Structural Models of Credit: A spectral Element Approach.

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Financial Mathematics Seminar
Florida State University
November 17, 2011

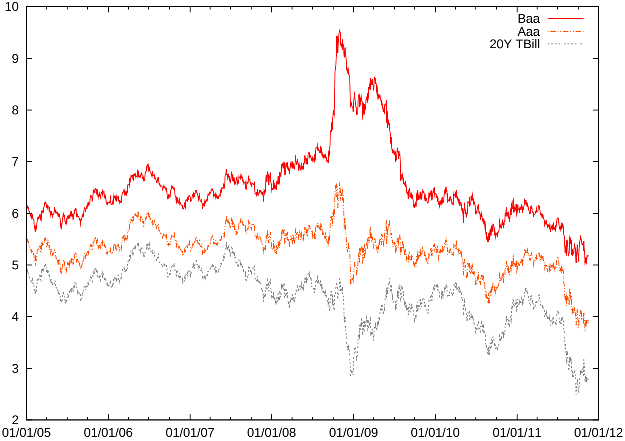


Figure: 20-Year Aaa and Baa bond spreads.

Definition

A credit spread is the premium paid to insure the risk of default of the underlying entity, i.e. it is the difference between the risk free rate, r_t, and the yield on the corporate bond, y_t.

$$
\begin{aligned}
B_t &= e^{-r_t t} \\
D_t &= L e^{-y_t t} \\
s_t &= -\frac{\ln D_t / L B_t}{t}
\end{aligned}
$$

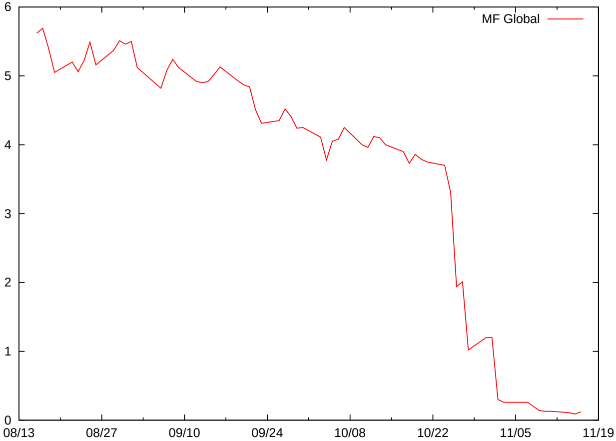
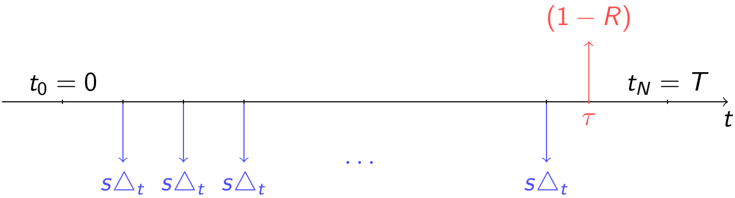


Figure: MF Global stock price.

Asset	Liabilities
V_t	S_t
	D_t

Table: Modigliani-Miller theorem


$(1-R)$
$t_0 = 0$
$t_N = T$
τ
t
$s\triangle_t$
$s\triangle_t$
$s\triangle_t$
$\dots$
$s\triangle_t$

$$DL = \mathbb{E}\left[(1-R)1_{\tau<T}e^{-\int_0^{\tau} r_s ds}\right]$$

$$FL = \mathbb{E}\left[\sum_{i=1}^{N} s\triangle_t 1_{\tau>t_i} e^{-\int_0^{t_i} r_s ds}\right]$$

$$DL = (1 - R) \int_0^T B(0, s) d\mathbb{P}(\tau \leq s)$$

$$FL = \sum_{i=1}^{N} s\triangle_t \mathbb{P}(\tau > t_i) B(0, t_i)$$

$$DL = (1-R)\left(1 - e^{-rT}\mathbb{P}(\tau > T) - r\int_0^T \mathbb{P}(\tau > s)e^{-rs}ds\right)$$

$$FL = \sum_{i=1}^{N} s\triangle_t \mathbb{P}(\tau > t_i)e^{-rt_i}$$

$$s = \frac{(1-R)\left(1 - e^{-rT}\mathbb{P}(\tau > T) - r\int_0^T \mathbb{P}(\tau > s)e^{-rs}ds\right)}{\sum_{i=1}^{N} \triangle_t \mathbb{P}(\tau > t_i)e^{-rt_i}}$$

