18. Two different ways to find the volume of a cone

Suppose you didn't already know that the volume of a solid cone of radius R and height H is $\frac{1}{3}\pi R^2 H$. How would you find out? One way would be to chop up the cone into lots and lots of thin coaxial cylindrical shells with sloping rooves, find the volume of each such shell and then sum the volumes to find the total. In Figure 1 I have drawn only eight such shells, and in Figure 2 I have suggested only twenty, but I want you to imagine that there are infinitely many of them; and because there are infinitely many of them, the thickness of each shell must be vanishingly small—otherwise, you couldn't possibly pack them all into the region occupied by the cone.

Let V denote the total volume, i.e., the volume of the cone; and let δV denote the infinitesimal *element* of volume–shown shaded in Figure 2e—that is added to the part of the cone whose perpendicular distance from the axis of symmetry does not exceed t when t increases infinitesimally to $t + \delta t$ (for $0 < t < R$). Observe that the direction in which t increases is perpendicular to the surface of the infinitesimal element of volume, and for that reason we refer to t as the *transverse* coordinate. So transverse equals radial.*.

Also observe that the INNER CIRCUMFERENCE of the infinitesimal volume element is $2\pi t$, the OUTER CIRCUMFERENCE of the element is $2\pi(t + \delta t)$ and the thickness of the element is δt. Let h denote the INNER HEIGHT of the infinitesimal volume element, so that—because it is infinitesimal—its OUTER HEIGHT must be $h + \delta h$, where $\delta h < 0$. Then whatever the magnitude of δV, it must exceed the volume of a cuboid of thickness δt with length INNER CIRCUMFERENCE and breadth OUTER HEIGHT, but it cannot exceed the volume of a cuboid of thickness δt with length OUTER CIRCUMFERENCE and breadth INNER HEIGHT; that is, $2\pi t(h + \delta h)\delta t < \delta V < 2\pi h(t + \delta t)\delta t$ or

$$2\pi th\,\delta t + 2\pi t\,\delta h\,\delta t \;<\; \delta V \;<\; 2\pi th\,\delta t + 2\pi h\,\delta t^2 \tag{1}$$

(see Figure 2f, where the bounding cuboids are sketched). But it is clear from Figures 1-2 that $\delta h \to 0$ as $\delta t \to 0$ in such a way that

$$\delta h \;=\; o(\delta t) \tag{2}$$

and (from the similar triangles in the vertical cross-section of Figure 2d)

$$\frac{H}{R} \;=\; \frac{h}{R-t}. \tag{3}$$

Thus, from (1)-(3),

$$\delta V \;=\; 2\pi th\delta t + o(\delta t) \tag{4}$$

with†

$$h \;=\; h(t) \;=\; \frac{H}{R}(R-t). \tag{5}$$

*In this particular case, but not always—see later

†In fact $\delta h = h'(t)\,\delta t = -\frac{H}{R}\delta t$ (because h is a linear function of t, the junk term $o(\delta t)$ in $\delta h = h'(t)\,\delta t + o(\delta t)$ is precisely zero), from which it can be shown that (1) is satisfied with $\delta V = 2\pi th\delta t + \pi(2h - H)\delta t^2 - \frac{2\pi H}{3R}\delta t^3$ *precisely*. But this is far more than we need to know to find the volume of the cone: (4) is quite enough.

