1) $q_x^{(\tau)} = 1 - p_x^{(\tau)} = 1 - .94 = .06$

$q_x^{(\tau)} = \sum q_x^{(j)} = q_x^{(1)} + q_x^{(2)}$

$\Rightarrow .06 = .04 + q_x^{(2)} \Rightarrow q_x^{(2)} = .02$

2) $p_x^{(\tau)} = \prod p_x^{\prime(j)}$

$q_x^{\prime(2)} = .1 \Rightarrow p_x^{\prime(2)} = .9$

$\therefore p_x^{(\tau)} = p_x^{\prime(1)} \cdot p_x^{\prime(2)} = (.95)(.9) = .855$

$\Rightarrow q_x^{(\tau)} = 1 - p_x^{(\tau)} = .145$

3) For (a), we use $q_x^{(2)} = q_x^{(\tau)} - q_x^{(1)}$

$q_x^{(\tau)} = 1 - p_x^{(\tau)} = .28 \Rightarrow q_x^{(2)} = .28 - .09 = .19$

For (b), we use $p_x^{\prime(2)} = \dfrac{p_x^{(\tau)}}{p_x^{\prime(1)}} = \dfrac{.72}{.9} = .8$

$\therefore q_x^{\prime(2)} = 1 - p_x^{\prime(2)} = .2$

4) (See Video Solution)

(a) ${}_{2}q_{31}^{(2)} = .3496$

(b) ${}_{1|2}q_{30}^{(1)} = .158064$

5) (See Video Solution)

(a) ${}_{5}q_{x}^{(\tau)} = 1 - e^{-3}$

(b) ${}_{5|10}q_{x}^{(2)} = \frac{1}{3}\left(e^{-3} - e^{-9}\right)$

6) (a) Since $\mu_x^{(\tau)} = .01 + .02 + .03 = .06$ (constant force)

$${}_{10}q_x^{(\tau)} = 1 - {}_{10}p_x^{(\tau)} = 1 - e^{-10(.06)} = 1 - e^{-.6}$$

(b) Since $\mu^{(2)} = \frac{1}{3}\mu^{(\tau)}$, ${}_{10}q_x^{(2)} = \frac{1}{3} \cdot {}_{10}q_x^{(\tau)}$

$$\Longrightarrow \; {}_{10}q_x^{(2)} = \frac{1}{3}\left(1 - e^{-.6}\right)$$

(c) $${}_{10|10}q_x^{(\tau)} = {}_{10}p_x^{(\tau)} - {}_{20}p_x^{(\tau)}$$

$$= e^{-10(.06)} - e^{-20(.06)} = e^{-.6} - e^{-1.2}$$

(d) Since $\mu^{(1)} = \frac{1}{6}\mu^{(\tau)}$, ${}_{10|10}q_x^{(1)} = \frac{1}{6} \cdot {}_{10|10}q_x^{(\tau)}$

$$\Longrightarrow \; {}_{10|10}q_x^{(1)} = \frac{1}{6}\left(e^{-.6} - e^{-1.2}\right)$$

(e) Since $\mu^{(\tau)} = .06$, (constant force) the random variable representing the time until departure follows an exponential distribution with mean $\frac{1}{\mu} = \frac{1}{.06}$

$$\therefore \; \mathring{e}_x^{(\tau)} = \frac{1}{.06}$$

7) (See Video Solution)

(a) ${}_5q_x^{(\tau)} = 1 - e^{-1.05}$

(b) ${}_5q_x^{(3)} = \frac{1}{2}\left(1 - e^{-1.05}\right)$

(c) $P = \dfrac{\mu_x^{(2)}(5)}{\mu_x^{(\tau)}(5)} = \dfrac{1}{3}$

(d) $P = \dfrac{{}_5q_x^{(1)}}{{}_5q_x^{(\tau)}} = \dfrac{1}{6}$

8) Use the given information to complete the table.

