14 An Approach to Modelling Aspects of Civil War

A class of games has been developed to describe human or non-human populations in which some individuals do productive work, such as searching for food or growing it, while other individuals do no such work but steal or otherwise reap the benefits of the first subpopulation's labors. In evolutionary biology, these games are usually known as producer-scrounger games (e.g., Broom and Rychtář, 2013, §17.7), whereas in economics the producers or scroungers are often given other names, e.g., peasants or bandits. To be as inclusive as possible here, I will refer to the scroungers as exploiters. Regardless of which labels are used, however, the important point from our perspective is that such models can be adapted to deal with aspects of civil war. The following discussion is based on Skaperdas (2008) and Konrad and Skaperdas (2012).

To exemplify the overall approach, consider a population of N individuals that has separated itself into N_p producers and N_e exploiters, so that

$$N_p + N_e \;=\; N. \tag{14.1}$$

Each producer has one unit of resources to divide between work and self-protection from the exploiters. Let $x \in [0,1]$ be the proportion assigned to self-protection, hence $1-x$ the proportion assigned to work. Let each unit of resources assigned to work yield Q units of output, so that each producer's output is $Q(1-x)$, and let $p(x)$ be the proportion of output protected from exploiters by investing x in self-protection. We assume

$$p(0) \;\geq\; 0, \quad p'(x) \;>\; 0, \quad p''(x) \;<\; 0, \quad p(1) \;\leq\; 1. \tag{14.2}$$

Then a producer's payoff is

$$U_p(x) \;=\; Q\{1-x\} \cdot p(x), \tag{14.3}$$

which has maximum

$$U_p^* \;=\; U_p(x^*) \tag{14.4}$$

at $x = x^*$, where x^* is the unique solution of the equation

$$\frac{d}{dx}\left\{\ln\bigl\{(1-x)p(x)\bigr\}\right\}\Bigg|_{x=x^*} \;=\; 0. \tag{14.5}$$

For example, if we satisfy (14.2) with

$$p(x) \;=\; x^\theta \tag{14.6}$$

for $\theta \in (0,1)$, so that θ is the elasticity[1] of p with respect to x, then

$$x^* \;=\; \frac{\theta}{1+\theta} \tag{14.7}$$

and

$$U_p^* \;=\; \frac{\theta^\theta}{(1+\theta)^{1+\theta}}\,Q. \tag{14.8}$$

[1] See Lecture 1.

For any level of self-protection x, the total production of all N_p producers is $N_p(1-x)Q$, of which proportion $1-p(x)$ is expropriated by the exploiters. Hence the total amount of output expropriated by all N_e exploiters is $N_p\{1-p(x)\}(1-x)Q$, implying that the payoff to each exploiter is $1/N_e$ of that amount or

$$U_e(x) \;=\; \frac{N_p}{N_e}\{1-p(x)\}(1-x)Q. \tag{14.9}$$

We assume, however, that producers optimize x. Hence $x = x^*$, and so the payoff to an exploiter is

$$U_e^* \;=\; U_e(x^*) \;=\; \frac{N_p}{N_e}(1-p^*)(1-x^*)Q \tag{14.10}$$

where

$$p^* \;=\; p(x^*) \tag{14.11}$$

is a measure of the security of property—the extent to which producers get to keep their output. For example, if (14.6) holds then

$$U_e^* \;=\; \frac{N_p}{N_e}\frac{(1+\theta)^\theta - \theta^\theta}{(1+\theta)^{1+\theta}}\,Q. \tag{14.12}$$

If $U_p^* > U_e^*$, then some exploiters to switch to producing. So N_e will fall, N_p will rise and, by (14.10), U_e^* will also rise, until eventually $U_p^* = U_e^*$. If $U_e^* > U_p^*$, on the other hand, then some producers will switch to exploiting. So N_p will fall, N_e will rise and, again by (14.10), U_e^* will fall, until eventually $U_p^* = U_e^*$. So, either way, eventually the population will settle down to an equilibrium consisting of N_p^* producers having payoff U_p^* and N_e^* exploiters having payoff U_e^*, as before, but now with

$$U_p^* \;=\; U_e^*, \tag{14.13}$$

so that no exploiter has an incentive to switch to production, and no producer has an incentive to switch to exploitation. Moreover,

