20. L'Hôpital's rule

You already know from Lecture 10 that any sequence $\{s_k\}$ induces a sequence of finite sums $\{S_n\}$ through

$$S_n = \sum_{k=1}^{n} s_k, \tag{1}$$

and that if $s_k \to 0$ as $k \to \infty$ then $\{S_n\}$ may converge to the limit

$$S_\infty = s_1 + s_2 + s_3 + s_4 + \ldots = \sum_{k=1}^{\infty} s_k. \tag{2}$$

Then the right-hand side of (2) is called an *infinite series*, and the left-hand side is the sum to which it converges.* You also know from Lecture 7 what a function sequence is, and that a function sequence may likewise converge. So it shouldn't surprise you that any function sequence $\{s_k(x)\}$ induces a function sequence of finite sums $\{S_n(x)\}$ through

$$S_n(x) = \sum_{k=1}^{n} s_k(x) \tag{3}$$

and that the corresponding infinite series may also converge; moreover, if it converges, then it defines a function S_∞ through its sum

$$S_\infty(x) = s_1(x) + s_2(x) + s_3(x) + s_4(x) + \ldots = \sum_{k=1}^{\infty} s_k(x). \tag{4}$$

The most important case is where s_k is a power function, defined by

$$s_k(x) = a_k x^k \tag{5}$$

(with $a_k \to 0$ as $k \to \infty$). Then the right-hand side of (4) is called—surprise, surprise!—a *power series*. If

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots = \sum_{k=0}^{\infty} a_k x^k \tag{6}$$

yields $g(x)$, where g is a known function, then we call (6) the *power-series representation* of g; and otherwise, $g(x)$ is just the sum of the power series. So, in effect, either the series yields an alternative representation of a function we already know, or else it defines a brand new function.

For example, recall from Lectures 7 and 9 that the exponential function is defined by

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n. \tag{7}$$

*The series converges only if $s_k \to 0$ as $k \to \infty$ because $s_k = S_{k+1} - S_k$, and so if $\{S_n\}$ converges then $\lim_{k\to\infty} s_k = \lim_{k\to\infty} (S_{k+1} - S_k) = \lim_{k\to\infty} S_{k+1} - \lim_{k\to\infty} S_k = S_\infty - S_\infty = 0$.

From the binomial theorem, however, we have

$$\begin{aligned}(1+a)^n &= 1 + na + \tfrac{n(n-1)}{1.2}a^2 + \tfrac{n(n-1)(n-2)}{1.2.3}a^3 \\ &\qquad + \tfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^4 + \tfrac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}a^5 + \ldots \\ &\qquad \text{plus a lot more terms (in fact, } n-5 \text{ of them).}\end{aligned} \tag{8}$$

If we set $a = x/n$ we get

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n &= 1 + n\left(\tfrac{x}{n}\right) + \tfrac{n(n-1)}{1.2}\left(\tfrac{x}{n}\right)^2 + \tfrac{n(n-1)(n-2)}{1.2.3}\left(\tfrac{x}{n}\right)^3 \\ &\qquad + \tfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}\left(\tfrac{x}{n}\right)^4 + \tfrac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\left(\tfrac{x}{n}\right)^5 + \ldots\end{aligned} \tag{9}$$

$$\begin{aligned}&= 1 + x + \left(1-\tfrac{1}{n}\right)\tfrac{x^2}{1.2} + \left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\tfrac{x^3}{1.2.3} \\ &\qquad + \left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\left(1-\tfrac{3}{n}\right)\tfrac{x^4}{1.2.3.4} + \left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\left(1-\tfrac{3}{n}\right)\left(1-\tfrac{4}{n}\right)\tfrac{x^5}{1.2.3.4.5} + \ldots\end{aligned}$$

so that, on using $\lim\limits_{n\to\infty}\frac{1}{n} = 0$, we have

$$e^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + x + \tfrac{x^2}{1.2} + \tfrac{x^3}{1.2.3} + \tfrac{x^4}{1.2.3.4} + \tfrac{x^5}{1.2.3.4.5} + \ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!} \tag{10}$$

which is a convergent power series.†

Power-series representations of functions can be extremely useful. For example, (10) yields two additional ways in which to show that the exponential is its own derivative. If we use the properties of the exponential function discovered in Lecture 9 to perturb $y = e^x$ to $y + \delta y = e^{x+\delta x}$, then we obtain $\delta y = e^{x+\delta x} - e^x = e^x\, e^{\delta x} - e^x = e^x(e^{\delta x} - 1)$. But replacing x by δx in (10) yields