(i) $q_{50}^{(\tau)} = .1 \Rightarrow p_{50}^{(\tau)} = .9 = \dfrac{\ell_{51}^{(\tau)}}{\ell_{50}^{(\tau)}} = \dfrac{900}{\ell_{50}^{(\tau)}} = \ell_{50}^{(\tau)} = 1000$

$\Longrightarrow d_{50}^{(2)} = 25$

(ii) ${}_2p_{50}^{(\tau)} = .825 = \dfrac{\ell_{52}^{(\tau)}}{\ell_{50}^{(\tau)}} = \dfrac{\ell_{52}^{(\tau)}}{1000} \Longrightarrow \ell_{52}^{(\tau)} = 825$

(iii) $d_{51}^{(1)} = 2 \cdot d_{51}^{(2)} \Longrightarrow d_{51}^{(\tau)} = 3d_{51}^{(2)}$

$d_{51}^{(\tau)} = \ell_{51}^{(\tau)} - \ell_{52}^{(\tau)} = 900 - 825 = 75$

$\Longrightarrow 75 = 3d_{51}^{(2)} \Longrightarrow d_{51}^{(2)} = 25$ and $d_{51}^{(1)} = 50$

(iv) ${}_{2|}q_{50}^{(1)} = \dfrac{d_{52}^{(1)}}{\ell_{50}^{(\tau)}} \Longrightarrow d_{52}^{(1)} = 1000(.025) = 25$

$\therefore d_{52}^{(\tau)} = d_{52}^{(1)} + d_{52}^{(2)} = 25 + 25 = 50$

$\Longrightarrow \ell_{53}^{(\tau)} = \ell_{52}^{(\tau)} - d_{52}^{(\tau)} = 825 - 50 = 775$

(a) $q_{50}^{(2)} = \dfrac{d_{50}^{(2)}}{\ell_{50}^{(\tau)}} = \dfrac{25}{1000} = .025$

(b) ${}_2q_{51}^{(\tau)} = 1 - {}_2p_{51}^{(\tau)} = 1 - \dfrac{\ell_{53}^{(\tau)}}{\ell_{51}^{(\tau)}} = 1 - \dfrac{775}{900} = \dfrac{125}{900} = \dfrac{5}{36}$

(c) ${}_{1|}q_{51}^{(1)} = \dfrac{d_{52}^{(1)}}{\ell_{51}^{(\tau)}} = \dfrac{25}{900} = \dfrac{1}{36}$

(d) ${}_{1|2}q_{50}^{(2)} = \dfrac{{}_2d_{51}^{(2)}}{\ell_{50}^{(\tau)}} = \dfrac{d_{51}^{(2)} + d_{52}^{(2)}}{\ell_{50}^{(\tau)}} = \dfrac{25+25}{1000} = .05$

9) We seek $d_x^{(2)} = l_x^{(\tau)} \cdot q_x^{(2)} = 1000 \cdot q_x^{(2)}$

$$\left.\begin{array}{l} \mu_x^{(1)} = .1 \\ \mu_x^{(2)} = .3 \\ \mu_x^{(3)} = .5 \end{array}\right\} \Longrightarrow \mu_x^{(\tau)} = .1 + .3 + .5 = .9 \text{ (constant)}$$

$$\Longrightarrow p_x^{(\tau)} = e^{-.9(1)} = e^{-.9}$$

$$\Longrightarrow q_x^{(\tau)} = 1 - e^{-.9}$$

$$\mu_x^{(2)} = \frac{1}{3} \mu_x^{(\tau)} \Longrightarrow q_x^{(2)} = \frac{1}{3} q_x^{(\tau)} = \frac{1}{3}\left(1 - e^{-.9}\right)$$

$$\therefore \ d_x^{(2)} = 1000 \cdot \frac{1}{3}\left(1 - e^{-.9}\right) \approx 197.81$$

10) (a) $\mu_x^{(\tau)} = .05$ (constant force)

$$\Longrightarrow {}_2p_x^{(\tau)} = e^{-.05(2)} = e^{-.1}$$

(b) $\mu^{(1)} = \frac{2}{5}\mu^{(\tau)} \Longrightarrow {}_2q_x^{(1)} = \frac{2}{5}\,{}_2q_x^{(\tau)}$

$${}_2p_x^{(\tau)} = e^{-.1} \Longrightarrow {}_2q_x^{(1)} = \frac{2}{5}(1-e^{-.1})$$