- Default time $\tau = \inf\{t : \min_{0\leq s\leq t} V_s \leq D\}$
- $\mathbb{P}(\tau > t) = \mathbb{P}(\min_{0\leq s\leq t} V_s > D) = \mathbb{E}\left[1_{\min_{0\leq s\leq t} V_s > D}\right]$
- $D_T = R_T.1_{\{\tau\leq T\}} + D.1_{\{\tau > T\}}$

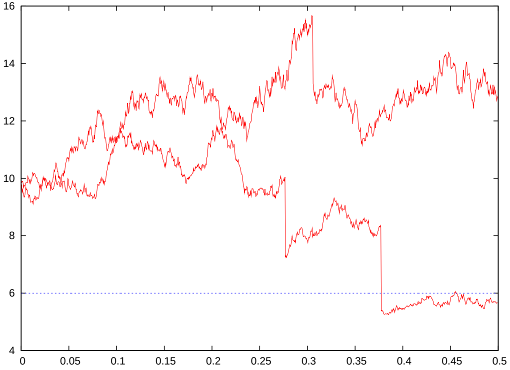


Figure: $V_t = V_0 e^{\mu t + \sigma W_t + \sum_{i=0}^{N_t} Y_i}$

- Default time $\tau = T.1_{\{V_T \leq D\}} + \infty 1_{\{V_T > D\}}$
- $\mathbb{P}(\tau > T) = \mathbb{P}(V_T > D)$
- $D_T = V_T.1_{\{V_T \leq D\}} + D.1_{\{V_T > D\}}$

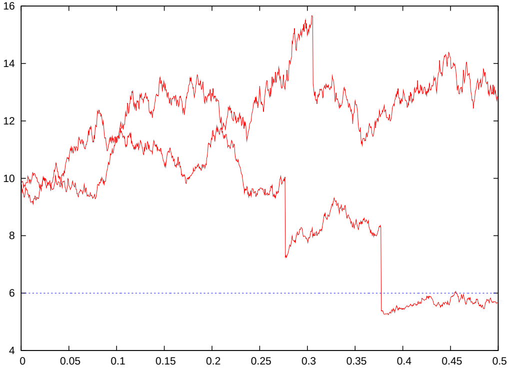


Figure: $V_t = V_0 e^{\mu t + \sigma W_t + \sum_{i=0}^{N_t} Y_i}$

- Default time $\tau = T.1_{\{V_T \le D\}} + \infty 1_{\{V_T > D\}}$
- $\mathbb{P}(\tau > T) = \mathbb{P}(V_T > D)$
- $D_T = D - (D - V_T)_+ \Rightarrow D_t = De^{-r(T-t)} - P_t(V_T, D)$

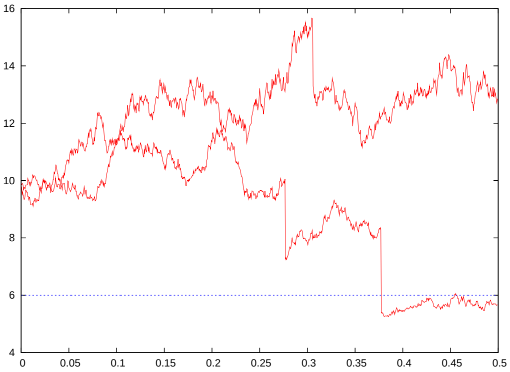
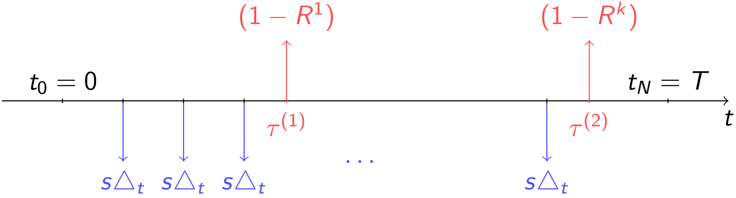
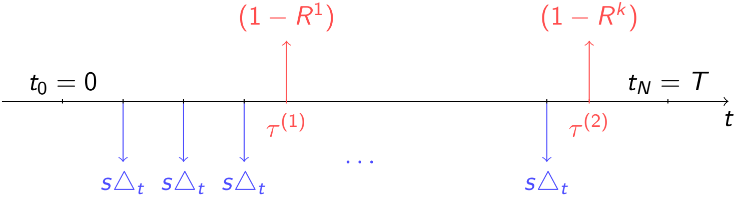


