CHAPTER 1

Introduction to Sets and Functions

1. Introduction to Sets

1.1. Basic Terminology. We begin with a refresher in the basics of set theory. Our treatment will be an informal one rather than taking an axiomatic approach at this time. Later in the semester we will revisit sets with a more formal approach.

A **set** is a collection or group of objects or **elements** or **members**. (Cantor 1895)

- A set is said to **contain** its elements.
- In each situation or context, there must be an underlying **universal set** U, either specifically stated or understood.

Notation:

- If x is a member or element of the set S, we write $x \in S$.
- If x is not an element of S we write $x \notin S$.

1.2. Notation for Describing a Set.

EXAMPLE 1.2.1. *List the elements between braces:*

- $S = \{a, b, c, d\} = \{b, c, a, d, d\}$

Specify by attributes:

- $S = \{x | x \geq 5 \text{ or } x < 0\}$, *where the universe is the set of real numbers.*

Use brace notation with ellipses:

- $S = \{\ldots, -3, -2, -1\}$, *the set of negative integers.*

Discussion

Sets can be written in a variety of ways. One can, of course, simply list the elements if there are only a few. Another way is to use set builder notation, which specifies the sets using a predicate to indicate the attributes of the elements of the set. For example, the set of even integers is

$$\{x|x = 2n, n \in \mathbb{Z}\}$$

or

$$\{\dots, -2, 0, 2, 4, 6, \dots\}.$$

The first set could be read as "the set of all x's such that x is twice an integer." The symbol | stands for "such that." A colon is often used for "such that" as well, so the set of even integers could also be written

$$\{x : x = 2n, n \in \mathbb{Z}\}.$$

1.3. Common Universal Sets. The following notation will be used throughout these notes.

- $\mathbb{R}$ = the real numbers
- $\mathbb{N}$ = the natural numbers = $\{0, 1, 2, 3, \dots\}$
- $\mathbb{Z}$ = the integers = $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- $\mathbb{Z}^+$ = the positive integers = $\{1, 2, 3, \dots\}$

Discussion

The real numbers, natural numbers, rational numbers, and integers have special notation which is understood to stand for these sets of numbers. Corresponding bold face letters are also a common notation for these sets of numbers. Some authors do not include 0 in the set of natural numbers. We will include zero.

EXERCISE 1.3.1. *Which of the following sets are equal to the set of all integers that are multiples of 5. There may be more than one or none.*

(1) $\{5n|n \in \mathbb{R}\}$
(2) $\{5n|n \in \mathbb{Z}\}$
(3) $\{n \in \mathbb{Z}|n = 5k \text{ and } k \in \mathbb{Z}\}$
(4) $\{n \in \mathbb{Z}|n = 5k \text{ and } n \in \mathbb{Z}\}$
(5) $\{-5, 0, 5, 10\}$

1.4. Complements and Subsets.

DEFINITION 1.4.1. *The* **complement** *of* A

$$\overline{A} = \{x \in U | x \notin A\}.$$

DEFINITION 1.4.2. *A set* A *is a* **subset** *of a set* B*, denoted*

$A \subseteq B$*, if and only if every element of* A *is also an element of* B*.*

DEFINITION 1.4.3. *If* $A \subseteq B$ *but* $A \neq B$ *then we say* A *is a* **proper subset** *of* B *and denote it by*

$$A \subset B.$$

DEFINITION 1.4.4. *The* **null** *set, or* **empty** *set, denoted* $\emptyset$*, is the set with no members.*

Note:

- $\emptyset$ *is a subset of every set.*
- *A set is always a subset of itself.*

Discussion

Please study the notation for elements, subsets, proper subsets, and the empty set. Two other common notations for the complement of a set, A, is A^c and A'. Notice that we make a notational distinction between subsets in general and proper subsets. Not all texts and/or instructors make this distinction, and you should check in other courses whether or not the notation $\subset$ really does mean proper as it does here.

1.5. Element v.s. Subsets.

Sets can be *subsets* and *elements* of other sets.

EXAMPLE 1.5.1. *Let* $A = \{\emptyset, \{\emptyset\}\}$*. Then* A *has two elements*

$$\emptyset \text{ and } \{\emptyset\}$$

and the four subsets

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}.$$

EXAMPLE 1.5.2. *Pay close attention to whether the symbols means "element" or "subset" in this example*

If $S = \{2, 3, \{2\}, \{4\}\}$*, then*

• $2 \in S$	• $3 \in S$	• $4 \notin S$
• $\{2\} \in S$	• $\{3\} \notin S$	• $\{4\} \in S$
• $\{2\} \subset S$	• $\{3\} \subset S$	• $\{4\} \not\subset S$
• $\{\{2\}\} \subset S$	• $\{\{3\}\} \not\subset S$	• $\{\{4\}\} \subset S$

EXERCISE 1.5.1. *Let* $A = \{1, 2, \{1\}, \{1, 2\}\}$. *True or false?*

(a) $\{1\} \in A$	*(c)* $\{\{1\}\} \in A$	*(e)* $2 \in A$	*(g)* $\{2\} \in A$
(b) $\{1\} \subseteq A$	*(d)* $\{\{1\}\} \subseteq A$	*(f)* $2 \subseteq A$	*(h)* $\{2\} \subseteq A$

1.6. Cardinality.

DEFINITION 1.6.1. *The number of (distinct) elements in a set* A *is called the* **cardinality** *of* A *and is written* $|A|$.