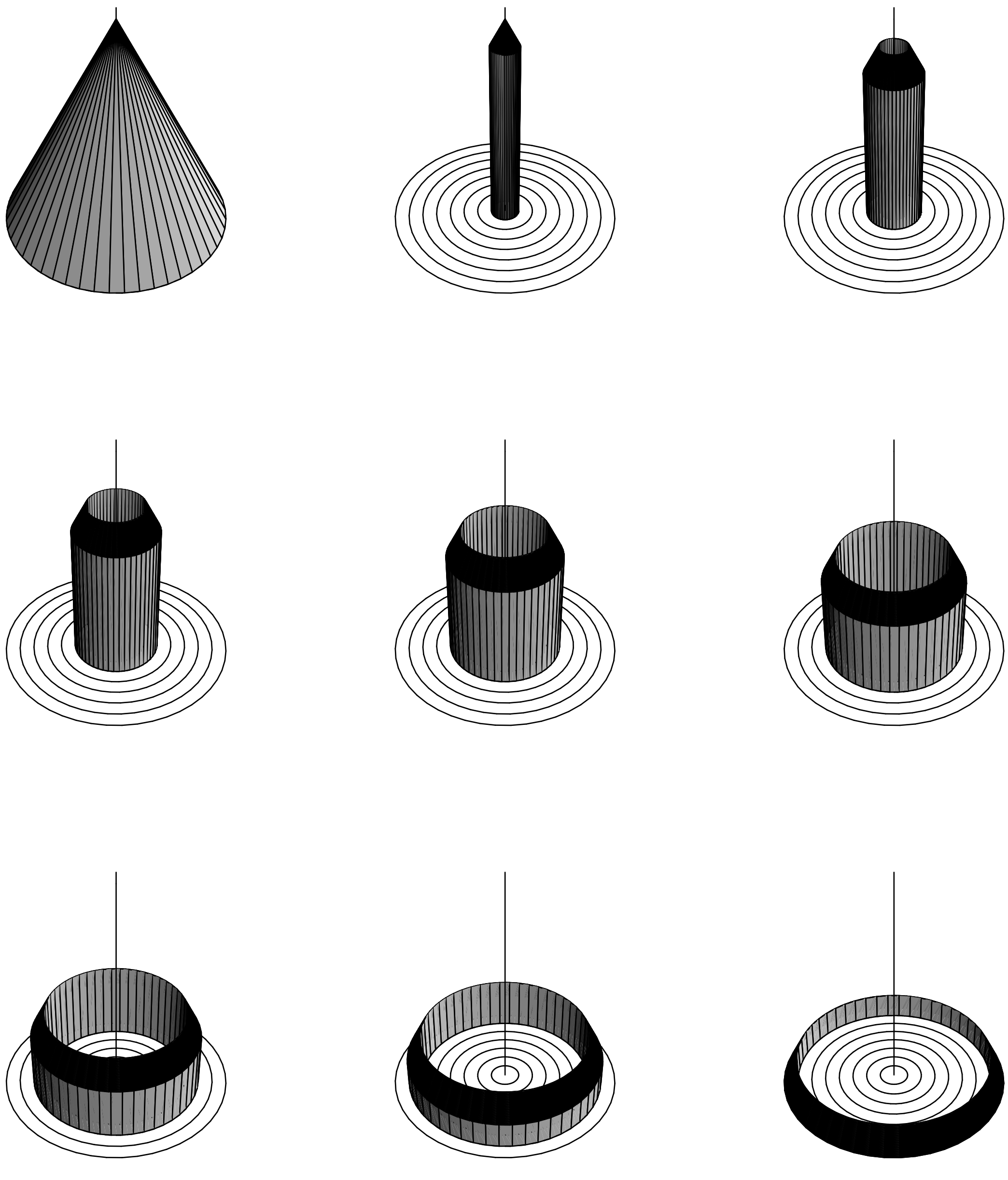


Figure 1: Using concentric cylindrical shells with sloping rooves for the volume of a (right circular) cone. The solids in the last eight panels fit inside one another to yield the cone in the first.

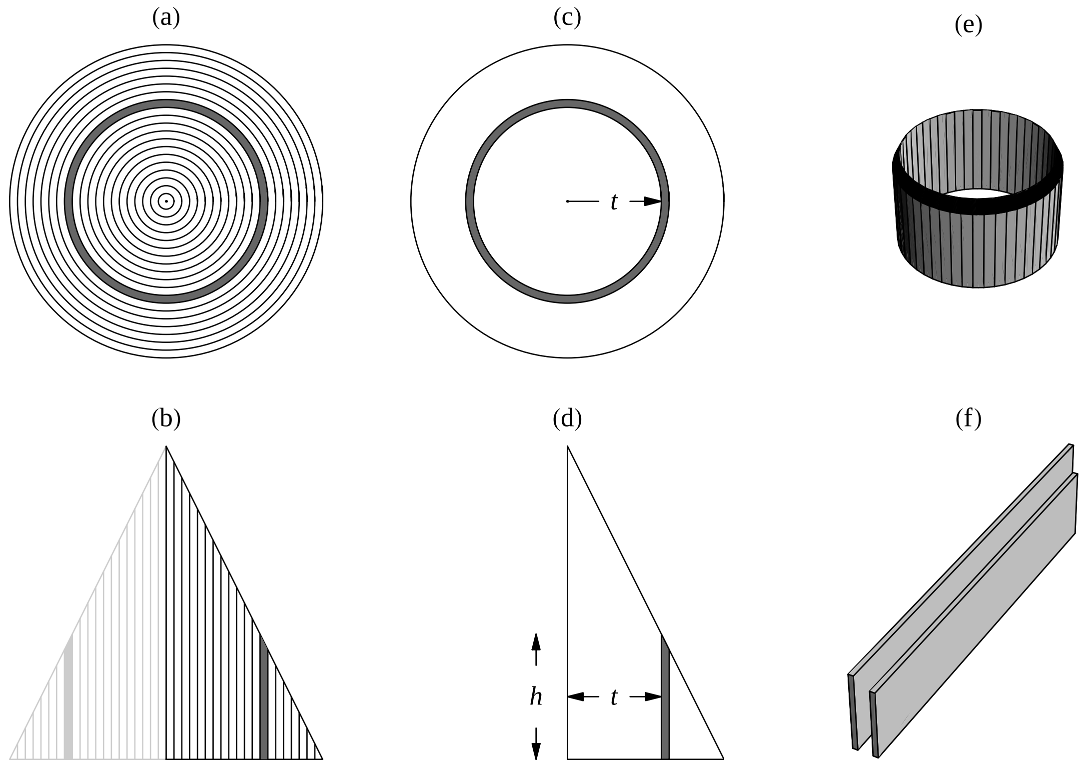


Figure 2: Using concentric cylindrical shells with sloping rooves for the volume of a (right circular) cone. (a) The sloping roof of the cone, viewed from above with a typical elementary volume shaded. (b) A vertical cross-section through the axis of symmetry, with a typical elementary volume (whose cross section is a vertical strip) shaded. (c) Horizontal cross-section of the generic elementary volume; the transverse coordinate t increases perpendicularly to its surface (i.e., radially). (d) Vertical cross-section of the generic elementary volume; the cone is traced out by rotating this triangle about the axis of symmetry through angle 2π or $360°$. (e) The generic elementary volume. (f) Rectangular slabs yielding upper (left) and lower (right) bounds on δV.

