2. Introduction to Functions

2.1. Function.

Definition 2.1.1. *Let A and B be sets. A* **function**

$$f\colon A \to B$$

is a rule which assigns to every element in A exactly one element in B.

If f assigns $a \in A$ to the element $b \in B$, then we write

$$f(a) = b,$$

and we call b the **image** *or* **value** *of f at a.*

Discussion

This is the familiar definition of a function f from a set A to a set B as a rule that assigns each element of A to exactly one element B. This is probably quite familiar to you from your courses in algebra and calculus. In the context of those subjects, the sets A and B are usually subsets of real numbers $\mathbb{R}$, and the *rule* usually refers to some concatenation of algebraic or transcendental operations which, when applied to a number in the set A, give a number in the set B. For example, we may define a function $f\colon \mathbb{R} \to \mathbb{R}$ by the formula (rule) $f(x) = \sqrt{1 + \sin x}$. We can then compute values of f – for example, $f(0) = 1$, $f(\pi/2) = \sqrt{2}$, $f(3\pi/2) = 0$, $f(1) = 1.357$ (approximately) – using knowledge of the sine function at special values of x and/or a calculator. Sometimes the rule may vary depending on which part of the set A the element x belongs. For example, the absolute value function $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

The *rule* that defines a function, however, need not be given by a formula of such as those above. For example, the rule that assigns to each resident of the state of Florida his or her last name defines a function from the set of Florida residents to the set of all possible names. There is certainly nothing formulaic about the rule that defines this function. At the extreme we could randomly assign everyone in this class one of the digits 0 or 1, and we would have defined a function from the set of students in the class to the set $\{0, 1\}$. We will see a more formal definition of a function later on that avoids the use of the term *rule*, but for now it will serve us reasonably well. We will instead concentrate on terminology related to the concept of a function, including special properties a function may possess.

2.2. Terminology Related to Functions. Let A and B be sets and suppose $f : A \to B$.

- The set A is called the **domain** of f.
- The set B is called the **codomain** of f.
- If $f(x) = y$, then x is a **preimage** of y. Note, there may be more than one preimage of y, but only one image (or value) of x.
- The set $f(A) = \{f(x)|x \in A\}$ is called the **range** of f.
- If $S \subseteq A$, then the **image** of S under f is the set
$$f(S) = \{f(s)|s \in S\}.$$
- If $T \subseteq B$, then the **preimage** of T under f is the set
$$f^{-1}(T) = \{x \in A|f(x) \in T\}.$$

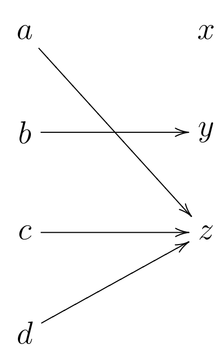
Discussion

Some of the fine points to remember:

- Every element in the domain must be assigned to exactly one element in the codomain.
- Not every element in the codomain is necessarily assigned to one of the elements in the domain.
- The range is the subset of the codomain consisting of all those elements that are the image of at least one element in the domain. It is the *image* of the domain.
- If a subset T of the codomain consists of a single element, say, $T = \{b\}$, then we usually write $f^{-1}(b)$ instead of $f^{-1}(\{b\})$. Regardless, $f^{-1}(b)$ is still a *subset* of A.

2.3. Example 2.3.1.

EXAMPLE 2.3.1. *Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$. The function f is defined by the relation pictured below:*



- $f(a) = z$
- *the image of* d *is* z
- *the domain of* f *is* $A = \{a, b, c, d\}$
- *the codomain is* $B = \{x, y, z\}$
- $f(A) = \{y, z\}$
- $f(\{c, d\}) = \{z\}$
- $f^{-1}(y) = \{b\}$.
- $f^{-1}(z) = \{a, c, d\}$
- $f^{-1}(\{y, z\}) = \{a, b, c, d\}$
- $f^{-1}(x) = \emptyset$.

Discussion

This example helps illustrate some of the differences between the codomain and the range. $f(A) = \{y, z\}$ is the range, while the codomain is all of $B = \{x, y, z\}$.

Notice also that the image of a single element is a single element, but the preimage of a single element may be more than one element. Here is another example.

EXAMPLE 2.3.2. *Let* $f : \mathbb{N} \to \mathbb{R}$ *be defined by* $f(n) = \sqrt{n}$.

- *The domain is the set of natural numbers.*
- *The codomain is the set of real numbers.*
- *The range is* $\{0, 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \dots\}$.
- *The image of* 5 *is* $\sqrt{5}$.
- *The preimage of* 5 *is* 25.
- *The preimage of* $\mathbb{N}$ *is the set of all perfect squares in* $\mathbb{N}$.