$$N_p^* + N_e^* \;=\; N \tag{14.14}$$

by (14.1), implying

$$N_p^* \;=\; p^* N, \qquad N_e^* \;=\; (1-p^*)N \tag{14.15}$$

and

$$\frac{N_e^*}{N_p^*} \;=\; \frac{1-p^*}{p^*} \tag{14.16}$$

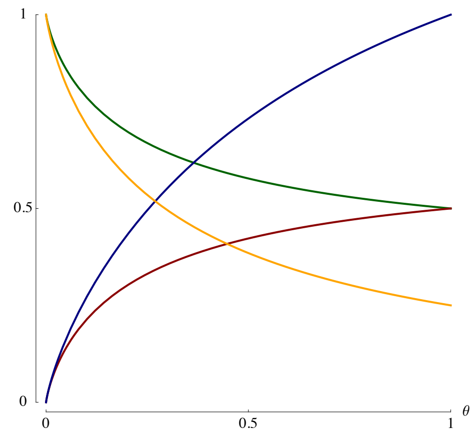
by (14.3) and (14.10). Since only producers produce, the total output at equilibrium is

$$N_p^* \cdot (1-x^*)Q \;=\; p^*(1-x^*)NQ \tag{14.17}$$

by (14.15); note that this expression also equals $N_p^* U_p^* + N_e^* U_e^* = NU_p^*$. In particular, if (14.6) holds, then at long-run equilibrium we obtain

$$N_p^* \;=\; \left(\frac{\theta}{1+\theta}\right)^{\theta} N \tag{14.18a}$$

Figure 14.1: Proportion of producers (N_p^*/N, green), proportion of exploiters (N_e^*/N, red), exploiter-producer ratio ($N_e^*/N_p^* = (1-p^*)/p^*$, blue) and scaled per capita output ($U_p^*/Q = U_e^*/Q$, orange) as a function of elasticity θ.



and

$$N_e^* \;=\; \left\{1 - \left(\frac{\theta}{1+\theta}\right)^{\theta}\right\} N \tag{14.18b}$$

with

$$U_p^* \;=\; U_e^* \;=\; \frac{\theta^{\theta}}{(1+\theta)^{1+\theta}}\, Q. \tag{14.19}$$

So the exploiter-producer ratio rises with elasticity at equilibrium while output falls; see Figure 14.1.

We now suppose that the population contains two categories of individual, a more productive or "high-level" type and a less productive or "low-level" type, either of which is a potential exploiter—so that high-level types are actually more productive only if they do indeed produce. Let there be N_h of the high-level types and N_l of the low-level types, with

$$N_l + N_h \;=\; N. \tag{14.20}$$

The population now has three subpopulations instead of two. Let there be N_{ph} high-level producers, N_{pl} low-level producers and N_e exploiters, so that

$$N_{pl} + N_{ph} \;=\; N_p \tag{14.21}$$

and (14.1) continues to hold. Note that, since the number of high-level producers cannot exceed the number of high-level individuals in the population, we must have

$$N_{ph} \;\le\; N_h. \tag{14.22}$$

A low-level producer will convert each unit of resources assigned to work into Q units of output, as before, whereas a high-level producer will convert each unit of resources assigned to work into AQ units of output, where

$$A \;>\; 1. \tag{14.23}$$

So the output of a low-level producer is still $Q\{1-x\} \cdot p(x) = U_p(x)$, as in (14.3), whereas the output of a high-level producer is $AQ\{1-x\} \cdot p(x) = AU_p(x)$. Nevertheless, we continue to assume that all producers optimize x, and because the second payoff is merely A times the first payoff, the optimal x^* is still determined by (14.5), for both types of producer. On the other hand, equilibrium payoffs differ. For low-level producers

$$U^*_{pl} \;=\; p^*(1-x^*)Q \tag{14.24}$$

by (14.4), whereas for high-level producers we now have

$$U^*_{ph} \;=\; p^*(1-x^*)AQ \;=\; AU^*_{pl} \tag{14.25}$$

instead.