From (4), i.e., $\delta V = 2\pi th\delta t + o(\delta t)$, we can now compute the volume as

$$\begin{aligned} V &= \lim_{\delta V\to 0}\sum \delta V = \lim_{\delta t\to 0}\sum_{t\in[0,R]}\{2\pi th\,\delta t + o(\delta t)\} \\ &= \lim_{\delta t\to 0}\sum_{t\in[0,R]} 2\pi th\,\delta t + \lim_{\delta t\to 0}\sum_{t\in[0,R]}\delta t \lim_{\delta t\to 0}\frac{o(\delta t)}{\delta t} \\ &= \int_{t=0}^{t=R} 2\pi th\,dt + \int_{t=0}^{t=R} 1\,dt \cdot 0 = \int_0^R 2\pi th\,dt \end{aligned} \tag{6}$$

or, on using $h = \frac{H}{R}(R-t)$ from (5),

$$V = \frac{\pi H}{R}\int_0^R 2t(R-t)\,dt = \frac{\pi H}{R}\int_0^R (2Rt - 2t^2)\,dt = \frac{\pi H}{R}\left\{Rt^2 - \frac{2}{3}t^3\right\}\Big|_0^R = \frac{1}{3}\pi R^2 H \tag{7}$$

as expected.

Whenever we use integration to calculate a volume V, there is always a transverse coordinate t satisfying

$$a \le t \le b \tag{8}$$

for suitable a, b and an element of volume of infinitesimal thickness δt whose (infinitesimal) volume δV is given with sufficient accuracy by an equation of the form

$$\delta V \;=\; f(t)\delta t + o(\delta t) \tag{9}$$

for suitable f, and V is always calculated as

$$V \;=\; \lim_{\delta V \to 0} \sum \delta V \;=\; \lim_{\delta t \to 0} \sum_{t \in [a,b]} \{f(t)\,\delta t + o(\delta t)\} \;=\; \int_a^b f(t)\,dt. \tag{10}$$

For example, with cylindrical shells we have $f(t) = 2\pi t h(t)$, from (7). Nevertheless, there is considerable choice over what to use for a transverse coordinate and, correspondingly, how to chop up the volume V into suitable infinitesimal elements: we do not have to use thin cylindrical shells.

In particular, we can use thin circular disks with sloping rims instead. In Figure 3 I have drawn only eight such disks, and in Figure 4 I have suggested only twenty, but I want you to imagine there are infinitely many of them; and because there are infinitely many of them, the thickness of each must be vanishingly small—otherwise, you couldn't possibly pack them all into the region occupied by the cone. Let δV now denote the infinitesimal element of volume that is added to the part of the cone whose perpendicular distance from the *base* does not exceed t when t increases infinitesimally to $t + \delta t$ (for $0 < t < H$). The direction in which t increases is still (as always) perpendicular to the surface of the infinitesimal element of volume, but now transverse means axial—not radial.

Let r denote the base radius of the infinitesimal volume element (Figure 4d) so that—because it is infinitesimal—its roof radius must be $r + \delta r$, where $\delta r < 0$; then the lower surface area of the volume element is πr^2, its upper surface area is $\pi(r + \delta r)^2$ and its thickness is δt (as always). Whatever the magnitude of δV, it must exceed the volume of a cylindrical disk (i.e., one having vertical rim) with thickness δt and cross-sectional area $\pi(r + \delta r)^2$, but it cannot exceed the volume of a cylindrical disk with thickness δt and cross-sectional area πr^2; that is, $\pi(r + \delta r)^2 \delta t < \delta V < \pi r^2 \delta t$ or

$$\pi r^2 \delta t + 2\pi r\,\delta r\,\delta t + \pi \delta r^2 \delta t \;<\; \delta V \;<\; \pi r^2 \delta t. \tag{11}$$

But it is clear from Figures 3-4 that $\delta r \to 0$ as $\delta t \to 0$ in such a way that

$$\delta r \;=\; o(\delta t) \tag{12}$$

and (from the similar triangles in the vertical cross-section of Figure 4d)