(c) $\mu^{(2)} = \frac{3}{5}\mu^{(\tau)} \Longrightarrow {}_2q_x^{(2)} = \frac{3}{5}\,{}_2q_x^{(\tau)} = \frac{3}{5}(1-e^{-.1})$

Note that ${}_2q_x^{(2)} + {}_2q_x^{(1)} = 1-e^{-.1} = {}_2q_x^{(\tau)}$

(d) $${}_2p_x'^{(1)} = e^{-\int_0^2 \mu_{x+t}^{(1)}\,dt} \underline{\underline{CF}}\ e^{-.02(2)} = e^{-.04}$$

$$\Longrightarrow {}_2q_x'^{(1)} = 1-e^{-.04}$$

(e) $${}_2p_x'^{(2)} = e^{-\int_0^2 \mu_{x+t}^{(2)}\,dt} \underline{\underline{CF}}\ e^{-.03(2)} = e^{-.06}$$

$$\Longrightarrow {}_2q_x'^{(2)} = 1-e^{-.06}$$

Note that ${}_2p_x'^{(1)} \cdot {}_2p_x'^{(2)} = e^{-.04}\cdot e^{-.06} = e^{-.1} = {}_2p_x^{(\tau)}$

11) (See Video Solution)

(a) ${}_{10}q_{50}^{(1)} = \frac{1}{3}\left(1 - e^{-1}\right)$

(b) $\mu_{50}^{(\tau)}(10) = .15$

12) (See Video Solution)

(a) ${}_{10}q_{30}^{(1)} = \frac{2}{7}\left(1 - e^{-.5}\right)$

(b) ${}_{10}q_{30}^{(2)} = \frac{5}{7} - \frac{4}{7}e^{-.5}$

13) (a) $\quad {}_tp_x'^{(j)} \overset{\text{MUDD}}{=} \left[{}_tp_x^{(\tau)}\right]^{q_x^{(j)}/q_x^{(\tau)}}$

$$p_x'^{(1)} = .9 \text{ and } p_x'^{(2)} = .8 \Longrightarrow p_x^{(\tau)} = .72 \Longrightarrow q_x^{(\tau)} = .28$$

$$\left.\begin{matrix} t=1 \\ j=1 \end{matrix}\right\} \Longrightarrow .9 = (.72)^{q_x^{(1)}/.28} \Longrightarrow q_x^{(1)} = .28\,\frac{\ln(.9)}{\ln(.72)} \approx .089804$$

$$\left.\begin{matrix} t=1 \\ j=2 \end{matrix}\right\} \Longrightarrow .8 = (.72)^{q_x^{(2)}/.28} \Longrightarrow q_x^{(2)} = .28\,\frac{\ln(.8)}{\ln(.72)} \approx .190196$$

<u>Note</u>: $q_x^{(1)} + q_x^{(2)} = .28 = q_x^{(\tau)}$

(b) (SUDD)

$$q_x^{(1)} = \int_0^1 {}_tp_x^{(\tau)} \cdot \mu_{x+t}^{(1)}\,dt = \int_0^1 \underline{{}_tp_x'^{(1)}} \cdot \underline{{}_tp_x'^{(2)}} \cdot \underline{\mu_{x+t}^{(1)}}\,dt$$

$${}_tp_x'^{(1)} \cdot \mu_{x+t}^{(1)} \overset{\text{SUDD}}{=} q_x'^{(1)} \qquad {}_tp_x'^{(2)} \overset{\text{SUDD}}{=} 1 - {}_tq_x'^{(2)} = 1 - t \cdot q_x'^{(2)}$$

$$\therefore\ q_x^{(1)} \overset{\text{SUDD}}{=} \int_0^1 \underline{q_x'^{(1)}} \cdot \left(\underline{1 - t\,q_x'^{(2)}}\right) dt = \int_0^1 .1\,(1 - .2t)\,dt$$

$$= .1(1) - .1\left(\tfrac{.2}{2}\right) = .09$$

Similarly,

$$q_x^{(2)} = \int_0^1 {}_tp_x^{(\tau)}\,\mu_{x+t}^{(2)}\,dt = \int_0^1 \underline{{}_tp_x'^{(1)}} \cdot \underline{{}_tp_x'^{(2)} \cdot \mu_{x+t}^{(2)}}\,dt$$

