# Frequency (Counting) Distributions

($a$,$b$,0) – class
(This is a class of 4 (actually only 3) discrete distributions. The values of $a$ and $b$ for the different distributions is given in the Exam M Tables.)

Notation: $p_k = \Pr(N = k)$ where $N$ is the frequency (counting) random variable

**Important Defining Relationship**: $\dfrac{p_k}{p_{k-1}} = a + \dfrac{b}{k}$ for $k$ = 1, 2, …

The "4" distributions are:

1. Poisson Distribution with parameter $\lambda$ : $P(\lambda)$
   $a$ = 0 and $b$ = $\lambda$
   $\lambda$ is a scale parameter

2. Binomial Distribution with parameters $m$ and $q$ : $B(m,q)$
   $a = -\dfrac{q}{1-q}$ and $b = \dfrac{(m+1)\cdot q}{1-q}$
   If $m$ = 1, this distribution is called Bernoulli with parameter $q$.

3. Negative Binomial Distribution with parameters $r$ and $\beta$ : $NB(r,\beta)$
   $a = \dfrac{\beta}{1+\beta}$ and $b = \dfrac{(r-1)\cdot\beta}{1+\beta}$
   $r$ is a scale parameter

4. Geometric Distribution with parameter $\beta$ : $G(\beta)$
   This distribution is the Negative Binomial Distribution with $r$ = 1.
   $NB(1,\beta) = G(\beta)$

Compound Counting Distribution $S = \sum_{i=1}^{N} M_i$

The distribution of the random variable *N* is called the primary distribution. The collection $\{M_i\}_1^N$ is independent and identically distributed to a given distribution *M*, called the secondary distribution, and also independent to *N*.

Probability Generating Function Relationship: $P_S(t) = P_N(P_M(t))$

**Important Formulas**: (Double Expectation D-E)
In each formula, the first equality follows from the D-E Theorem and the second equality follows from the facts $E[S \mid N] = N \cdot E[M]$ and $Var(S \mid N) = N \cdot Var(M)$

1. $E[S] = E\big[E[S \mid N]\big] = E[N] \cdot E[M]$

2. $Var(S) = E\big[Var(S \mid N)\big] + Var\big(E[S \mid N]\big) = E[N] \cdot Var(M) + \big(E[M]\big)^2 \cdot Var(N)$

**Often Tested Special Case of Compound Counting Distributions**:
$N \sim P(\lambda)$: This case is called the **compound Poisson distribution**.

The D-E formulas simplify to the following:

1. $E[S] = \lambda \cdot E[M]$

2. $Var(S) = \lambda \cdot E[M^2]$

# Tweaks to Severity Distributions

## Basic Random Variables

$X$ – ground up loss random variable (severity distribution)

Apply (regular) deductible, $d$

$Y^L = (X-d)_+ = \begin{cases} 0 \cdots X \le d \\ X-d \cdots X > d \end{cases}$ (payment per loss random variable)

$Y^P = Y^L \mid Y > 0 = X - d \mid X > d$ (payment per payment random variable)
(recognize $X - d \mid X > d$ as $T(d)$ in survival analysis)

Expectations: (continuous case; similar summations in the discrete case)

$$E[(Y^L)] = E[(X-d)_+] = \int_d^\infty (x-d)\cdot f_X(x)dx = \int_d^\infty [1-F_X(x)]dx$$

$$E[Y^P] = E[Y^L \mid Y^L > 0] = \frac{E[Y^L]}{\Pr(Y^L > 0)} = \frac{E[Y^L]}{\Pr(X > d)}$$

(Called expected payment per payment, mean excess loss or the mean residual loss)

Anytime using deductibles, for the payment per payment random variable,

$$E[(Y^P)^k] = \int_d^\infty (x-d)^k \cdot \frac{f_X(x)}{\Pr(X>d)}dx = \frac{\int_d^\infty (x-d)^k \cdot f_X(x)dx}{\Pr(X>d)} = \frac{E[(Y^L)^k]}{\Pr(X>d)}$$

## Often Tested Severity Distributions

1. $X \sim U(0,\omega) \Rightarrow X - d \mid X > d \sim U(0, \omega - d)$ (DML in survival analysis)

2. $X \sim EX(mean = \theta) \Rightarrow X - d \mid X > d \sim EX(mean = \theta)$ (CF in survival analysis)

3. $X \sim Par(\alpha, \theta) \Rightarrow X - d \mid X > d \sim Par(\alpha, \theta + d)$

Apply Policy Limit, *u*

$$Y = X \wedge u = \begin{cases} X \cdots X \le u \\ u \cdots X > u \end{cases}$$ = payment per loss random variable

Expectations: (continuous case; similar summations in the discrete case)

$$E[(X \wedge u)^k] = \int_0^u x^k f_X(x)dx + u^k \cdot \Pr(X > u)$$

Comments and Concepts:

1. $X = (X - d)_+ + (X \wedge d)$
   It may be easier to calculate $E[(X-d)_+]$ as $E[(X-d)_+] = E[X] - E[X \wedge d]$

2. $LER = \dfrac{E[X \wedge d]}{E[X]}$ (loss elimination ratio)

3. A **franchise deductible**, *d*, is applies as follows:

$$Y^L = \begin{cases} 0 \cdots X \le d \\ X \cdots X > d \end{cases} = (X-d)_+ + \begin{cases} 0 \cdots X \le d \\ d \cdots X > d \end{cases}$$

$$E[Y^L] = E[(X-d)_+] + d \cdot \Pr(X > d)$$

4. Apply deductible, *d*, along with **maximum covered loss**, *u*, as follows:

$$Y^L = \begin{cases} 0 \cdots X \le d \\ X - d \cdots d < X \le u \\ u - d \cdots X > u \end{cases} = (X \wedge u) - (X \wedge d)$$ (**policy limit** = *u* – *d*)

$$E[Y^L] = E[X \wedge u] - E[X \wedge d]$$

$$E[(Y^L)^2] = E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\big(E[X \wedge u] - E[X \wedge d]\big)$$

Applying this equation with no maximum covered loss, in other words replacing $(X \wedge u)$ by $X$, gives another formula for $E[(X-d)_+^2]$.