Figure: $V_t = V_0 e^{\mu t + \sigma W_t + \sum_{i=0}^{N_t} Y_i}$

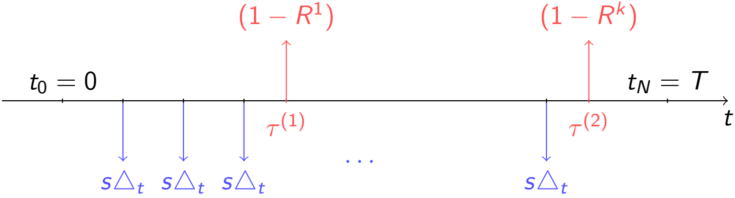

$(1 - R^1)$
$(1 - R^k)$
$t_0 = 0$
$t_N = T$
t
$\tau^{(1)}$
$\tau^{(2)}$
$s\triangle_t$
$s\triangle_t$
$s\triangle_t$
$\dots$
$s\triangle_t$

$$DL = \mathbb{E}\left[(1-R)1_{\tau^{(k)}<T}e^{-\int_0^{\tau^{(k)}} r_s ds}\right]$$



$$FL = \mathbb{E}\left[\sum_{i=1}^{N} s\triangle_t 1_{\tau^{(k)}>t_i} e^{-\int_0^{t_i} r_s ds}\right]$$

$$DL = \mathbb{E}\left[(1-R)1_{\min(\tau_1,\tau_2)<T}\, e^{-\int_0^{\min(\tau_1,\tau_2)} r_s ds}\right]$$



$$FL = \mathbb{E}\left[\sum_{i=1}^{N} s\triangle_t 1_{\min(\tau_1,\tau_2)>t_i}\, e^{-\int_0^{t_i} r_s ds}\right]$$

- First to default time $\tau = \inf\{t : \min_{0\le s\le t} V^i_s \le D^i, i = 1,2\}$
- $\mathbb{P}(\tau > t) = \mathbb{E}\left[1_{\{\min_{0\le s\le t} V^1_s > D^1, \min_{0\le s\le t} V^2_s > D^2\}}\right]$
- $D_T = V^i_{\tau^{(1)}}.1_{\{\tau^{(1)}\le T\}} + D^i.1_{\{\tau^{(1)}> T\}}$

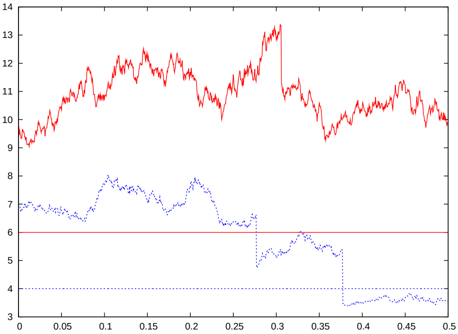


Figure: $V^i_t = e^{\mu^i t + \sigma^i W^i_t + \sum_{k=0}^{N^i_t} Y^i_k}$

- First to default time $\tau = T.1_{\min(V_T^1, V_T^2) \leq D\}} + \infty 1_{\{\min(V_T^1, V_T^2) > D\}}$
- $\mathbb{P}(\tau > T) = \mathbb{P}(\min(V_T^1, V_T^2) > D)$
- $D_T = \min(V_T^1, V_T^2).1_{\{\min(V_T^1, V_T^2) \leq D\}} + D.1_{\{\min(V_T^1, V_T^2) > D\}}$