$$\frac{H}{R} \;=\; \frac{H - t}{r}. \tag{13}$$

Thus, from (11)-(13),

$$\delta V \;=\; \pi r^2 \delta t + o(\delta t) \tag{14}$$

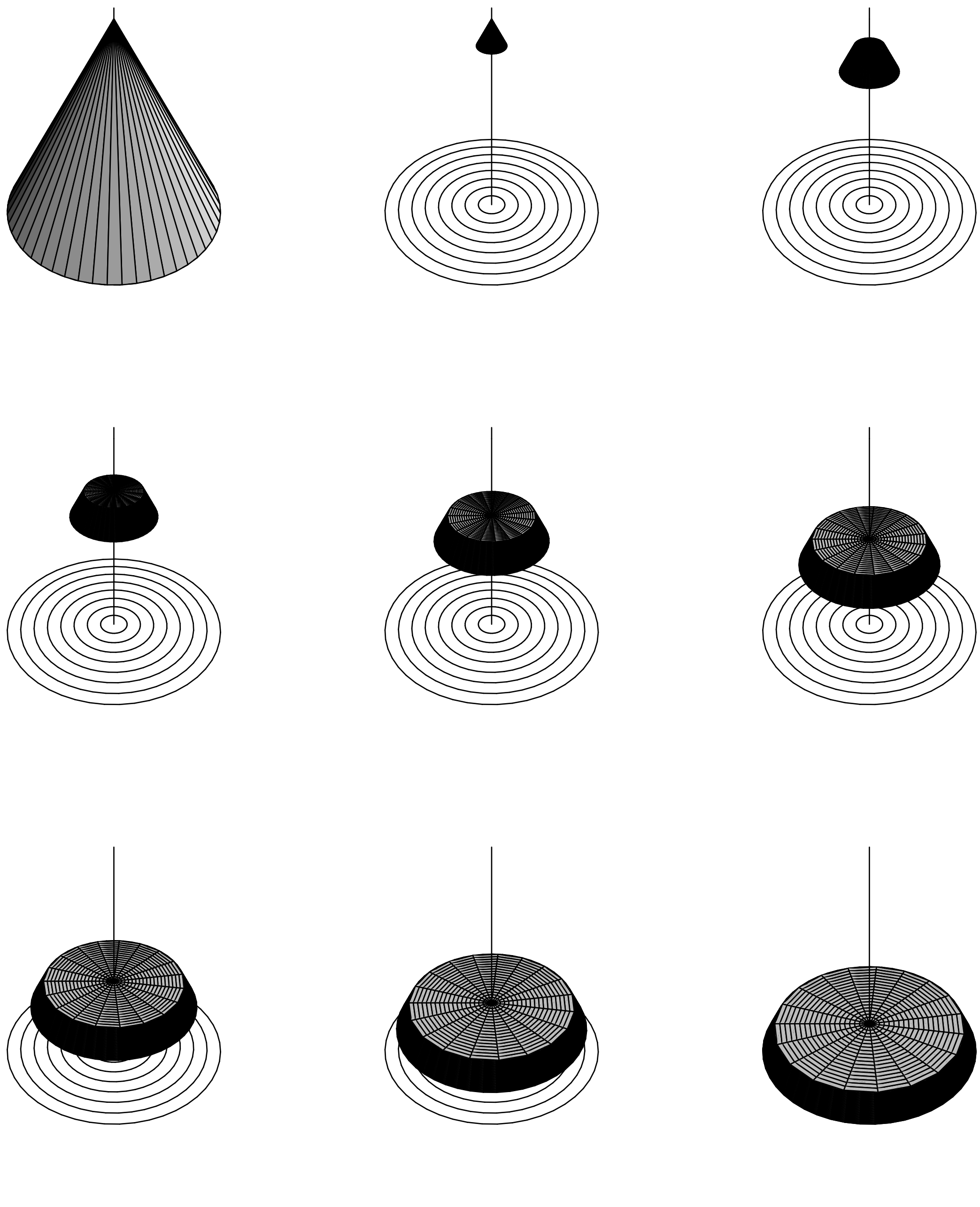


Figure 3: Using coaxial circular disks with sloping rims for the volume of a (right circular) cone. The solids in the last eight panels stack upon one another to produce the cone in the first panel.

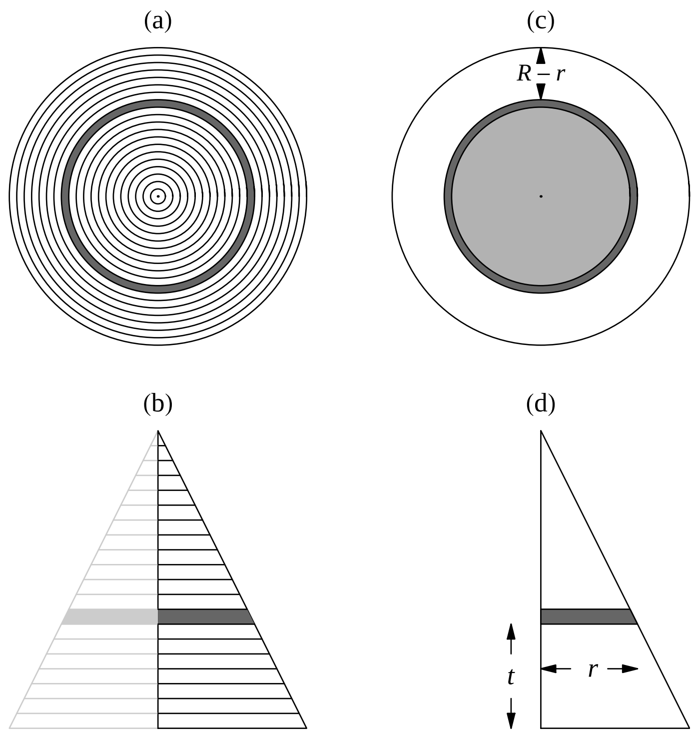


Figure 4: Using coaxial circular disks with sloping rims for the volume of a (right circular) cone. (a) The sloping roof of the cone, viewed from above with a typical elementary volume shaded. The view is identical to Figure 2a but the interpretation is completely different because the shading no longer represents the sloping roof of a cylindrical shell—instead it represents the sloping rim of a thin circular disk perpendicular to the cone's axis of symmetry. (b) A vertical cross-section through the axis of symmetry, with a typical elementary volume (whose cross section is a horizontal strip) shaded. (c) Horizontal cross-section of the generic elementary volume, viewed from above; the transverse coordinate t increases perpendicularly to its surface, and hence to the page (i.e., axially). (d) Vertical cross-section of the generic elementary volume; the cone is traced out by rotating this triangle about the axis of symmetry through angle 2π or $360°$.

with‡

$$r \;=\; r(t) \;=\; \frac{R}{H}(H-t). \tag{15}$$

From (14) we now compute the volume as

$$\begin{aligned} V \;&=\; \lim_{\delta V \to 0} \sum \delta V \;=\; \lim_{\delta t \to 0} \sum_{t \in [0,H]} \{\pi r^2\, \delta t + o(\delta t)\} \\ &=\; \lim_{\delta t \to 0} \sum_{t \in [0,H]} \pi r^2\, \delta t \;+\; \lim_{\delta t \to 0} \sum_{t \in [0,H]} \delta t \, \lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} \\ &=\; \int_{t=0}^{t=H} \pi r^2\, dt \;+\; \int_{t=0}^{t=H} 1\, dt \cdot 0 \;=\; \int_0^H \pi r^2\, dt \end{aligned} \tag{16}$$

or, on using (15),

$$V \;=\; \tfrac{\pi R^2}{H^2} \int_0^H (H-t)^2\, dt \;=\; \tfrac{\pi R^2}{H^2} \left\{-\tfrac{1}{3}(H-t)^3\right\}\Big|_0^H \;=\; \tfrac{1}{3}\pi R^2 H \tag{17}$$

as expected.