For any level of self-protection x, the total output of all $N_p = N_{pl} + N_{ph}$ producers is $N_{pl} \cdot (1-x)Q + N_{ph} \cdot (1-x)AQ$, of which a proportion $1-p(x)$ is expropriated by the exploiters. Hence the total amount of output expropriated by all N_e exploiters is

$$\{1-p(x)\}\{N_{pl} \cdot (1-x)Q + N_{ph} \cdot (1-x)AQ\} \;=\; \{N_{pl} + AN_{ph}\}\{1-p(x)\}(1-x)Q,$$

implying that the payoff to each exploiter is $1/N_e$ of that amount or

$$U_e(x) \;=\; \frac{N_p}{N_e}\{1-p(x)\}(1-x)\,\overline{A}\,Q, \tag{14.26}$$

where

$$\overline{A} \;=\; \frac{N_{pl} + AN_{ph}}{N_p} \;=\; 1 \cdot \frac{N_{pl}}{N_{pl}+N_{ph}} + A \cdot \frac{N_{ph}}{N_{pl}+N_{ph}} \tag{14.27}$$

denotes the average per capita output or productivity of a producer, when output is scaled with respect to Q. Note that

$$1 \;\leq\; \overline{A} \;\leq\; A. \tag{14.28}$$

But $x = x^*$, because all producers optimize. Hence the payoff to an exploiter is

$$U^*_e \;=\; U_e(x^*) \;=\; \frac{N_p}{N_e}(1-p^*)(1-x^*)\,\overline{A}\,Q \tag{14.29}$$

by (14.11) and (14.26), whereas the payoffs to a low-level and a high-level producer are

$$U^*_{pl} \;=\; p^*(1-x^*)Q \tag{14.30a}$$

and

$$U^*_{ph} \;=\; Ap^*(1-x^*)Q, \tag{14.30b}$$

respectively, by (14.24)–(14.25).

As before, the population will eventually settle down to a long-run equilibrium at which no exploiter has an incentive to switch to production, while no producer has an incentive to switch to exploitation. At this equilibrium, N^*_{pl} low-level producers, N^*_{ph} high-level producers and N^*_e exploiters obtain payoffs U^*_{pl}, U^*_{ph} and U^*_e, respectively, where

$$N^*_{pl} + N^*_{ph} + N^*_e \;=\; N^*_p + N^*_e \;=\; N \tag{14.31}$$

by (14.1) and (14.21), with U_{pl}^* given by (14.30a), U_{ph}^* by (14.30b) and

$$U_e^* \;=\; \frac{N_p^*}{N_e^*}(1-p^*)(1-x^*)\,\overline{A}^*\,Q \tag{14.32}$$

from (14.29). However, there are now three forms that the equilibrium can take—whereas previously there was only one.

Before proceeding, we define the proportion of high-level individuals to be

$$\alpha \;=\; \frac{N_h}{N} \tag{14.33}$$

to facilitate further analysis.

Case I: All high-level types produce, some low-level types exploit

In this case we have

$$N_{ph}^* \;=\; N_h \tag{14.34}$$

with

$$U_{ph}^* \;>\; U_{pl}^* \;=\; U_e^*. \tag{14.35}$$

So from (14.27) we obtain

$$\begin{aligned}\overline{A}^* \;&=\; \frac{N_{pl}^* + AN_{ph}^*}{N_p^*} \;=\; \frac{N_{pl}^*}{N_p^*} + \frac{AN_{ph}^*}{N_p^*} \;=\; \frac{N_p^* - N_{ph}^*}{N_p^*} + \frac{AN_{ph}^*}{N_p^*}\\ &=\; \frac{N_p^* - N_h}{N_p^*} + \frac{AN_h}{N_p^*} \;=\; \frac{(A-1)N_h}{N_p^*} + 1 \;=\; \frac{(A-1)\alpha N}{N_p^*} + 1\end{aligned} \tag{14.36}$$

by (14.33). Substituting from (14.36) into $U_{pl}^* \;=\; U_e^*$ and rearranging, we obtain

$$p^* N_e^* - (1-p^*)N_p^* \;=\; (1-p^*)(A-1)\alpha N.$$

Solving together with $N_p^* + N_e^* = N$ yields

$$N_p^* \;=\; \{p^* - (A-1)\alpha(1-p^*)\}N \tag{14.37a}$$
$$N_e^* \;=\; (1-p^*)\{(A-1)\alpha + 1\}N. \tag{14.37b}$$