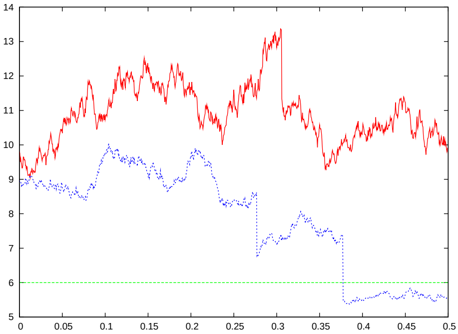


Figure: $V_t^i = e^{\mu^i t + \sigma^i W_t^i + \sum_{k=0}^{N_t^i} Y_k^i}$

- First to default time $\tau = T.1_{\min(V_T^1, V_T^2) \leq D\}} + \infty 1_{\{\min(V_T^1, V_T^2) > D\}}$
- $\mathbb{P}(\tau > T) = \mathbb{P}(\min(V_T^1, V_T^2) > D)$
- $D_T = D - (D - \min(V_T^1, V_T^2))_+ = De^{-r(T-t)} - P_t^{min}(V_T^1, V_T^2, D)$

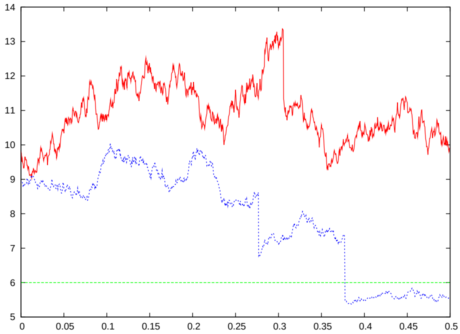


Figure: $V_t^i = e^{\mu^i t + \sigma^i W_t^i + \sum_{k=0}^{N_t^i} Y_k^i}$

Summary

- One period models, default only happens at T.
- Default is triggered by crossing a level, recovery happens at maturity.
- In 2 dimensions, we are assuming rescaling.
- Pricing a 0-coupon bond / computing survival probabilities

One dimension	Two dimensions
$\mathbb{E}\left[1_{\min_{0\leq s\leq t} V_s > D}\right]$	$\mathbb{E}\left[1_{\{\min_{0\leq s\leq t} V_s^1 > D^1, \min_{0\leq s\leq t} V_s^2 > D^2\}}\right]$
$\mathbb{E}\left[1_{\{V_T > D\}}\right]$	$\mathbb{E}\left[1_{\{\min(V_T^1, V_T^2) > D\}}\right]$
$P_t(V_T, D)$	$P_t^{min}(V_T^1, V_T^2, D)$

- Black Scholes framework !!

Pricing problem

Theorem (Feyman-Kac)

$$F(t,x) = \mathbb{E}\left[e^{-r(T-t)} f(S_T^{t,x}) | \mathcal{F}_t\right]$$

$$\Leftrightarrow \begin{cases} (\partial_u + \mathcal{A}_u - r)\, F(u,x) = 0 & t < u < T, x \in (0,\infty)^n \\ F(T,x) = f(x) & \forall x \in (0,\infty)^n \end{cases}$$

Pricing problem

Theorem (Feyman-Kac)

$$F(t,x) = \mathbb{E}\left[e^{-r(T-t)} f(S_T^{t,x}) | \mathcal{F}_t \right]$$

$$\Leftrightarrow \begin{cases} (\partial_u + \mathcal{A}_u - r)\, F(u,x) = 0 & t < u < T, x \in (0,\infty)^n \\ F(T,x) = f(x) & \forall x \in (0,\infty)^n \end{cases}$$

Proof.

$$\begin{aligned} \tilde{V}_t &= e^{-rt} F(t,S_t) = \tilde{F}(t,S_t) \\ &= \tilde{F}(0,S_0) + \int_0^t \phi_u d\tilde{S}_u + \int_0^t (\partial_u + \mathcal{A}_u - r) F du \end{aligned}$$