Both methods generalize to other volumes with an axis of circular symmetry (which need not be vertical); and with non-circular slices as elementary volumes, the second method generalizes to arbitrary volumes. As a general rule, however, we try to avoid drawing three-dimensional diagrams like Figures 1 and 3 because a cross-section through the axis of symmetry–like those in Figures 2 and 4—usually suffices. Indeed we can use the very same diagram that we would use to calculate area: it is necessary only to re-interpret it appropriately.

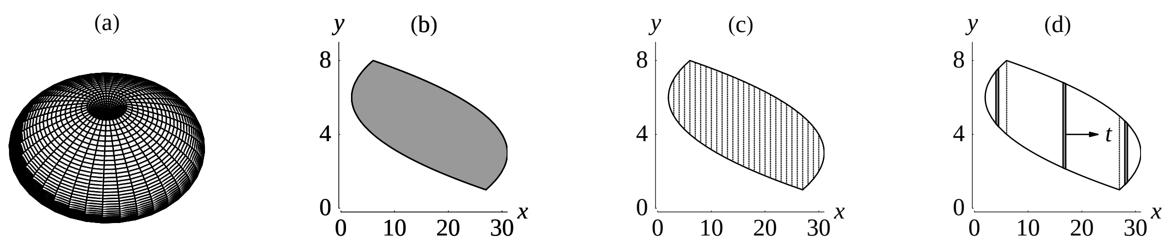


Figure 5: Using thin coaxial cylindrical shells for a volume of revolution. The transverse coordinate is $t = x$; (b)-(d) show vertical cross-sections through the axis of symmetry.

Consider, for example, the volume in Figure 5a, generated when the region shaded in Figure 5b (as opposed to the triangle in Figure 2d or Figure 4d) is rotated through $360°$ about the y-axis (which is therefore in this case the axis of symmetry). If we use cylindrical shells as elementary volumes, then the transverse coordinate is perpendicular

‡In fact $\delta r = r'(t)\,\delta t = -\frac{R}{H}\delta t$ (again, because r is a linear function of t, the junk term $o(\delta t)$ in $\delta r = r'(t)\,\delta t + o(\delta t)$ is precisely zero), from which it can be shown that (11) is satisfied with $\delta V = \pi r^2 \delta t + \pi r \delta r \delta t + \frac{\pi R^2}{3H^2}\delta t^3$ *precisely*. But again this is far more than we need to know to find the volume of the cone: (14) suffices.

to the axis of symmetry, hence in this case perpendicular to the y-axis. Because of the circular symmetry, it does not matter in which direction we measure distance from the axis of symmetry; accordingly, we choose the direction parallel to the x-axis by setting

$$t = x, \qquad 2 \le x \le 31. \tag{18}$$

Then, because Figures 5b-5d are identical to Figure 5 of Lecture 17, we already know from (21) of that lecture that the height of the generic element (Figure 5d) is

$$h(x) = \begin{cases} 2\sqrt{x-2} & \text{if } 2 \le x < 6 \\ \sqrt{31-x} + \sqrt{x-2} - 3 & \text{if } 6 \le x < 27 \\ 2\sqrt{31-x} & \text{if } 27 \le x \le 31. \end{cases} \tag{19}$$

But we interpret $h(x)$ as the height of a cylindrical shell of radius x (as opposed to the height of a planar strip). Now, from (18) by analogy with (4) and (6), we obtain

$$\delta V = 2\pi x h(x)\delta x + o(\delta x) \Longrightarrow V = \int_2^{31} 2\pi x h(x)\, dx = 2\pi \int_2^{31} x h(x)\, dx. \tag{20}$$

Hence, from (19),

$$V = 2\pi \left\{ \int_2^6 x h(x)\, dx + \int_6^{27} x h(x)\, dx + \int_{27}^{31} x h(x)\, dx \right\} = 2\pi\{2I_1 + I_2 + 2I_3\} \tag{21}$$

where I_1, I_2 and I_3 are defined by

$$\begin{aligned} I_1 &= \int_2^6 x\sqrt{x-2}\, dx \\ I_2 &= \int_6^{27} \left\{x\sqrt{31-x} + x\sqrt{x-2} - 3x\right\}\, dx \\ I_3 &= \int_{27}^{31} x\sqrt{31-x}\, dx. \end{aligned} \tag{22}$$

Each of these integrals is readily evaluated. For example, the substitution $u = \sqrt{x-2} \Longrightarrow x = 2 + u^2 \Longrightarrow \frac{dx}{du} = 0 + 2u \Longrightarrow$

$$I_1 = \int_{u=\sqrt{2-2}}^{u=\sqrt{6-2}} (2+u^2)u\frac{dx}{du}\, du = 2\int_0^2 (2u^2 + u^4)\, du = 2\left(\tfrac{2}{3}u^3 + \tfrac{1}{5}u^5\right)\Big|_0^2 = \tfrac{352}{15} \tag{23}$$

and from Exercise 1 we similarly find that $I_2 = \frac{3069}{2}$ and $I_3 = \frac{2288}{15}$. So, from (21),

$$V = 2\pi\left(2 \cdot \tfrac{352}{15} + \tfrac{3069}{2} + 2 \cdot \tfrac{2288}{15}\right) = 3773\pi. \tag{24}$$