If the cardinality is a natural number, then the set is called **finite***, otherwise it is called* **infinite**.

EXAMPLE 1.6.1. *Suppose* $A = \{a, b\}$. *Then*

$$|A| = 2,$$

EXAMPLE 1.6.2. *The cardinality of* $\emptyset$ *is 0, but the cardinality of* $\{\emptyset, \{\emptyset\}\}$ *is 2.*

EXAMPLE 1.6.3. *The set of natural numbers is infinite since its cardinality is not a natural number. The cardinality of the natural numbers is a* **transfinite cardinal number**.

Discussion

Notice that the real numbers, natural numbers, integers, rational numbers, and irrational numbers are all infinite. Not all infinite sets are considered to be the same "size." The set of real numbers is considered to be a much larger set than the set of integers. In fact, this set is so large that we cannot possibly list all its elements in any organized manner the way the integers can be listed. We call a set like the real numbers that has too many elements to list *uncountable* and a set like the integers that can be listed is called *countable*. We will not delve any deeper than this into the study of the relative sizes of infinite sets in this course, but you may find it interesting to read further on this topic.

EXERCISE 1.6.1. *Let* $A = \{1, 2, \{1\}, \{1, 2\}\}$, $B = \{1, \{2\}\}$, $C = \{1, 2, 2, 2\}$, $D = \{5n | n \in \mathbb{R}\}$ *and* $E = \{5n | n \in \mathbb{Z}\}$.

Find the cardinality of each set.

1.7. Set Operations.

DEFINITION 1.7.1. *The* **union** *of sets* A *and* B, *denoted by* $A \cup B$ *(read "A union B"), is the set consisting of all elements that belong to either* A *or* B *or both. In symbols*

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

DEFINITION 1.7.2. *The* **intersection** *of sets A and B, denoted by $A \cap B$ (read "A intersection B"), is the set consisting of all elements that belong both A and B. In symbols*

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

DEFINITION 1.7.3. *The* **difference** *or* **relative compliment** *of two sets A and B, denoted by $A - B$ is the set of all elements in A that are not in B.*

$$A - B = \{x | x \in A \textit{ and } x \notin B\}$$

Discussion

The operations of union and intersection are the basic operations used to combine two sets to form a third. Notice that we always have $A \subseteq A \cup B$ and $A \cap B \subseteq A$ for arbitrary sets A and B.

1.8. Example 1.8.1.

EXAMPLE 1.8.1. *Suppose*
$A = \{1, 3, 5, 7, 9, 11\}$,
$B = \{3, 4, 5, 6, 7\}$ *and*
$C = \{2, 4, 6, 8, 10\}$.
Then

(a) $A \cup B = \{1, 3, 4, 5, 6, 7, 9, 11\}$
(b) $A \cap B = \{3, 5, 7\}$
(c) $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
(d) $A \cap C = \emptyset$
(e) $A - B = \{1, 9, 11\}$
(f) $|A \cup B| = 8$
(g) $|A \cap C| = 0$

EXERCISE 1.8.1. *Let $A = \{1, 2, \{1\}, \{1, 2\}\}$ and $B = \{1, \{2\}\}$. True or false:*

(a) $2 \in A \cap B$	*(b)* $2 \in A \cup B$	*(c)* $2 \in A - B$
(d) $\{2\} \in A \cap B$	*(e)* $\{2\} \in A \cup B$	*(f)* $\{2\} \in A - B$

1.9. Product.

DEFINITION 1.9.1. *The* **(Cartesian) Product** *of two sets, A and B, is denoted $A \times B$ and is defined by*

$$A \times B = \{(a, b) | a \in A \textit{ and } b \in B\}$$

Discussion

A cartesian product you have used in previous classes is $\mathbb{R} \times \mathbb{R}$. This is the same as the real plane and is shortened to $\mathbb{R}^2$. Elements of $\mathbb{R}^2$ are the points in the plane.

Notice the notation for an element in $\mathbb{R} \times \mathbb{R}$ is the same as the notation for an open interval of real numbers. In other words, $(3, 5)$ could mean the ordered pair in $\mathbb{R} \times \mathbb{R}$ or it could mean the interval $\{x \in \mathbb{R} | 3 < x < 5\}$. If the context does not make it clear which $(3, 5)$ stands for you should make it clear.

EXAMPLE 1.9.1. *Let* $A = \{a, b, c, d, e\}$ *and let* $B = \{1, 2\}$. *Then*

(1) $A \times B = \{(a, 1), (b, 1), (c, 1), (d, 1), (e, 1), (a, 2), (b, 2), (c, 2), (d, 2), (e, 2)\}$.
(2) $|A \times B| = 10$
(3) $\{a, 2\} \notin A \times B$
(4) $(a, 2) \notin A \cup B$

EXERCISE 1.9.1. *Let* $A = \{a, b, c, d, e\}$ *and let* $B = \{1, 2\}$. *Find*

(1) $B \times A$.
(2) $|B \times A|$
(3) Is $(a, 2) \in B \times A$*?*
(4) Is $(2, a) \in B \times A$*?*
(5) Is $2a \in B \times A$*?*