$$\overset{\text{SUDD}}{=} \int_0^1 \underline{q_x'^{(2)}} \cdot \left(\underline{1 - t \cdot q_x'^{(1)}}\right) dt = \int_0^1 .2\,(1 - .1t)\,dt$$

$$= .2\left[1 - \tfrac{.1}{2}\right] = .19$$

<u>Note</u>: $q_x^{(1)} + q_x^{(2)} = .09 + .19 = .28 = q_x^{(\tau)}$

14) $q_x^{(\tau)} = .1 + .2 = .3 \implies p_x^{(\tau)} = .7$

(a) (MUDD) $\quad {}_tp_x'^{(j)} = \left[{}_tp_x^{(\tau)}\right]^{q_x^{(j)}/q_x^{(\tau)}}$

$\left.\begin{matrix} t=1 \\ j=1 \end{matrix}\right\}$ $p_x'^{(1)} = (.7)^{.1/.3} = (.7)^{1/3} \implies q_x'^{(1)} = 1-(.7)^{1/3}$

$\left.\begin{matrix} t=1 \\ j=2 \end{matrix}\right\}$ $p_x'^{(2)} = (.7)^{.2/.3} = (.7)^{2/3} \implies q_x'^{(2)} = 1-(.7)^{2/3}$

<u>Note</u>: $p_x'^{(1)} \cdot p_x'^{(2)} = (.7)^{1/3} \cdot (.7)^{2/3} = .7 = p_x^{(\tau)}$

(b) (SUDD)

$$0.1 = q_x^{(1)} = \int_0^1 {}_tp_x^{(\tau)} \cdot \mu_{x+t}^{(1)}\, dt = \int_0^1 \underline{{}_tp_x'^{(1)}} \cdot \underline{\underline{{}_tp_x'^{(2)}}} \cdot \underline{\mu_{x+t}^{(1)}}\, dt$$

$$= \int_0^1 \underline{q_x'^{(1)}} \cdot \underline{\underline{(1 - t \cdot q_x'^{(2)})}}\, dt$$

$$\left.\begin{aligned} \therefore \quad 0.1 = q_x^{(1)} &= q_x'^{(1)}\left[1 - \frac{q_x'^{(2)}}{2}\right] \\ \text{Likewise} \quad 0.2 = q_x^{(2)} &= q_x'^{(2)}\left[1 - \frac{q_x'^{(1)}}{2}\right] \end{aligned}\right\} \begin{matrix} \text{2 equations} \\ \text{2 unknowns} \end{matrix}$$

To simplify appearance, let $q_x'^{(1)} = a$ and $q_x'^{(2)} = b$

$$\left.\begin{aligned} \therefore \quad 0.1 &= a\left(1 - \tfrac{b}{2}\right) \\ 0.2 &= b\left(1 - \tfrac{a}{2}\right) \end{aligned}\right\}$$

Solving for a in the first equation, we get $a = \frac{0.2}{2-b}$, and substituting into the second equation results in a quadratic in b, $b^2 - 2.1b + .4 = 0$.

$\therefore b = q_x'^{(2)} \approx .211847 \implies a = q_x'^{(1)} = \frac{0.2}{2-b} \approx .111847$

Note: $p_x'^{(1)} \cdot p_x'^{(2)} = (1 - q_x'^{(1)})(1 - q_x'^{(2)}) = .7 = p_x^{(\tau)}$

15) (See Video Solution)