EXERCISE 2.3.1. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is defined by* $f(x) = |x|$. *Find*

(1) *the range of* f.
(2) *the image of* $\mathbb{Z}$, *the set of integers.*
(3) $f^{-1}(\pi)$.
(4) $f^{-1}(-1)$.
(5) $f^{-1}(\mathbb{Q})$, *where* $\mathbb{Q}$ *is the set of rational numbers.*

2.4. Floor and Ceiling Functions.

DEFINITIONS 2.4.1. *Floor and ceiling functions:*

- *The* **floor function**, *denoted*

$$f(x) = \lfloor x \rfloor$$

or

$$f(x) = \mathit{floor}(x),$$

is the function that assigns to x the greatest integer less than or equal to x.

- *The* **ceiling function**, *denoted*
$$f(x) = \lceil x \rceil$$
or
$$f(x) = \textit{ceiling}(x),$$
is the function that assigns to x the smallest integer greater than or equal to x.

Discussion

These two functions may be new to you. The floor function, $\lfloor x \rfloor$, also known as the "greatest integer function", and the ceiling function, $\lceil x \rceil$, are assumed to have domain the set of all reals, unless otherwise specified, and range is then the set of integers.

EXAMPLE 2.4.1. *(a)* $\lfloor 3.5 \rfloor = 3$
(b) $\lceil 3.5 \rceil = 4$
(c) $\lfloor -3.5 \rfloor = -4$
(d) $\lceil -3.5 \rceil = -3$
(e) notice that the floor function is the same as truncation for positive numbers.

EXERCISE 2.4.1. *Suppose x is a real number. Do you see any relationships among the values $\lfloor -x \rfloor$, $-\lfloor x \rfloor$, $\lceil -x \rceil$, and $-\lceil x \rceil$?*

EXERCISE 2.4.2. *Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \lfloor x \rfloor$. Find*

(1) the range of f.
(2) the image of $\mathbb{Z}$, the set of integers.
(3) $f^{-1}(\pi)$.
(4) $f^{-1}(-1.5)$.
(5) $f^{-1}(\mathbb{N})$, *where $\mathbb{N}$ is the set of natural numbers.*
(6) $f^{-1}([2.5, 5.5])$.
(7) $f([2.5, 5.5])$.
(8) $f(f^{-1}([2.5, 5.5]))$.
(9) $f^{-1}(f([2.5, 5.5]))$.

EXAMPLE 2.4.2. *Let $h : [0, \infty) \to \mathbb{R}$ be defined by $h(x) = \lfloor 3x - 1 \rfloor$. Let $A = \{x \in \mathbb{R} | 4 < x < 10\}$.*

(1) The domain is $[0, \infty)$
(2) The codomain is $\mathbb{R}$
(3) The range is $\{-1, 0, 1, 2, \dots\}$
(4) $g(A) = \{11, 12, 13, 14, \dots, 28\}$.

(5) $g^{-1}(A) = [2, 11/3)$.

EXERCISE 2.4.3. *Let* $g : \mathbb{R} \to [0, \infty)$ *be defined by* $g(x) = \lceil x^2 \rceil$. *Let* $A = \{x \in [0, \infty) | 3.2 < x < 8.9\}$.

(1) domain
(2) codomain
(3) range
(4) Find $g(A)$.
(5) Find $g^{-1}(A)$.

2.5. Characteristic Function.

Definition: Let U be a universal set and $A \subseteq U$. The **Characteristic Function** of A is defined by

$$\chi_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

Discussion

The Characteristic function is another function that may be new to you.

EXAMPLE 2.5.1. *Consider the set of integers as a subset of the real numbers. Then*

$$\chi_{\mathbb{Z}}(y)$$

will be 1 when y is an integer and will be zero otherwise.

EXERCISE 2.5.1. *Graph the function*

$$\chi_{\mathbb{Z}}$$

given in the previous example in the plane.

EXERCISE 2.5.2. *Find*

(1) $\chi_{\mathbb{Z}}(0)$
(2) $\chi_{\mathbb{Z}}^{-1}(0)$
(3) $\chi_{\mathbb{Z}}([3, 5])$
(4) $\chi_{\mathbb{Z}}^{-1}([3, 5])$

EXERCISE 2.5.3. *Let* $E = \{4n | n \in \mathbb{N}\}$ *and consider the characteristic function* $\chi_E : \mathbb{Z} \to \mathbb{Z}$. *What is the ...*

(1) domain
(2) codomain
(3) range

(4) $\chi_E(\{2n|n \in \mathbb{N}\})$
(5) $\chi_E^{-1}(\{2n|n \in \mathbb{N}\})$