17. Four different ways to find the area of a circle

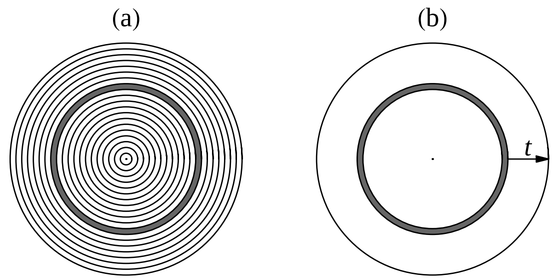

Figure 1: Using concentric annuli for the area of a circular disk. The transverse coordinate, denoted by t, increases in the radial direction.

Suppose you didn't already know that the area enclosed by a circle of radius r is πr^2. How would you find out? One way would be to chop up the region inside the circle into lots and lots of concentric annuli, find the area of each annulus and sum these areas to find the total. In Figure 1a I have drawn only twenty such annuli, but I want you to imagine that there are infinitely many of them; and because there are infinitely many of them, the thickness of each annulus must be vanishingly small—otherwise, you couldn't possibly pack them all into the region enclosed by the circle.

Let A denote the total area, i.e., the area of the circle; and let δA denote the infinitesimal *element* of area—shown shaded in Figure 1b—that is added to a circle of radius t when its radius increases infinitesimally to $t + \delta t$ (for $0 < t < r$). Observe that the direction in which t increases is perpendicular to the direction in which you would have to be headed if you were a tiny creature travelling along the infinitesimal area element, and for that reason we refer to t as the *transverse* coordinate (to the element of area).

Also observe that the inner circumference of the element is $2\pi t$, the outer circumference of the element is $2\pi(t+\delta t)$ and the thickness of the element is δt. Therefore, whatever the magnitude of δA, it must exceed the area of a rectangle with length $2\pi t$ and thickness δt, but it cannot exceed the area of a rectangle with length $2\pi(t + \delta t)$ and thickness δt; i.e., $2\pi t\delta t < \delta A < 2\pi(t + \delta t)\delta t$ or

$$2\pi t\delta t \;<\; \delta A \;<\; 2\pi t\delta t + 2\pi\delta t^2. \tag{1}$$

In other words, it must be true that

$$\delta A \;=\; 2\pi t\delta t + o(\delta t). \tag{2}$$

Hence we can compute the area as

$$\begin{aligned} A \;&=\; \lim_{\delta A\to 0}\sum \delta A \;=\; \lim_{\delta t\to 0}\sum_{t\in[0,r]}\{2\pi t\,\delta t + o(\delta t)\} \\ &=\; \lim_{\delta t\to 0}\sum_{t\in[0,r]} 2\pi t\,\delta t \;+\; \lim_{\delta t\to 0}\sum_{t\in[0,r]}\delta t\,\lim_{\delta t\to 0}\frac{o(\delta t)}{\delta t} \\ &=\; \int_{t=0}^{t=r} 2\pi t\,dt \quad+\quad \int_{t=0}^{t=r} 1\,dt \;\cdot\; 0 \end{aligned}$$

or

$$A \;=\; \int_0^r 2\pi t\,dt \;=\; \pi\int_0^r 2t\,dt \;=\; \pi t^2|_0^r \;=\; \pi\{r^2-0^2\} \;=\; \pi r^2. \tag{3}$$

Now here comes a very important point. Once you know that the area enclosed by a circle of radius r is πr^2, you know that the area enclosed by a circle of radius t is πt^2 and that the area enclosed by a circle of radius $t+\delta t$ is $\pi(t+\delta t)^2$, and hence you also know that the area of the infinitesimal element shaded in Figure 1b is *precisely*

$$\delta A \;=\; \pi(t+\delta t)^2 - \pi t^2 \;=\; 2\pi t\delta t + \pi\delta t^2, \tag{4}$$

which clearly satisfies (1). However—and this is the very important point—you did not need to know (4) in order to calculate the area of the disk: all you needed was $\delta A = 2\pi t\delta t + o(\delta t)$. More generally, to use integration to calculate areas, you do not need to know the element of area precisely: all you need is an approximation with error $o(\delta t)$.