$$e^{\delta x} = 1 + \delta x + \tfrac{\delta x^2}{1.2} + \tfrac{\delta x^3}{1.2.3} + \tfrac{\delta x^4}{1.2.3.4} + \tfrac{\delta x^5}{1.2.3.4.5} + \ldots = 1 + \delta x + o(\delta x), \tag{11}$$

and so

$$\begin{aligned}\frac{d}{dx}\{e^x\} &= \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{e^x(e^{\delta x}-1)}{\delta x} = e^x\lim_{\delta x\to 0}\frac{(e^{\delta x}-1)}{\delta x} \\ &= e^x\lim_{\delta x\to 0}\frac{(1+\delta x + o(\delta x) - 1)}{\delta x} = e^x\lim_{\delta x\to 0}\left\{1 + \frac{o(\delta x)}{\delta x}\right\} = e^x\{1+0\} \\ &= e^x.\end{aligned} \tag{12}$$

†It converges for all values of x. In general, a power series will converge for some values of x and diverge for other values of x; however, the power series of three of our most important functions, namely, $\exp$, $\sin$ and $\cos$ have the very nice property that they always converge.

Or else we can proceed as follows[‡]:

$$
\begin{aligned}
\frac{d}{dx}\left(e^x\right) &= \frac{d}{dx}\left(1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \ldots\right) \\
&= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{x^2}{1.2}\right) + \frac{d}{dx}\left(\frac{x^3}{1.2.3}\right) + \frac{d}{dx}\left(\frac{x^4}{1.2.3.4}\right) + \frac{d}{dx}\left(\frac{x^5}{1.2.3.4.5}\right) + \ldots \\
&= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{1}{1.2}\frac{d}{dx}\left(x^2\right) + \frac{1}{1.2.3}\frac{d}{dx}\left(x^3\right) + \frac{1}{1.2.3.4}\frac{d}{dx}\left(x^4\right) + \frac{1}{1.2.3.4.5}\frac{d}{dx}\left(x^5\right) + \ldots \\
&= 0 + 1 + \frac{1}{1.2}\cdot 2x + \frac{1}{1.2.3}\cdot 3x^2 + \frac{1}{1.2.3.4}\cdot 4x^3 + \frac{1}{1.2.3.4.5}\cdot 5x^4 + \ldots \\
&= 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \ldots \\
&= e^x.
\end{aligned}
$$

Any other power series can likewise be differentiated term by term.[§] So let us consider the power series of an arbitrary function g:

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots = \sum_{k=0}^{\infty} a_k x^k \tag{13}$$

Differentiating once yields

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}\left\{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\right\} \\
&= \frac{d}{dx}\{a_0\} + a_1\frac{d}{dx}\{x\} + a_2\frac{d}{dx}\left\{x^2\right\} + a_3\frac{d}{dx}\left\{x^3\right\} + a_4\frac{d}{dx}\left\{x^4\right\} + a_5\frac{d}{dx}\left\{x^5\right\} + \ldots \\
&= 0 + a_1\cdot 1 + a_2\cdot 2x + a_3\cdot 3x^2 + a_4\cdot 4x^3 + a_5\cdot 5x^4 + \ldots \\
&= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots
\end{aligned}
\tag{14}
$$

Differentiating again yields

$$
\begin{aligned}
g''(x) = \frac{d}{dx}\{g'(x)\} &= \frac{d}{dx}\left\{a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots\right\} \\
&= \frac{d}{dx}\{a_1\} + 2a_2\frac{d}{dx}\{x\} + 3a_3\frac{d}{dx}\left\{x^2\right\} + 4a_4\frac{d}{dx}\left\{x^3\right\} + 5a_5\frac{d}{dx}\left\{x^4\right\} + \ldots \\
&= 0 + 2a_2\cdot 1 + 3a_3\cdot 2x + 4a_4\cdot 3x^2 + 5a_5\cdot 4x^3 + \ldots \\
&= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots
\end{aligned}
\tag{15}
$$

and so on. Setting $x = 0$ in (13) yields $g(0) = a_0$; setting $x = 0$ in (14) yields $g'(0) = a_1$; setting $x = 0$ in (15) yields $g''(0) = 2a_2$; and so on. In other words, $a_0 = g(0)$; $a_1 = g'(0)$; $a_2 = \frac{1}{2}g''(0)$; and so on. Substituting back into (13) yields

$$g(x) = g(0) + g'(0)x + \tfrac{1}{2}g''(0)x^2 + \ldots \tag{16a}$$

Clearly, the name we gave our arbitrary function itself is arbitrary, and so if (16) holds then just as surely does

$$h(x) = h(0) + h'(0)x + \tfrac{1}{2}h''(0)x^2 + \ldots \tag{16b}$$

[‡]Making various assumptions whose validity we have no choice but to take for granted at this stage in our study of the calculus.