Nevertheless, it would have been easier to use disks instead. Then the transverse coordinate is aligned with (as opposed to perpendicular to) the axis of symmetry; accordingly, we now choose

$$t = y, \qquad 1 \le y \le 8 \tag{25}$$

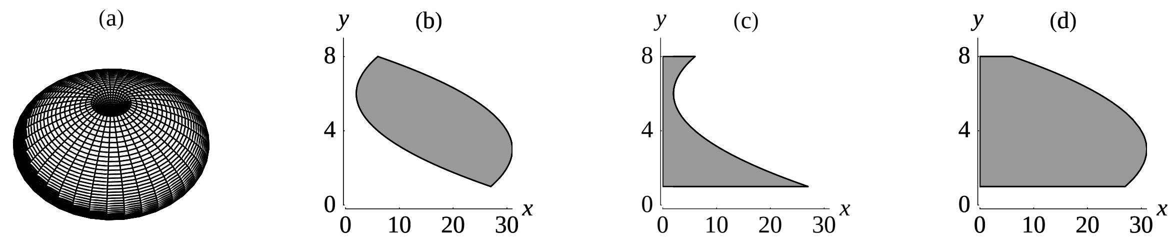


Figure 6: Using thin coaxial circular disks for a volume of revolution. The transverse coordinate is $t = y$, and (b)-(d) show vertical cross-sections through the axis of symmetry. The planar region (b) that generates the solid is the difference between planar region (d) and planar region (c).

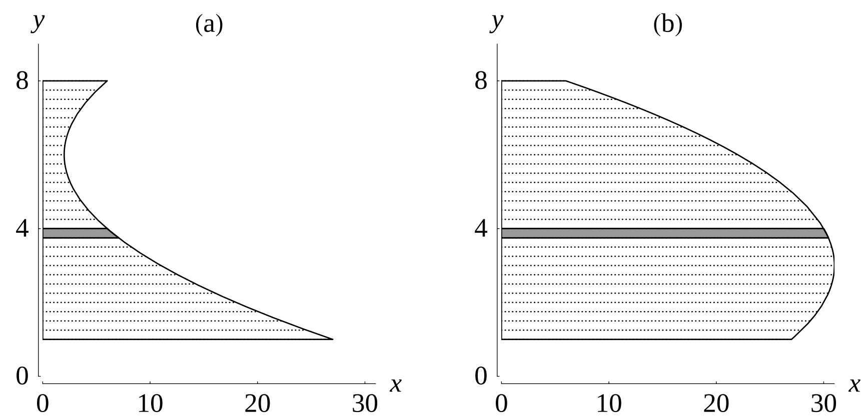


Figure 7: Using thin coaxial circular disks for a volume of revolution. The diagrams show a vertical cross section through the axis of symmetry of the two elementary volumes used.

so that perpendicular distance from axis of symmetry is

$$r = x. \tag{26}$$

Observe from Figure 6 that the planar region that generates the solid of revolution when rotated about the y-axis (i.e., the region shaded in Figure 6b) is the difference between the region shaded in Figure 6d and the region shaded in Figure 6c. So we can use disks as elementary volumes to calculate the volumes generated by rotating the planar regions in Figure 6c and Figure 6d about the y-axis, and then subtract the first from the second to deduce the volume generated by rotating the planar region in Figure 6b.

From Lecture 17, the curved boundary of the region shaded in Figure 6c and sliced into disks in Figure 7a is given by

$$x = L(y) = 38 - 12y + y^2. \tag{27}$$

Hence, for the volume of revolution it generates, we obtain

$$\begin{aligned}
\delta V &= \pi r^2\,\delta t + o(\delta t) \;=\; \pi x^2\,\delta y + o(\delta y) \Longrightarrow \\
V &= \int_1^8 \pi r^2\,dt \;=\; \int_1^8 \pi x^2\,dy \;=\; \pi\int_1^8 \{L(y)\}^2\,dy \;=\; \pi\int_1^8 \{38 - 12y + y^2\}^2\,dy \\
&= \pi\int_1^8 \{1444 - 912y + 220y^2 - 24y^3 + y^4\}\,dy \\
&= \pi\left\{1444y - 456y^2 + \tfrac{220}{3}y^3 - 6y^4 + \tfrac{1}{5}y^5\right\}\Big|_1^8 \;=\; \pi\left\{\tfrac{28384}{15} - \tfrac{15833}{15}\right\} \;=\; \tfrac{12551\pi}{15}
\end{aligned} \tag{28}$$

from (25)-(26) by analogy with (14) and (16). Also from Lecture 17, the curved boundary of the region shaded in Figure 6d and sliced into disks in Figure 7b is given by

$$x \;=\; R(y) \;=\; 22 + 6y - y^2. \tag{29}$$

Hence, for the volume of revolution it generates, we obtain

$$\begin{aligned}
\delta V &= \pi r^2\,\delta t + o(\delta t) \;=\; \pi x^2\,\delta y + o(\delta y) \Longrightarrow \\
V &= \int_1^8 \pi r^2\,dt \;=\; \int_1^8 \pi x^2\,dy \;=\; \pi\int_1^8 \{R(y)\}^2\,dy \;=\; \pi\int_1^8 \{22 + 6y - y^2\}^2\,dy \\
&= \pi\int_1^8 \{484 + 264y - 8y^2 - 12y^3 + y^4\}\,dy \\
&= \pi\left\{484y + 132y^2 - \tfrac{8}{3}y^3 - 3y^4 + \tfrac{1}{5}y^5\right\}\Big|_1^8 \;=\; \pi\left\{\tfrac{78304}{15} - \tfrac{9158}{15}\right\} \;=\; \tfrac{69146\pi}{15}
\end{aligned} \tag{30}$$

again from (25)-(26) by analogy with (14) and (16). Hence the volume of revolution generated by the planar region in Figure 6a is

$$\tfrac{69146\pi}{15} - \tfrac{12551\pi}{15} \;=\; 3773\pi \tag{31}$$

in agreement with (24).