Whenever we use integration to calculate areas, there is always a transverse coordinate t satisfying

$$a \le t \le b \tag{5}$$

for suitable a and b, there is always an element of area of infinitesimal thickness δt whose (infinitesimal) area is given with sufficient accuracy by an equation of the form

$$\delta A \;=\; f(t)\delta t + o(\delta t) \tag{6}$$

for suitable f, and A is always calculated as

$$A \;=\; \lim_{\delta A\to 0}\sum \delta A \;=\; \lim_{\delta t\to 0}\sum_{t\in[a,b]}\{f(t)\,\delta t + o(\delta t)\} \;=\; \int_a^b f(t)\,dt. \tag{7}$$

Nevertheless, there still exists choice over what to use for a transverse coordinate and, correspondingly, how to chop up the area A into suitable infinitesimal elements.

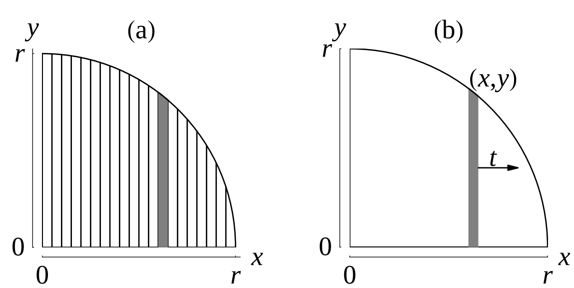

Figure 2: Using vertical strips for the area of a circular disk. The transverse coordinate is $t = x$.

For example, we can instead calculate the area enclosed by a circle of radius r using vertical slices, in which case the transverse coordinate is *horizontal*: $t = x$, and so $\delta t = \delta x$. Let the circle have its center at the origin, so that its equation is

$$x^2 + y^2 \;=\; r^2. \tag{8}$$

The first quadrant of the disk is shown in Figure 2. It will simplify matters if we calculate only the area A of this quadrant (and multiply by 4 for the area of the circle itself).

It is clear from Figure 2 that $\delta A < y\,\delta t$ or $\delta A > y\,\delta t$ according to whether the point (x, y) is at upper left-hand corner or the upper right-hand corner of the shaded elementary area. But it does not matter which: in either case we have

$$\delta A \;=\; y\,\delta t + o(\delta t) \;=\; y\,\delta x + o(\delta x) \tag{9}$$

because $t = x$ is the transverse coordinate. Thus

$$A \;=\; \int_{t=0}^{t=r} y\,dt \;=\; \int_0^r y\,dx. \tag{10}$$

Of course, we cannot integrate y with respect to x without rewriting y in terms of x. From (8), we have $y = \sqrt{r^2 - x^2}$. So

$$\begin{aligned} A \;=\; \int_0^r \sqrt{r^2 - x^2}\,dx \;&=\; \int_0^r \frac{d}{dx}\Big\{r^2 \arctan\Big(\tfrac{x}{\sqrt{r^2-x^2}}\Big) + x\sqrt{r^2 - x^2}\Big\}\,dx \\ &=\; \Big\{r^2 \arctan\Big(\tfrac{x}{\sqrt{r^2-x^2}}\Big) + x\sqrt{r^2 - x^2}\Big\}\Big|_0^r \\ &=\; r^2 \arctan(1) + r \cdot 0 - r^2 \arctan(0) - 0 \\ &=\; r^2 \cdot \tfrac{\pi}{4} \;=\; \tfrac{1}{4}\pi r^2 \end{aligned} \tag{11}$$

as expected.*

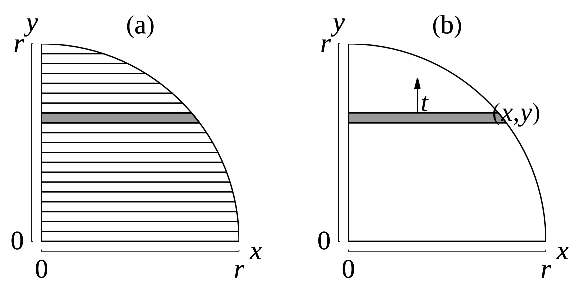

Figure 3: Using horizontal strips for the area of a circular disk. The transverse coordinate is $t = x$.