[§]Provided the answer converges, which naturally we assume.

Hence

$$\frac{g(x)}{h(x)} = \frac{g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + \dots}{h(0) + h'(0)x + \frac{1}{2}h''(0)x^2 + \dots} \tag{17}$$

There is a problem with (17), however: by virtue of all those dots, it is less precise than we like our equations to be. We therefore define "$O(x^n)$" (called "big oh of x^n") for $m \geq 1$ to mean any infinite power series in which n is the lowest (integer) exponent: intuitively, $O(x^n) = Ax^n + Bx^{n+1} + Cx^{n+2} + \dots$, where A, B and C are independent of x (and hence constants in the limit as $x \to 0$). Thus, by inspection, $O(x^n)$ has the following two properties:¶

$$\lim_{x\to 0} O(x^n) = 0 \quad \text{for} \quad n \geq 1, \qquad \frac{O(x^n)}{x^m} = O(x^{n-m}) \quad \text{for} \quad n \geq m+1. \tag{18}$$

With this new notation we can be more precise about (17):

$$\frac{g(x)}{h(x)} = \frac{g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + O(x^3)}{h(0) + h'(0)x + \frac{1}{2}h''(0)x^2 + O(x^3)}. \tag{19}$$

Now, suppose that you wish to calculate the limit of the quotient $\frac{g(x)}{h(x)}$ as $x \to 0$. By our general rule about limits, the limit of this quotient is the quotient of the limits, *provided the answer exists*. So if $g(0)$ and $h(0)$ are both nonzero then

$$\lim_{x\to 0} \frac{g(x)}{h(x)} = \frac{g(0)}{h(0)}. \tag{20}$$

If $g(0) = h(0) = 0$, however, then (19) implies

$$\lim_{x\to 0} \frac{g(x)}{h(x)} = \lim_{x\to 0} \frac{g'(0)x + O(x^2)}{h'(0)x + O(x^2)} = \lim_{x\to 0} \frac{g'(0) + O(x)}{h'(0) + O(x)} = \frac{g'(0)}{h'(0)} \tag{21}$$

after dividing both numerator and denominator by x and using (18); and if $g'(0)$ and $h'(0)$ are also both zero, i.e., if $g(0) = h(0) = g'(0) = h'(0) = 0$, then (19) implies

$$\lim_{x\to 0} \frac{g(x)}{h(x)} = \lim_{x\to 0} \frac{\frac{1}{2}g''(0)x^2 + O(x^3)}{\frac{1}{2}h''(0)x^2 + O(x^3)} = \lim_{x\to 0} \frac{\frac{1}{2}g''(0) + O(x)}{\frac{1}{2}h''(0) + O(x)} = \frac{g''(0)}{h''(0)} \tag{22}$$

after dividing both numerator and denominator by x^2 and using (18) again. And so on.‖

A slight generalization of this result is sometimes useful. Suppose that G and H are related to g and h by

$$g(x) = G(x+a), \qquad h(x) = H(x+a). \tag{23}$$

Then, from the chain rule, we have $g'(x) = G'(x+a)\frac{d}{dx}\{x+a\} = G'(x+a)\{1+0\} = G'(x+a)$, $g''(x) = \frac{d}{dx}\{G'(x+a)\} = G''(x+a)\frac{d}{dx}\{x+a\} = G''(x+a)\{1+0\} = G''(x+a)$, etc.,

¶So $O(x^n)$ is a new kind of junk term, related to $o(x)$ by $O(x^n) = o(x)$ if $n \geq 2$, but dwelling on this relationship will serve no purpose here.