Because all high-level types produce, we must have

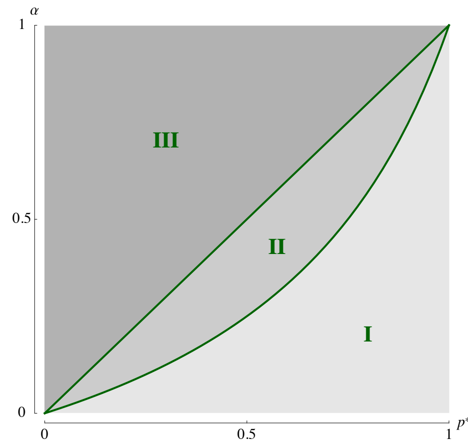
$$N_p^* \;\geq\; N_{ph}^*$$

or

$$\alpha \;\leq\; \frac{p^*}{p^* + A(1-p^*)} \tag{14.38}$$

from (14.33)–(14.34) and (14.37a). So (p, α) must lie in Region I of Figure 14.2.

Figure 14.2: How the long-run equilibrium depends on (p^*, α). Case X corresponds to (p^*, α) lying in Region X for $X = I, \ldots, III$. The area of Region II increases with A (the diagram being drawn for $A = 3$).



Case II: All high-level types produce, all low-level types exploit

In this case we have

$$N_{ph}^* \;=\; N_h, \qquad N_{pl}^* \;=\; 0 \tag{14.39}$$

so that

$$N_p^* \;=\; \alpha N \tag{14.40a}$$

$$N_e^* \;=\; (1-\alpha)N \tag{14.40b}$$

from (14.31) and (14.33) and $N_p^* = N_{ph}^* + N_{pl}^* = N_h + 0 = N_h$ implying

$$\overline{A}^* \;=\; \frac{N_{pl}^* + AN_{ph}^*}{N_p^*} \;=\; A \tag{14.41}$$

by (14.27). Substituting into (14.30b) and (14.32) and simplifying, we now obtain

$$\frac{U_{ph}^*}{U_e^*} \;=\; \frac{p^*(1-\alpha)}{(1-p^*)\alpha}. \tag{14.42}$$

This long-run equilibrium is stable only if both

$$U_{ph}^* \;>\; U_e^* \tag{14.43}$$

(for otherwise some high-level types would switch to exploitation) and

$$U_e^* \;>\; U_{pl}^* \tag{14.44}$$

(for otherwise some low-level types would switch to production). But $U_{pl}^* = U_{ph}^*/A$ by (14.30). So we require $U_e^* < U_{ph}^* < AU_e^*$ or

$$1 \;<\; \frac{U_{ph}^*}{U_e^*} \;<\; A, \tag{14.45}$$

which reduces to

$$\frac{p^*}{p^* + A(1-p^*)} \;<\; \alpha \;<\; p^* \tag{14.46}$$

by (14.42). So (p, α) must lie in Region II of Figure 14.2.

Case III: Some high-level types produce, all low-level types exploit

In this case we have

$$N^*_{pl} \;=\; 0 \tag{14.47}$$

with

$$U^*_{ph} \;=\; U^*_e. \tag{14.48}$$

So $N^*_p = N^*_{ph} + N^*_{pl} = N^*_{ph} + 0 = N^*_{ph}$ implying

$$\overline{A}^* \;=\; \frac{N^*_{pl} + AN^*_{ph}}{N^*_p} \;=\; A \tag{14.49}$$

by (14.27). Substituting into $U^*_{pl} = U^*_e$ and rearranging, we obtain $(1-p^*)N^*_p = p^*N^*_e$; and solving together with $N^*_p + N^*_e = N$ yields

$$N^*_p \;=\; p^*N \tag{14.50a}$$

$$N^*_e \;=\; (1-p^*)N. \tag{14.50b}$$

Because only high-level types produce, we must have

$$N^*_p \;\le\; N_h$$

or

$$\alpha \;\ge\; p^* \tag{14.51}$$

from (14.33) and (14.50a). So (p, α) must lie in Region III of Figure 14.2.