Apply Coinsurance With Factor $\alpha$ $(0 < \alpha < 1)$, Deductible *d*, and Maximum Covered Loss *u*. (Insurer pays $\alpha$ of the amount calculated *after* application of deductible and policy limit.)

$$Y^L = \alpha \cdot \left((X \wedge u) - (X \wedge d)\right) = \begin{cases} 0 \cdots X \le d \\ \alpha \cdot (X - d) \cdots d < X \le u \\ \alpha \cdot (u - d) \cdots X > u \end{cases}$$

Expectations: (Compare this $Y^L$ to the $Y^L$ in Comment 4. above.)

$$E[Y^L] = \alpha \cdot \left(E[X \wedge u] - E[X \wedge d]\right)$$

$$E[(Y^L)^2] = \alpha^2 \cdot \left(E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\left(E[X \wedge u] - E[X \wedge d]\right)\right)$$

These are just the modified versions of the expectations in Comment 4.

Apply Inflation Adjustment With Rate *r*, Deductible *d*, and Maximum Covered Loss *u*. (This concept is similar to coinsurance except that inflation is usually applied to losses only, keeping the maximum covered loss, *u*, and the deductible, *d*, the same from one year to the next.)

After inflation:

$$Y^L = ((1+r) \cdot X \wedge u) - ((1+r) \cdot X \wedge d) = \begin{cases} 0 \cdots (1+r) \cdot X \le d \\ (1+r) \cdot X - d \cdots d < (1+r) \cdot X \le u \\ u - d \cdots (1+r) \cdot X > u \end{cases}$$

Expectation:

**Useful Fact:** $E[c \cdot X \wedge d] = c \cdot E\left[X \wedge \frac{d}{c}\right]$

Applied to an inflation adjustment, this fact gives

$$E[Y^L] = (1+r) \cdot \left(E\left[X \wedge \frac{u}{1+r}\right] - E\left[X \wedge \frac{d}{1+r}\right]\right)$$

## Aggregate Loss Model

This concept is the same as the compound counting distribution above except the primary distribution is a frequency distribution and the secondary distribution is a severity distribution.

$$S = \sum_{i=1}^{N} X_i$$

**Often Tested Process**:
Many questions ask to use the normal approximation to perform a calculation. The normalization of the random variable *S* is

$$Z = \frac{S - E[S]}{\sqrt{Var(S)}}$$

Applying a deductible, *d*, to individual losses in the aggregate loss model (This concept is sometimes called **excess of loss insurance**.)

Method 1:

Leave the frequency distribution $N$ unchanged and replace the severity distribution $X$ by $Y^L = (X - d)_+$.

$S = \sum_{i=1}^{N} Y_i^L$ where each $Y_i^L$ has the same distribution as $Y^L = (X - d)_+$

Method 2:

Replace $N$ by $M$, the random variable representing the number of losses that exceed the deductible *d*, and replace $X$ by $Y^P = X - d \mid X > d$.

$S = \sum_{i=1}^{M} Y_i^P$ where each $Y_i^P$ has the same distribution as $Y^P = X - d \mid X > d$

## Often Tested Special Cases of Excess of Loss Insurance

1. $N \sim P(\lambda) \Rightarrow M \sim P(\lambda \cdot \Pr(X > d))$

2. $N \sim B(m,q) \Rightarrow M \sim B(m, q \cdot \Pr(X > d))$

3. $N \sim NB(r,\beta) \Rightarrow M \sim NB(r, \beta \cdot \Pr(X > d))$

## Stop-Loss Reinsurance

(This is insurance for the insurance company, from the reinsurer's perspective. Apply deductible, *d*, to aggregate losses, *not* to individual losses)

$(S-d)_+ = S-(S \wedge d)$ = payment per loss random variable for reinsurer

$E[(S-d)_+] = \int_d^\infty (1-F_S(s))ds = E[S]-E[S \wedge d]$ = stop-loss premium

Comments and Concepts:

1. If $S$ is discrete, then $1-F_S(s)$ is a step function, starting at $1-F_S(0)$ and usually stepping down to 0. The above integral reduces to finding the sum of the areas of rectangles.

2. An **often tested case** is $a \le d \le b$ and $\Pr(a < S < b) = 0$ ($S$ is discrete). In this case,

$$E[(S-d)_+] = E[(S-a)_+] - (d-a)\cdot(1-F_S(a)) \quad \text{(draw a picture)}$$

$$= \frac{b-d}{b-a}E[(S-a)_+] + \frac{d-a}{b-a}E[(S-b)_+] \quad \text{(linear interpolation)}$$

3. Adding a maximum covered loss, *u*, along with an aggregate loss deductible, *d*, results in a payment per loss to the reinsurer of

$$(S \wedge u)-(S \wedge d) = (S-d)_+ - (S-u)_+ = \begin{cases} 0 \cdots S \le d \\ S-d \cdots d < S \le u \\ u-d \cdots S > u \end{cases}$$