$\square$

One dimension

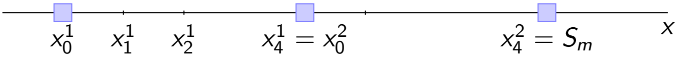
Problem (1d Black-Scholes PDE)

$$\begin{cases} \varphi_t = \partial_x(\nu(x)\varphi_x) + u(x)\varphi_x + \rho(x)\varphi & 0 < x < L \\ \varphi(0,t) = \beta_l(t),\ \varphi(L,t) = \beta_r(t) & t > 0 \\ \varphi(x,0) = \varphi_0(x) & 0 \le x \le L \end{cases}$$

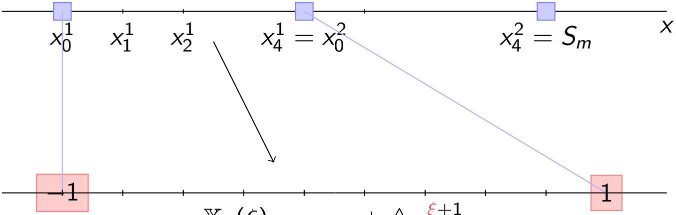
$$\nu(x) = \sigma^2 x^2/2, \quad u(x) = (r - \sigma^2)x, \quad \rho(x) = -r$$

$$L = S_{max}, \quad \varphi_0(x) = (K - x)_+$$

$$\beta_r = 0$$

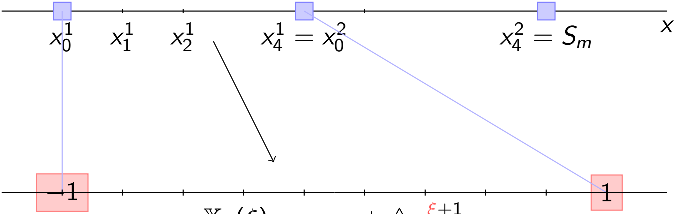


$$
\begin{aligned}
\int_0^L \varphi_t \phi dx &= \varphi_x \phi \,|_0^L - \int_0^L \nu(x) \varphi_x \phi_x dx \\
&+ \int_0^L u(x) \varphi_x \phi dx + \int_0^L \rho(x) \phi dx
\end{aligned}
$$



$$\mathbb{X}_k(\xi) = x_{k-1} + \triangle_k \frac{\xi+1}{2}$$

$$\begin{aligned}\sum_{k=1}^{K}\int_{x_{k-1}}^{x_k} \varphi_t^k \phi^k dx &= \varphi_x^k \phi^k \big|_0^L - \sum_{k=1}^{K}\int_{x_{k-1}}^{x_k} \nu(x)\varphi_x^k \phi_x^k dx \\ &+ \sum_{k=1}^{K}\int_{x_{k-1}}^{x_k} u(x)\varphi_x^k \phi^k dx + \sum_{k=1}^{K}\int_{x_{k-1}}^{x_k} \rho(x)\phi^k dx\end{aligned}$$



$$\mathbb{X}_k(\xi) = x_{k-1} + \triangle_k \frac{\xi+1}{2}$$

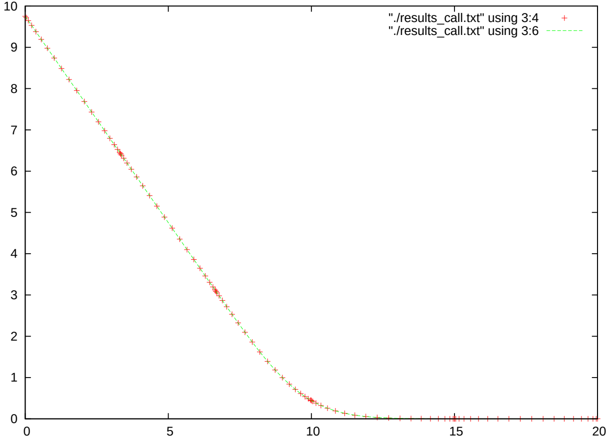
- $k = 1, \ldots, K \quad j \in \{1, \ldots, N-1\}$

$$\frac{\triangle_k}{2} \dot{\Phi}_j^k w_n = \langle \nabla \cdot (\nu \nabla \Phi), \phi_j^k \rangle_N + \langle u \cdot \nabla \Phi, \phi_j^k \rangle_N + \langle \rho \cdot \Phi, \phi_j^k \rangle_N$$