Needless to say, we can calculate the same area just as easily by using horizontal slices, in which case the transverse coordinate is *vertical*: $t = y$, and so $\delta t = \delta y$. From Figure 3 we see that $\delta A < x\,\delta t$ or $\delta A > x\,\delta t$ according to whether the point (x, y) is at lower right-hand corner or the upper right-hand corner of the shaded elementary area; but again it does not matter which, because in either case we have

$$\delta A \;=\; x\,\delta t + o(\delta t) \;=\; x\,\delta y + o(\delta y). \tag{12}$$

Thus

$$A \;=\; \int_{t=0}^{t=r} x\,dt \;=\; \int_0^r x\,dy. \tag{13}$$

*A in (11) is also readily evaluated by using the substitution $x = r\sin(u)$.

Of course, we cannot integrate x with respect to y without rewriting x in terms of y, for which we use (8): $x = \sqrt{r^2 - y^2}$. Thus, recycling (11), we obtain

$$A = \int_0^r \sqrt{r^2 - y^2}\, dy = \int_0^r \sqrt{r^2 - x^2}\, dx = \tfrac{1}{4}\pi r^2. \tag{14}$$

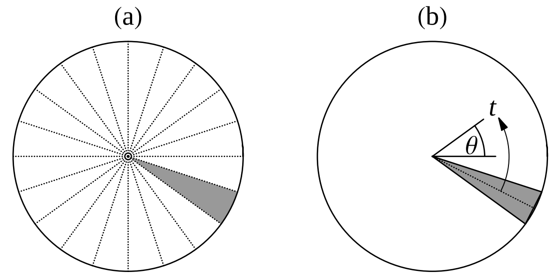

Figure 4: Using sectors for the area of a circular disk. The transverse coordinate is $t = \theta$.

There is even another way in which to chop up a disk into elements of area: use sectors. In Figure 4a I have drawn only twenty such sectors, but as usual there are really infinitely many of them, so that the thickness of each must be vanishingly small (yet still greater at the circumference than at the center). Because motion along such an infinitesimal sector is radial, the transverse direction must be azimuthal, i.e., in the direction of increase of the polar angle θ. The generic such sector is represented by the shading in Figure 4a and corresponds to an increase of azimuth from θ to $\theta + \delta\theta$, so that the angle at the center is $\delta\theta$. Comparing with Figure 4b, we see that the area of the sector is approximated with negligible error by the sum of the areas of two congruent right-angled triangles, each with hypotenuse r, altitude $r\cos\left(\frac{1}{2}\delta\theta\right)$ and base $r\sin\left(\frac{1}{2}\delta\theta\right)$, so that

$$\begin{aligned} \delta A &= 2 \cdot \tfrac{1}{2} r \cos\left(\tfrac{1}{2}\delta\theta\right) \cdot r \sin\left(\tfrac{1}{2}\delta\theta\right) + o(\delta\theta) \\ &= \tfrac{1}{2} r^2 \cdot \sin(\delta\theta) + o(\delta\theta) \end{aligned} \tag{15}$$

on using the trigonometric identity $\sin(2A) = 2\cos(A)\sin(A)$. But from Exercise 1 of Lecture 20 we have $\sin(\delta\theta) = \delta\theta + o(\delta\theta)$. Hence for $0 \le \theta < 2\pi$ we have†

$$\delta A = \tfrac{1}{2} r^2\, \delta\theta + o(\delta\theta) \Longrightarrow A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \tfrac{1}{2} r^2 \int_0^{2\pi} 1\, d\theta = \tfrac{1}{2} r^2 \cdot 2\pi = \pi r^2. \tag{16}$$