Exercises

1. For the integrals defined by (22):

(a) Use the substitution $u = \sqrt{31 - x}$ to show that $I_3 = \frac{2288}{15}$.

(b) Show that $I_2 = \frac{3069}{2}$.

2. The region R is bounded above by the line $y = \frac{1}{2}x + 1$, to the right by the parabola $x = \frac{1}{3}y^2 + 1$ and below by the line $y = 0$.

(a) Use integration with respect to x to find the volume generated by rotating R about the x-axis.

(b) Use integration with respect to y to find the volume generated by rotating R about the x-axis.

(c) Use integration with respect to x to find the volume generated by rotating R about $x = 4$.

(d) Use integration with respect to y to find the volume generated by rotating R about $x = 4$.

Suitable problems from standard calculus texts

Stewart (2003): pp. 452-453, ## 1-36 and 48-49; pp. 452-453, ## 1-26 and 37-46.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

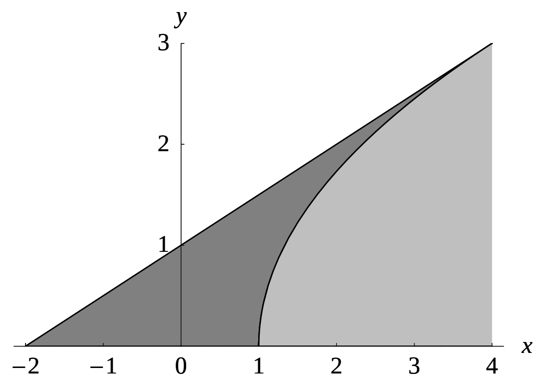
Solutions or hints for selected exercises

2. (a) Here the transverse coordinate $t = x$ is aligned with the axis of symmetry. We must therefore use disks. To find the volume V generated by rotating R, i.e., the darker shaded region in the diagram below, we subtract the volume V_1 generated by rotating the lighter shaded region from the volume V_2 of the cone generated by rotating both regions together. This cone has "base radius" 3 and "height" 6. Hence $V_2 = \frac{1}{3}\pi 3^2 \times 6 = 18\pi$. The element of volume for V_1 is $\delta V_1 = \pi r^2 \delta t + o(\delta t) = \pi r^2 \delta x + o(\delta x)$, where r is the distance from the axis of symmetry to the circumference of the disk, and hence to the curve $x = \frac{1}{3}y^2 + 1$ or $y = \sqrt{3(x-1)}$. Hence $r = y =$

$\sqrt{3(x-1)} \Longrightarrow r^2 = 3(x-1)$. Now we have

$$\begin{aligned} V_1 &= \lim_{\delta V_1 \to 0} \sum \delta V_1 \;=\; \lim_{\delta t \to 0} \sum_{t \in [\text{LOWEST}, \text{HIGHEST}]} \{\pi r^2\, \delta t + o(\delta t)\} \;=\; \int_{t=\text{LOWEST}}^{t=\text{HIGHEST}} \pi r^2\, dt \\ &= \int_{x=1}^{x=4} \pi r^2\, dx \;=\; 3\pi \int_1^4 (x-1)\, dx \;=\; 3\pi\, \tfrac{1}{2}(x-1)^2\Big|_1^4 \;=\; 3\pi \left\{\tfrac{1}{2}(4-1)^2 - \tfrac{1}{2}0^2\right\} \end{aligned}$$

or $V_1 = \frac{27\pi}{2}$. Thus $V = V_2 - V_1 = 18\pi - \frac{27\pi}{2} = \frac{9\pi}{2}$.



(b) The transverse coordinate $t = y$ is now perpendicular to the axis of symmetry. We must therefore use cylindrical shells. The generic shell is obtained by rotating the generic strip, which is aligned with the axis of symmetry and therefore stretches from the point (x_1, y) to the point (x_2, y), where (x_1, y) lies on the line $y = \frac{1}{2}x + 1$ or $x = 2(y-1)$, and (x_2, y) lies on the parabola $x = \frac{1}{3}y^2 + 1$. In other words, $x_1 = 2(y-1)$ and $x_2 = \frac{1}{3}y^2 + 1$. So the "height" of the strip—and hence of the generic shell—is

$$h \;=\; h(y) \;=\; |x_2 - x_1| \;=\; x_2 - x_1 \;=\; \tfrac{1}{3}y^2 + 1 - 2(y-1) \;=\; 3 - 2y + \tfrac{1}{3}y^2$$