‖Provided, of course, that all the derivatives exist, which we assume.

and similarly for h. So, in particular, $g'(0) = G'(a)$, $g''(0) = G''(a)$, etc.; and similarly, $h'(0) = H'(a)$, $h''(0) = H''(a)$, etc. Also, $g(0) = G(a)$ and $h(0) = H(a)$, from (23). So (20)-(23) yield the following:

$$\begin{aligned} G(a) = H(a) = 0 \text{ AND } H'(a) \neq 0 \Longrightarrow \lim_{x\to a} \frac{G(x)}{G(x)} &= \frac{G'(a)}{H'(a)} \\ G(a) = H(a) = G'(a) = H'(a) = 0 \text{ AND } H''(a) \neq 0 \Longrightarrow \lim_{x\to a} \frac{G(x)}{G(x)} &= \frac{G''(a)}{H''(a)} \end{aligned} \tag{24a}$$

and so on. Moreover,** we even have

$$G(\infty) = H(\infty) = \infty \text{ AND } G'(\infty), H'(\infty) \text{ FINITE} \Longrightarrow \lim_{x\to\infty} \frac{G(x)}{G(x)} = \frac{G'(\infty)}{H'(\infty)} \tag{24b}$$

and so on. We refer to (24) as *L'Hôpital's rule*.

We can often use L'Hôpital's rule to determine the limiting behavior of "indeterminate quotients," or even—after manipulation—indeterminate products or differences. It even works for infinite limits.†† Nevertheless, L'Hôpital's rule is also often unnecessary, and need not yield the most elegant approach, even where it works. Consider, e.g.,

$$L = \lim_{x\to 0} \frac{g(x)}{h(x)} \qquad \text{where } g(x) = x^3 \text{ and } h(x) = e^x - 1 - x - \tfrac{1}{2}x^2. \tag{25}$$

Because $g'(x) = 3x^2$, $g''(x) = 6x$, $g'''(x) = 6$, $h'(x) = e^x - 0 - 1 - \frac{1}{2}\cdot 2x = e^x - 1 - x$, $h''(x) = e^x - 0 - 1 = e^x - 1$ and $h'''(x) = e^x - 0 = e^x$, we have both $g(0) = g'(0) = g''(0) = 0$ and $h(0) = h'(0) = h''(0) = 0$, so that the "and so on" of L'Hôpital's rule yields

$$L = \frac{g'''(0)}{h'''(0)} = \lim_{x\to 0} \frac{g'''(x)}{h'''(x)} = \frac{6}{e^0} = 6. \tag{26}$$

It would have been simpler, however, to observe from (10)

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + O(x^4), \tag{27}$$

so that (25) implies

$$L = \lim_{x\to 0} \frac{x^3}{e^x - 1 - x - \frac{1}{2}x^2} = \lim_{x\to 0} \frac{x^3}{\frac{1}{6}x^3 + O(x^4)} = \lim_{x\to 0} \frac{1}{\frac{1}{6} + O(x)} = \frac{1}{\frac{1}{6} + 0} = 6, \tag{28}$$

after division by x^3 and use of (18).

On the other hand, there are cases where L'Hôpital's rule is undeniably the most efficient tool for the job. Consider, e.g.,

$$L = \lim_{x\to 0} \frac{G(x)}{H(x)} \qquad \text{where } G(x) = x \text{ and } H(x) = \ln(1 + 2e^x). \tag{29}$$

**By supposing instead that G and H are related to g and h by $g(1/x) = G(x)$, $h(1/x) = H(x)$ and noting that $\lim\limits_{x\to\infty} G(x) = \lim\limits_{x\to 0^+} g(x)$, $\lim\limits_{x\to\infty} H(x) = \lim\limits_{x\to 0^+} h(x)$ and $g'(1/x)/h'(1/x) = G'(x)/H'(x)$ by the chain rule, etc.

††Although in this course we have a strong preference for thinking that anything that deserves the name of limit should at least have the goodness to be finite.

Because $G(\infty) = \lim_{x\to\infty} G(x) = \infty$ and $H(\infty) = \lim_{x\to\infty} H(x) = \infty$, and because $G'(x) = 1$ (implying at once that $G'(\infty) = 1$) and, from the chain rule,

$$H'(x) = \frac{1}{1+2e^x}\frac{d}{dx}\{1+2e^x\} = \frac{1}{1+2e^x}\{0+2e^x\} = \frac{2e^x}{1+2e^x} = \frac{1}{\frac{1}{2}e^{-x}+1} \tag{30}$$

so that $H'(\infty) = \lim_{x\to\infty} H'(x) = \frac{1}{\frac{1}{2}\cdot 0+1} = 1$, from (24b) we obtain

$$L = \frac{G'(\infty)}{H'(\infty)} = \lim_{x\to\infty}\frac{G'(x)}{H'(x)} = \frac{1}{1} = 1. \tag{31}$$