Let us now follow Skaperdas (2008) in supposing that a territory in which all producers reside is taken over by $n \ge 2$ warlords, who provide collective security to all producers in exchange for tribute; and that although the warlords are completely effective in eliminating other exploiters, the new rate of tribute equals the old security rate, so that each producer still retains proportion p^* of output, and is therefore neither worse off nor better off than before (according to our model). Let us also suppose that warlords can extract additional benefits that are not available to individual producers, and that their total value is bQ. We further suppose that each warlord hires the same fixed number $\overline{g}$ of guards to protect individual producers, and that the i-th warlord hires u_i soldiers to compete for the total prize, which is

$$B \;=\; bQ + (1-p^*)(1-x^*)N_pQ \tag{14.52}$$

(where N_p is the number of producers, as before). We will regard the i-th warlord as Player i in a noncooperative game. In accordance with CSF (11.11), the share of the prize to Player i is[2]

$$\frac{u_i}{\sum_{j=1}^{n} u_j} \;=\; \frac{u_i}{u_i + u_{-i}}$$

[2]In the special case where $\gamma = 1$.

where we define

$$u_{-i} \;=\; \sum_{\substack{j=1\\ j\neq i}}^{n} u_j \tag{14.53}$$

to be the sum of the sizes of the other warlords' armies. Soldiers and guards are assumed to receive in pay exactly the value of the amount that they would keep for themselves if they were independent producers, which is $p^*(1-x^*)Q$. So the payoff to Player i is

$$f_i \;=\; \frac{Bu_i}{u_i+u_{-i}} - Cu_i - C\,\overline{g} \tag{14.54}$$

where B, u_{-i} and

$$C \;=\; p^*(1-x^*)Q \tag{14.55}$$

are all independent of u_i. Because

$$\frac{\partial f_i}{\partial u_i} \;=\; \frac{Bu_{-i}}{(u_i+u_{-i})^2} - C \tag{14.56}$$

with

$$\frac{\partial^2 f_i}{\partial u_i^2} \;=\; \frac{-2Bu_{-i}}{(u_i+u_{-i})^3} \;<\; 0, \tag{14.57}$$

Player i's best reply to the other players' collective u_{-i} satisfies $\partial f_i/\partial u_i = 0$ or

$$\frac{Bu_{-i}}{(u_i+u_{-i})^2} \;=\; C \tag{14.58}$$

from (14.56). The strategy combination $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ is a strong Nash equilibrium if each u_i^* is the best reply to the other players' u_{-i}^*. So for any $i \neq j$, at the Nash equilibrium we must have

$$\frac{Bu_{-i}^*}{(u_i^*+u_{-i}^*)^2} \;=\; \frac{Bu_{-j}^*}{(u_j^*+u_{-j}^*)^2} \;=\; C. \tag{14.59}$$

But $u_i^* + u_{-i}^* = u_j^* + u_{-j}^* = \sum_{k=1}^n u_k^*$. Hence (14.59) implies $u_{-i}^* = u_{-j}^* \implies \sum_{k=1}^n u_k^* - u_{-i}^* = \sum_{k=1}^n u_k^* - u_{-j}^* \implies u_i = u_j$. We conclude that

$$u_i^* \;=\; u^* \tag{14.60}$$

for $i = 1, \ldots, n$ where (14.58) implies $B(n-1)u^*/\{nu^*\}^2 = C$ or

$$u^* \;=\; \frac{(n-1)B}{Cn^2} \tag{14.61}$$

The associated payoff is

$$f_i^* \;=\; \frac{B}{n^2} - C\,\overline{g}. \tag{14.62}$$

Because the soldiers and guards are taken from the same population as the producers, and because each of n warlords hires u^* soldiers and $\overline{g}$ guards, consistency requires that

$$N_p \;\leq\; N - n(u^* + \overline{g}) \tag{14.63}$$

(where N is the size of the population, as before). Substituting from (14.61) for u^*, and in turn from (14.52) and (14.55) for B and C, we reduce (14.63) to

$$N_p \;\leq\; \frac{np^*(N - n\overline{g})}{n-1+p^*} - \frac{(n-1)b}{(n-1+p^*)(1-x^*)} \tag{14.64}$$

with equality if there is "full employment" (as a producer, a soldier or a guard).

Skaperdas (2008) proceeds to tease a handful of insights on civil war out of the model developed above. Although he shows that his approach has potential, I think it is fair to say that civil-war models—at least the mathematical kind—are still very much in their infancy. If you are interested, then his paper is available from the course page. For the rest of us, however, this is as far as we have time to go.