All four methods generalize to other planar regions, although in practice the second and third are most commonly used. In each of these cases, we slice the region into (infinitely) many strips, identify a transverse coordinate t (either $t = x$ or $t = y$) such that the region is covered for suitable $a \le t \le b$, find the area of an elementary strip in the form

$$\delta A = h(t)\delta t + o(\delta t) \tag{17}$$

†In fact $\delta A = \frac{1}{2} r^2\, \delta\theta$ *precisely*—but as before, that information is more accurate than is necessary.

where h stands for the "height" of the strip (regardless of whether it is vertical or horizontal) and thus obtain

$$A \;=\; \int_a^b h(t)\,dt. \tag{18}$$

In the case of a circular disk, it makes no difference whether $t = x$ or $t = y$: as we have seen, both approaches yield the very same integral. In other cases, however, one approach may require significantly less calculation than the other.

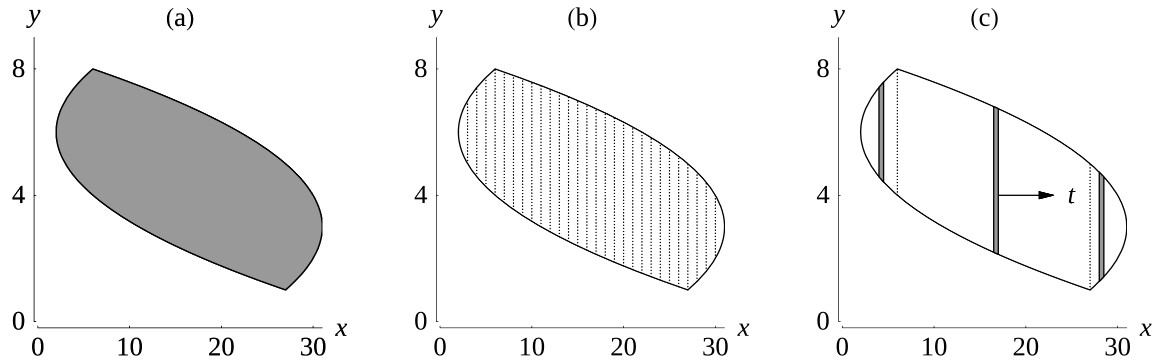

Figure 5: Using vertical strips for the area of a planar region. The transverse coordinate is $t = x$.

For example, to find the area of the region shaded in Figure 5a with top boundary

$$y \;=\; T(x) \;=\; \begin{cases} 6+\sqrt{x-2} & \text{if } 2 \le x < 6 \\ 3+\sqrt{31-x} & \text{if } 6 \le x \le 31 \end{cases} \tag{19}$$

rising from $(2,6)$ to $(6,8)$ before falling from $(6,8)$ to $(31,3)$ and bottom boundary

$$y \;=\; B(x) \;=\; \begin{cases} 6-\sqrt{x-2} & \text{if } 2 \le x < 27 \\ 3-\sqrt{31-x} & \text{if } 27 \le x \le 31 \end{cases} \tag{20}$$

falling from $(2,6)$ to $(27,1)$ before rising from $(27,1)$ to $(31,3)$, we slice the region into vertical strips (Figure 5b) and observe that there are three generic types (Figure 5c). The transverse coordinate is $t = x$, and the height of a strip is given by $h(t) =$

$$h(x) \;=\; T(x)-B(x) \;=\; \begin{cases} 2\sqrt{x-2} & \text{if } 2 \le x < 6 \\ \sqrt{31-x}+\sqrt{x-2}-3 & \text{if } 6 \le x < 27 \\ 2\sqrt{31-x} & \text{if } 27 \le x \le 31 \end{cases} \tag{21}$$