Or consider

$$L = \lim_{x\to 1}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right), \tag{32}$$

which is an indeterminate difference (of type "$\infty - \infty$"). Noting that

$$\frac{1}{\ln(x)} - \frac{1}{x-1} = \frac{x-1-\ln(x)}{(x-1)\ln(x)}, \tag{33}$$

we can exploit (24) with $a = 1$, $G(x) = x - 1 - \ln(x) \Longrightarrow G'(x) = 1 - 0 - \frac{1}{x} = 1 - 1/x$, $G''(x) = 0 - \frac{d}{dx}\{x^{-1}\} = x^{-2}$ and $H(x) = (x-1)\ln(x) \Longrightarrow H'(x) = (1-0)\ln(x) + (x-1)\frac{1}{x} = \ln(x) + 1 - 1/x$, $H''(x) = 1/x + 0 + x^{-2} = 1/x + x^{-2}$ to observe that, because $G(1) = 1-1-\ln(1) = 0$, $G'(1) = 1-1/1 = 0$ and $H(1) = (1-1)\ln(1) = 0$, $H'(1) = \ln(1)+1-1/1 = 0$,

$$L = \frac{G''(1)}{H''(1)} = \lim_{x\to 1}\frac{G''(x)}{H''(x)} = \frac{1^{-2}}{\frac{1}{1}+1^{-2}} = \tfrac{1}{2}. \tag{34}$$

Whether, however, L'Hôpital's rule yields the neatest method for obtaining this limit is moot. An alternative approach is to note that with $g(x) = \ln(1+x)$ in (16) we obtain

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + O(x^3). \tag{35}$$

Now, substituting $u = x - 1$ (or $x = 1 + u$) in (32) and recalling that the definition of "$O(u^n)$" does not distinguish between $-O(u^n)$ and $O(u^n)$, we obtain

$$\begin{aligned} L &= \lim_{u\to 0}\left(\frac{1}{\ln(1+u)} - \frac{1}{u}\right) = \lim_{u\to 0}\frac{u-\ln(1+u)}{u\ln(1+u)} = \lim_{u\to 0}\frac{u - \{u - \frac{1}{2}u^2 + O(u^3)\}}{u\{u-\frac{1}{2}u^2+O(u^3)\}} \\ &= \lim_{u\to 0}\frac{\frac{1}{2}u^2 + O(u^3)}{u^2 - \frac{1}{2}u^3 + O(u^4)} = \lim_{u\to 0}\frac{\frac{1}{2}+O(u)}{1-\frac{1}{2}u+O(u^2)} = \frac{\frac{1}{2}+0}{1-\frac{1}{2}\cdot 0+0} = \tfrac{1}{2} \end{aligned} \tag{36}$$

after division by u^2 and use of (18) with u in place of x. Note that, in the second line, we could have replaced $-\frac{1}{2}u^3 + O(u^4)$ by $O(u^3)$ and $-\frac{1}{2}u + O(u^2)$ by $O(u)$ in the denominator, without affecting the result in any way.

I prefer the second approach. Which do you prefer?

Exercise

1. Show that $\sin(\delta x) = \delta x + o(\delta x)$ and $\cos(\delta x) = 1 - \frac{1}{2}\delta x^2 + o(\delta x)$.

2. The line $y = ax + b$ is said to be an *oblique asymptote* for the rational function f as $x \to \infty$ if $f(x)$ is of type $O(x^{n+1})/O(x^n)$ for some $n \geq 2$ and

$$\lim_{x \to \infty} \{f(x) - ax - b\} \;=\; 0$$

(and similarly for $x \to -\infty$). Find the oblique asymptote for f defined on $(1, \infty)$ by

$$f(x) \;=\; \frac{4x^3 - 2x^2 + 5}{(x-1)(2x+3)}.$$

Suitable problems from standard calculus texts

Stewart (2003): pp. 313-314, ## 5-62; p. 324, ## 55-61, 63 and 64 (see Exercise 2 and its solution).

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

Solutions to selected exercises

1. Use (16) with $g(x) = \sin(x)$ and $h(x) = \cos(x)$ to obtain $\sin(x) = x + O(x^3)$ and $\cos(x) = 1 - \frac{1}{2}x^2 + O(x^3)$; then replace x by δx.

2. Here $n = 2$. After simplification we have

$$f(x) - ax - b \;=\; \frac{5 + 3b + (3a - b)x - (2 + a + 2b)x^2 + 2(2 - a)x^3}{(x-1)(2x+3)}$$

This expression approaches zero as $x \to \infty$ if and only if the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. So the coefficients of x^2 and x^3 must both be zero in the numerator. That is, $2+a+2b = 0$ and $2-a = 0$ or $a = 2, b = -2$. So the oblique asymptote is $y = 2(x-1)$.