + u = u(x+1)$, the inverse substition is

$$u = \phi(x) = \frac{16x}{x+1}. \tag{18}$$

So, from (15) with $\alpha = 0$, $\beta = \frac{1}{63}$ and $g(x) = \frac{64\sqrt{x}}{(1+x)^{\frac{7}{2}}}$, we obtain

$$I = \int_{x=0}^{x=\frac{1}{63}} \frac{64\sqrt{x}}{(1+x)^{\frac{7}{2}}}\,dx = \int_{u=\phi(0)}^{u=\phi(\frac{1}{63})} \frac{64\sqrt{\zeta(u)}}{(1+\zeta(u))^{\frac{7}{2}}}\,\zeta'(u)\,du. \tag{19}$$

At first, perhaps, this looks worse. But (17) $\Longrightarrow \zeta'(u) = \frac{d}{du}\left\{\frac{16}{16-u} - 1\right\} = 16\frac{d}{du}\left\{(16-u)^{-1}\right\} - 0 = 16\{-(16-u)^{-2}\}\frac{d}{du}\{16-u\} = 16\{-(16-u)^{-2}\}\{-1\} = \frac{16}{(16-u)^2}$ and $1+\zeta(u) = \frac{16}{16-u}$. Also (17) $\Longrightarrow \phi(0) = 0$ and $\phi\left(\frac{1}{63}\right) = \frac{16}{1+63} = \frac{1}{4}$. So (19) reduces to

$$\begin{aligned}
I &= \int_{u=0}^{u=\frac{1}{4}} \frac{64\sqrt{\frac{u}{16-u}}}{\left(\frac{16}{16-u}\right)^{\frac{7}{2}}}\frac{16}{(16-u)^2}\,du = \int_0^{\frac{1}{4}} 64\sqrt{\frac{u}{16-u}}\left(\frac{16-u}{16}\right)^{\frac{7}{2}}\frac{16}{(16-u)^2}\,du \\
&= \frac{64\cdot 16}{16^{\frac{7}{2}}}\int_0^{\frac{1}{4}} \sqrt{u}(16-u)\,du = \frac{64}{16^{\frac{5}{2}}}\int_0^{\frac{1}{4}}\left\{16u^{\frac{1}{2}} - u^{\frac{3}{2}}\right\}du \\
&= \frac{1}{16}\int_0^{\frac{1}{4}} \frac{d}{du}\left\{\frac{32}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right\}du = \frac{1}{16}\left\{\frac{32}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right\}\Big|_0^{\frac{1}{4}} \\
&= \frac{1}{16}\left\{\frac{32}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}} - \frac{2}{5}\left(\frac{1}{4}\right)^{\frac{5}{2}} - \frac{32}{3}0^{\frac{3}{2}} + \frac{2}{5}0^{\frac{5}{2}}\right\} = \frac{1}{16}\left\{\frac{32}{3}\cdot\frac{1}{8} - \frac{2}{5}\cdot\frac{1}{32} - 0 + 0\right\} \\
&= \frac{317}{3840}\ (\approx 0.0825521).
\end{aligned} \tag{20}$$

Exercises

1. Calculate $\int_0^1 x(x^2+1)^4\,dx$ by

(a) using the substitution $x = \sqrt{u-1}$ and

(b) some other method.

2. Use the substitution $x = 1 + u^2$ to show that $\int_1^5 x\sqrt{x-1}\,dx = \frac{272}{15}$.

3. Use the substitution $x = \sqrt{1+u^2}$ to show that $\int_1^{\sqrt{5}} x^3\sqrt{x^2-1}\,dx = \frac{136}{15}$.

4. Use the substitution $x = 1 + u^2$ to show that

(a) $\int_1^2 (x+2)\sqrt{x-1}\,dx = \frac{12}{5}$

(b) $\int_1^2 (2x+1)\sqrt{x-1}\,dx = \frac{14}{5}$.

Suitable problems from standard calculus texts

Stewart (2003): pp. 420-421, ## 1-74.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.