- $k = 1, \ldots, K-1$

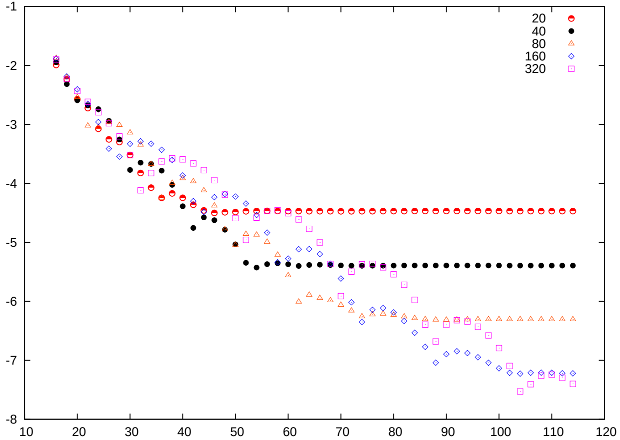
$$\frac{\triangle_k}{2} \dot{\Phi}_N^k w_N + \frac{\triangle_{k+1}}{2} \dot{\Phi}_0^{k+1} w_0 = \text{rhs}_N^k + \text{rhs}_0^{k+1}$$

- $\Phi_0^1 = \beta_l, \Phi_N^K = \beta_r$



(a) Approximated vs Exact: Put

Figure: $X = S_0 = 10, r = 8\%, \sigma = 30\%, T = 1$



(a) Log-Error in terms of total mesh points N, for several number of time steps

Figure: $X = S_0 = 10, r = 8\%, \sigma = 30\%, T = 1$

Two dimensions

Problem (2d Black-Scholes)

$$\begin{cases} \frac{\partial\varphi}{\partial t} = \nabla\cdot(\nu\cdot\nabla\varphi) + \mathbf{q}\cdot\nabla\varphi + \rho\cdot\varphi, & (x,y)\in\mathcal{D}, 0<t<T \\ \varphi(x,y,0) = \varphi_0(x,y), & (x,y)\in\mathcal{D} \\ \varphi(x,y,t) = \varphi_b(x,y,t), & (x,y)\in\partial\mathcal{D} \end{cases}$$

$$\nu = \frac{1}{2}\begin{bmatrix} \sigma_1^2 s_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 s_2^2 \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} (r-\sigma_1^2-\frac{1}{2}\rho\sigma_1\sigma_2)s_1 \\ (r-\sigma_2^2-\frac{1}{2}\rho\sigma_1\sigma_2)s_2 \end{bmatrix} \quad \rho = -r$$

$$\varphi_0(x,y) = (k-\min(x,y))_+$$

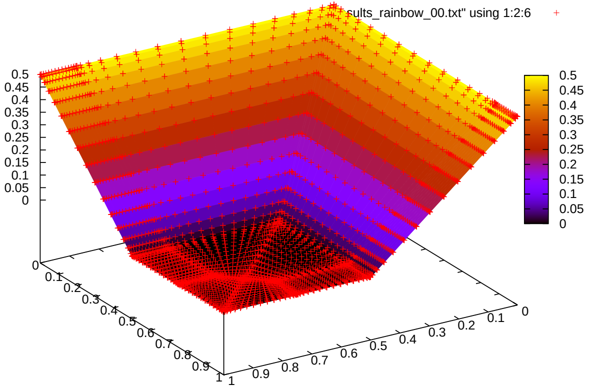
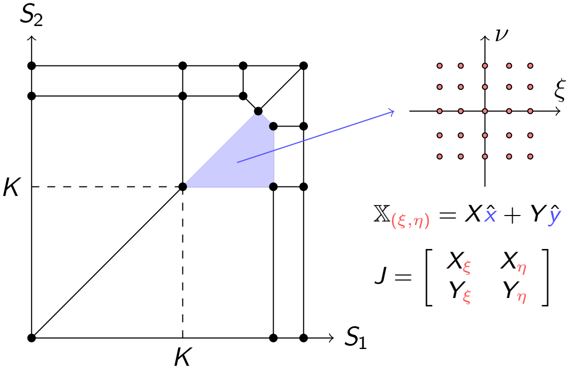