so that h is a join of three components. Hence, from (18), we obtain

$$\begin{aligned} A &= \int_2^{31} h(x)\,dx \;=\; \int_2^{6} h(x)\,dx + \int_6^{27} h(x)\,dx + \int_{27}^{31} h(x)\,dx \\ &= 2\int_2^6 (x-2)^{\frac12}\,dx + \int_6^{27} \left\{(31-x)^{\frac12} + (x-2)^{\frac12} - 3\right\}dx + 2\int_{27}^{31} (31-x)^{\frac12}\,dx \\ &= \tfrac43 (x-2)^{\frac32}\Big|_2^6 \;+\; \left\{-\tfrac23(31-x)^{\frac32} + \tfrac23(x-2)^{\frac32} - 3x\right\}\Big|_6^{27} \;-\; \tfrac43(31-x)^{\frac32}\Big|_{27}^{31} \\ &= \tfrac43\{8-0\} + \left\{-\tfrac{16}{3} + \tfrac{250}{3} - 81\right\} - \left\{-\tfrac{250}{3} + \tfrac{16}{3} - 18\right\} - \tfrac43\{0-8\} \;=\; \tfrac{343}{3}. \end{aligned} \tag{22}$$

But it would have been easier to use horizontal strips instead, because then there is only a single generic type stretching from the left-hand boundary

$$x = L(y) = 38 - 12y + y^2 \tag{23}$$

to the right-hand boundary

$$x = R(y) = 22 + 6y - y^2 \tag{24}$$

for $1 \leq y \leq 8$ (Figure 6c).

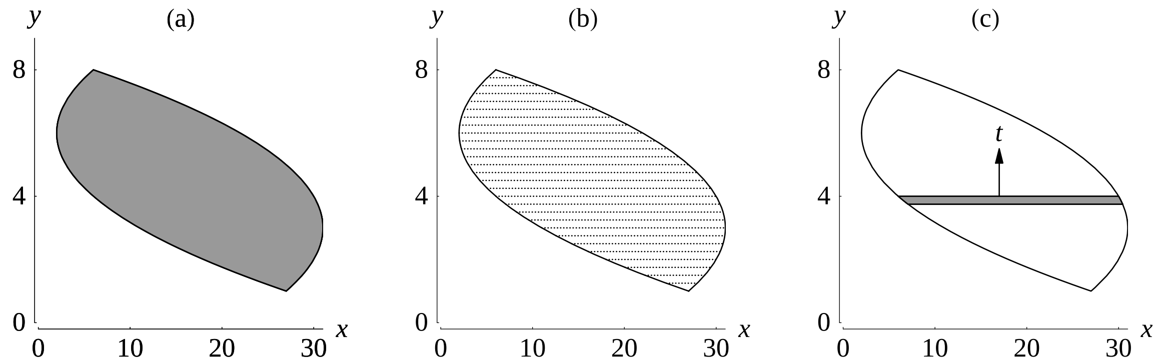

Figure 6: Using vertical strips for the area of a planar region. The transverse coordinate is $t = y$.

So the elementary area is given by

$$\delta A = h(y)\,\delta y + o(\delta y) \tag{25}$$

where h denotes the "height" of the generic strip; i.e., from (23)-(24),

$$h(y) = R(y) - L(y) = 2\{9y - y^2 - 8\}. \tag{26}$$

The limits of integration are determined by $L(y) = R(y)$ or $h(y) = 0$, which implies $y = 1$ or $y = 8$. Hence

$$\begin{aligned} A &= \int_1^8 h(y)\,dy = 2\int_1^8 \{9y - y^2 - 8\}\,dy = 2 \cdot \left\{\tfrac{9}{2}y^2 - \tfrac{1}{3}y^3 - 8y\right\}\Big|_1^8 \\ &= 2 \cdot \left\{\tfrac{160}{3} - \left(-\tfrac{23}{6}\right)\right\} = \tfrac{343}{3} \end{aligned} \tag{27}$$

as before.

Suitable problems from standard calculus texts

Stewart (2003): pp. 442-443, ## 1-30, 38 and 44-49.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.