(note that $x_2 > x_1 \Longrightarrow |x_2 - x_1| = x_2 - x_1$). The volume element is $\delta V = 2\pi r h \delta t + o(\delta t) = 2\pi r h(y)\delta y + o(\delta y)$, where r is the distance from the axis of symmetry to the cylindrical wall of the shell. Hence $r = y$. Now we have

$$\begin{aligned} V &= \lim_{\delta V \to 0} \sum \delta V \;=\; \lim_{\delta t \to 0} \sum_{t \in [\text{LOWEST}, \text{HIGHEST}]} \{2\pi r\, h\, \delta t + o(\delta t)\} \;=\; \int_{t=\text{LOWEST}}^{t=\text{HIGHEST}} 2\pi r\, h\, dt \\ &= \int_{y=0}^{y=3} 2\pi y\, h(y)\, dy \;=\; 2\pi \int_0^3 \left\{3y - 2y^2 + \tfrac{1}{3}y^3\right\} dy \;=\; 2\pi \left\{\tfrac{3}{2}y^2 - \tfrac{2}{3}y^3 + \tfrac{1}{12}y^4\right\}\Big|_0^3 \\ &= 2\pi \left\{\tfrac{3}{2}3^2 - \tfrac{2}{3}3^3 + \tfrac{1}{12}3^4 - \tfrac{3}{2}0^2 + \tfrac{2}{3}0^3 - \tfrac{1}{12}0^4\right\} \;=\; \tfrac{9\pi}{2} \end{aligned}$$

as before.

(c) The transverse coordinate $t = x$ is again perpendicular to the axis of symmetry (which is now vertical), and so again we must use cylindrical shells. To find the new volume V generated by rotating R about this new axis of symmetry, we subtract the volume V_3 generated by rotating the lighter shaded region about $x = 4$

from the volume V_4 of the cone generated by rotating both regions together. The new cone has "base radius" 6 and "height" 3. Hence $V_4 = \frac{1}{3}\pi 6^2 \times 3 = 36\pi$. The generic (lighter shaded) shell is obtained by rotating the generic (lighter shaded) strip, which is aligned with the axis of symmetry and stretches from the x-axis to the parabola $y = \sqrt{3(x-1)}$. In other words, the height of the strip—and hence of the generic shell—is

$$h = h(x) = \sqrt{3(x-1)}.$$

Thus the element of volume for V_3 is $\delta V_3 = 2\pi r h \delta t + o(\delta t) = 2\pi r h(x)\delta x + o(\delta x)$, where r is the distance from the axis of symmetry to the cylindrical wall of the shell. Hence $r = 4 - x$. Now we have

$$\begin{aligned} V_3 &= \lim_{\delta V_3 \to 0} \sum \delta V_3 = \lim_{\delta t \to 0} \sum_{t \in [\text{LOWEST,HIGHEST}]} \{2\pi r\, h\, \delta t + o(\delta t)\} = \int_{t=\text{LOWEST}}^{t=\text{HIGHEST}} 2\pi r\, h\, dt \\ &= \int_{x=1}^{x=4} 2\pi(4-x)\, h(x)\, dx = 2\pi \int_1^4 (4-x)\sqrt{3(x-1)}\, dx. \end{aligned}$$

This integral is best evaluated by means of the substitution $u = \sqrt{3(x-1)}$ or $x = \frac{1}{3}u^2 + 1$, so that $\frac{dx}{du} = \frac{2}{3}u$ and

$$\begin{aligned} V_3 &= 2\pi \int_{x=1}^{x=4} (4-x)\sqrt{3(x-1)}\, dx = 2\pi \int_{u=\sqrt{3(1-1)}}^{u=\sqrt{3(4-1)}} (4-x)\sqrt{3(x-1)}\, \frac{dx}{du}\, du \\ &= 2\pi \int_{u=0}^{u=3} \left(4 - \{\tfrac{1}{3}u^2 + 1\}\right) u \cdot \tfrac{2}{3}u\, du = \tfrac{4}{3}\pi \int_0^3 \left(3 - \tfrac{1}{3}u^2\right) u^2\, du \\ &= \tfrac{4}{3}\pi \int_0^3 \left(3u^2 - \tfrac{1}{3}u^4\right)\, du = \tfrac{4}{3}\pi \left(u^3 - \tfrac{1}{15}u^5\right)\Big|_0^3 = \tfrac{4}{3}\pi \left(3^3 - \tfrac{1}{15}3^5\right) - 0 = \tfrac{72\pi}{5}. \end{aligned}$$

Thus $V = V_4 - V_3 = 36\pi - \frac{72\pi}{5} = \frac{108\pi}{5}$.

(d) The transverse coordinate $t = y$ is aligned with the axis of symmetry. We must therefore use disks as in (a), and so the element of volume for V_3 is $\delta V_3 = \pi r^2 \delta t + o(\delta t) = \pi r^2 \delta y + o(\delta y)$, where r is the distance from the axis of symmetry $x = 4$ to the curve $x = \frac{1}{3}y^2 + 1$. Hence $r = 4 - x = 3 - \frac{1}{3}y^2$, and

$$\begin{aligned} V_3 &= \lim_{\delta V_3 \to 0} \sum \delta V_3 = \lim_{\delta t \to 0} \sum_{t \in [\text{LOWEST,HIGHEST}]} \{\pi r^2\, \delta t + o(\delta t)\} = \int_{t=\text{LOWEST}}^{t=\text{HIGHEST}} \pi r^2\, dt \\ &= \int_{y=0}^{y=3} \pi r^2\, dy = \pi \int_0^3 \left(3 - \tfrac{1}{3}y^2\right)^2\, dy = \pi \int_0^3 \left(9 - 2y^2 + \tfrac{1}{9}y^4\right)\, dy \\ &= \pi \left(9y - \tfrac{2}{3}y^3 + \tfrac{1}{45}y^5\right)\Big|_0^3 = \pi\left(27 - \tfrac{2}{3}3^3 + \tfrac{1}{45}3^5\right) - \pi \cdot 0 = \tfrac{72\pi}{5}. \end{aligned}$$

So again we have $V = V_4 - V_3 = 36\pi - \frac{72\pi}{5} = \frac{108\pi}{5}$