Figure: Rainbow Payoff $K = 0.5, S^1_{max} = S^2_{max} = 1$

S_1	S_2	$(K - \min(S_1, S_2))_+$
0	y	K
x	0	K
S^1_{max}	y	$(K - y)_+$
x	S^2_{max}	$(K - x)_+$

Table: Boundary for a put option on the minimum



$$\mathbb{X}_{(\xi,\eta)} = X\hat{x} + Y\hat{y}$$

$$J = \begin{bmatrix} X_\xi & X_\eta \\ Y_\xi & Y_\eta \end{bmatrix}$$

- $k = 1, \ldots, K,\ \{(i,j)\}_{i,j=0}^{N}$,

$$\dot{\Phi}_{ij}^{k}\,\omega_i\omega_j J_{ij}^{k} - \langle \nabla \cdot \mathbf{F}^k, \phi_{ij}^k \rangle_N = \langle \mathbf{q}^k \cdot \nabla\Phi, \phi_{ij}^k \rangle_N + \langle \rho^k \cdot \nabla\Phi, \phi_{ij}^k \rangle_N$$

- $i \in \partial\mathcal{D}_k(\xi^k),\ j \in \partial\mathcal{D}_k(\eta^k)$,

$$\begin{aligned}\sum_{k\in\mathcal{K}(\partial\mathcal{D}_k)} \left\{ \dot{\Phi}_{ij}^{k}\,\omega_i\omega_j J_{ij}^{k} - \langle \nabla \cdot \mathbf{F}^k, \phi_{ij}^k \rangle_N \right\} &= \sum_{k\in\mathcal{K}(\partial\mathcal{D}_k)} \Big\{ \langle \mathbf{q}^k \cdot \nabla\Phi, \phi_{ij}^k \rangle_N \\ &+ \langle \rho^k \cdot \nabla\Phi, \phi_{ij}^k \rangle_N \Big\}\end{aligned}$$

- $k \in \mathcal{K}(\partial\mathcal{D}),\ i \in \partial\mathcal{D}(\xi^k),\ j \in \partial\mathcal{D}(\eta^k)$,

$$\Phi_{ij}^{n+1,k} = \varphi_b(x_i, y_j, t_{n+1})$$

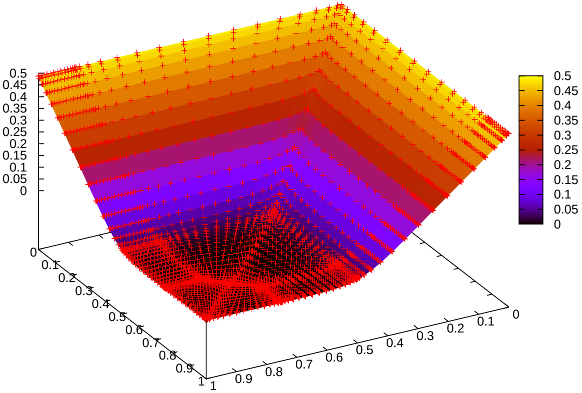
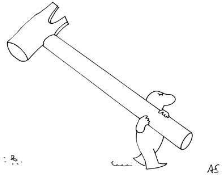


Figure:
$K = S_0^1 = S_0^2 = 0.5, r = 2\%, \sigma_1 = 20\%, \sigma_2 = 30\%, T = 0.5, N_e = 10, N = 16$

Future research

- Current
  - Convergence analysis of Spectral Element methods in 2-d.
  - Add jumps components in the PDE: integral term.
  - Add dependence structure on jump sizes / frequency.
  - Solve problem on $\min_{0 \leq s \leq T} S_s$ rather than S_T.
- Prospect
  - Multi-periods.
  - More that two assets: Monte Carlo.
  - Comparison to intensity models !



Merci !