(a) ${}_{.3}q_x^{(2)} = .057$

(b) ${}_{.5|.3}q_x^{(2)} = .057$

(c) ${}_{.3}q_{x+.5}^{(2)} = \frac{.057}{.86}$

16) $${}_{1.5}q_{40}^{(1)} = q_{40}^{(1)} + {}_{1|.5}q_{40}^{(1)} = q_{40}^{(1)} + p_{40}^{(\tau)} \cdot {}_{.5}q_{41}^{(1)}$$

$$p_{40}^{(\tau)} = p_{40}^{\prime(1)} \cdot p_{40}^{\prime(2)} = (.9)(.8) = .72$$

$$q_{40}^{(1)} = \int_0^1 {}_tp_{40}^{(\tau)} \cdot \mu_{40+t}^{(1)}\, dt = \int_0^1 \underline{{}_tp_{40}^{\prime(1)}} \cdot \underline{\underline{{}_tp_{40}^{\prime(2)}}} \cdot \underline{\mu_{40+t}^{(1)}}\, dt$$

$$\underline{\underline{\text{SUDD}}}\quad \underline{q_{40}^{\prime(1)}} \int_0^1 (\underline{\underline{1 - t\, q_{40}^{\prime(2)}}})\, dt$$

$$\therefore q_{40}^{(1)} = q_{40}^{\prime(1)}\left[1 - \frac{q_{40}^{\prime(2)}}{2}\right] = .1\left[1 - \frac{.2}{2}\right] = .09$$

$$\overset{x=41}{}$$

$${}_{.5}q_{41}^{(1)} = \int_0^{0.5} {}_tp_{41}^{(\tau)}\, \mu_{41+t}^{(1)}\, dt = \int_0^{0.5} \underline{{}_tp_x^{\prime(1)}} \cdot \underline{\underline{{}_tp_x^{\prime(2)}}} \cdot \underline{\mu_{x+t}^{(1)}}\, dt$$

$$\underline{\underline{\text{SUDD}}}\quad \int_0^{.5} \underline{q_{41}^{\prime(1)}}\, (\underline{\underline{1 - t \cdot q_{41}^{\prime(2)}}})\, dt$$

$$= .1 \int_0^{.5} (1 - .2t)\, dt$$

$$= .1\left[0.5 - .1(.5)^2\right] = .0475$$

$$\therefore\ {}_{1.5}q_x^{(1)} = .09 + .72(.0475) = .1242$$

17) (See Video Solution)

(a) $q_x'^{(1)} = .1$

(b) $q_x'^{(2)} = \frac{2}{9}$

18) Since decrement 3 is EOY, ignore it until the end. We have $q_x'^{(1)} = .2$, $q_x'^{(2)} = .4$, and $\underbrace{q_x'^{(3)} = .6}_{\text{ignoring until the end}}$

Then

(a) $q_x^{(1)} = \int_0^1 \underbrace{{}_tp_x^{(\tau)}}_{{}_tp_x'^{(1)} \cdot {}_tp_x'^{(2)}} \cdot \mu_{x+t}^{(1)}\, dt \overset{\text{SUDD}}{=} \int_0^1 q_x'^{(1)}(1 - t \cdot q_x'^{(2)})\,dt$

$\Longrightarrow q_x^{(1)} = q_x'^{(1)}\left[1 - \frac{q_x'^{(2)}}{2}\right] = .2\left[1 - \frac{.4}{2}\right] = .16$

(b) As in (a), $q_x^{(2)} \overset{\text{SUDD}}{=} q_x'^{(2)}\left[1 - \frac{q_x'^{(1)}}{2}\right] = .4\left[1 - \frac{.2}{2}\right] = .36$

(c) Method 1: $q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)} + q_x^{(3)} = .16 + .36 + q_x^{(3)} = .52 + q_x^{(3)}$

$q_x^{(\tau)} = 1 - p_x^{(\tau)} = 1 - p_x'^{(1)} p_x'^{(2)} p_x'^{(3)} = 1 - (.8)(.6)(.4) = .808$

$\Longrightarrow .808 = .52 + q_x^{(3)} \Longrightarrow q_x^{(3)} = .288$

Method 2: In order to depart by decrement 3, one must survive to the end of the year. Therefore

$q_x^{(3)} = p_x'^{(1)} \cdot p_x'^{(2)} \cdot q_x'^{(3)} = (.8)(.6)(.6) = .288$

19) (See Video Solution)

(a) $q_x^{(1)} = .085$

(b) $q_x^{(2)} = .285$

20) (See Video Solution)

(a) $q_x^{(1)} = .168$

(b) $q_x^{(2)} = .352$