3. Predicates and Quantifiers

3.1. Predicates and Quantifiers.

Definition 3.1.1. *A* **predicate** *or* **propositional function** *is a description of the property (or properties) a variable or subject may have. A proposition may be created from a propositional function by either assigning a value to the variable or by quantification.*

Definition 3.1.2. *The independent variable of a propositional function must have a* **universe of discourse***, which is a set from which the variable can take values.*

Discussion

Recall from the introduction to logic that the sentence "$x + 2 = 2x$" is not a proposition, but if we assign a value for x then it becomes a proposition. The phrase "$x + 2 = 2x$" can be treated as a function for which the input is a value of x and the output is a proposition.

Another way we could turn this sentence into a proposition is to quantify its variable. For example, "for every real number x, $x + 2 = 2x$" is a proposition (which is, in fact, false, since it fails to be true for the number $x = 0$).

This is the idea behind *propositional functions* or *predicates*. As stated above a *predicate* is a property or attribute assigned to elements of a particular set, called the *universe of discourse*. For example, the predicate "$x + 2 = 2x$", where the universe for the variable x is the set of all real numbers, is a property that some, but not all, real numbers possess.

In general, the set of all x in the universe of discourse having the attribute $P(x)$ is called the **truth set of** P. That is, the truth set of P is

$$\{x \in U | P(x)\}.$$

3.2. Example of a Propositional Function.

Example 3.2.1. *The propositional function $P(x)$ is given by "$x > 0$" and the universe of discourse for x is the set of integers. To create a proposition from P, we may assign a value for x. For example,*

- *setting $x = -3$, we get $P(-3)$: "$-3 > 0$", which is false.*
- *setting $x = 2$, we get $P(2)$: "$2 > 0$", which is true.*

Discussion

In this example we created propositions by choosing particular values for x.

Here are two more examples:

EXAMPLE 3.2.2. *Suppose $P(x)$ is the sentence "x has fur" and the universe of discourse for x is the set of all animals. In this example $P(x)$ is a true statement if x is a cat. It is false, though, if x is an alligator.*

EXAMPLE 3.2.3. *Suppose $Q(y)$ is the predicate "y holds a world record," and the universe of discourse for y is the set of all competitive swimmers. Notice that the universe of discourse must be defined for predicates. This would be a different predicate if the universe of discourse is changed to the set of all competitive runners.*

Moral: Be very careful in your homework to specify the universe of discourse precisely!

3.3. Quantifiers. A **quantifier** turns a propositional function into a proposition without assigning specific values for the variable. There are primarily two quantifiers, the

universal quantifier

and the

existential quantifier.

DEFINITION 3.3.1. *The* **universal quantification** *of $P(x)$ is the proposition*

"$P(x)$ is true for all values x in the universe of discourse."

Notation: "For all x $P(x)$" or "For every x $P(x)$" is written

$$\forall x P(x).$$

DEFINITION 3.3.2. *The* **existential quantification** *of $P(x)$ is the proposition*

"There exists an element x in the universe of discourse such that $P(x)$ is true."

Notation: "There exists x such that $P(x)$" or "There is at least one x such that $P(x)$" is written

$$\exists x P(x).$$

Discussion

As an alternative to assigning particular values to the variable in a propositional function, we can turn it into a proposition by *quantifying* its variable. Here we see the two primary ways in which this can be done, the universal quantifier and the existential quantifier.

In each instance we have created a proposition from a propositional function by *binding its variable.*

3.4. Example 3.4.1.

EXAMPLE 3.4.1. *Suppose* $P(x)$ *is the predicate* $x + 2 = 2x$*, and the universe of discourse for* x *is the set* $\{1, 2, 3\}$*. Then...*

- $\forall x P(x)$ *is the proposition "For every* x *in* $\{1, 2, 3\}$ $x + 2 = 2x$*." This proposition is false.*
- $\exists x P(x)$ *is the proposition "There exists* x *in* $\{1, 2, 3\}$ *such that* $x + 2 = 2x$*." This proposition is true.*

EXERCISE 3.4.1. *Let* $P(n, m)$ *be the predicate* $mn > 0$*, where the domain for* m *and* n *is the set of integers. Which of the following statements are true?*

(1) $P(-3, 2)$
(2) $\forall m P(0, m)$
(3) $\exists n P(n, -3)$

3.5. Converting from English.

EXAMPLE 3.5.1. *Assume*

$F(x)$*: x is a fox.*

$S(x)$*: x is sly.*

$T(x)$*: x is trustworthy.*

and the universe of discourse for all three functions is the set of all animals.

- *Everything is a fox:* $\forall x F(x)$
- *All foxes are sly:* $\forall x[F(x) \to S(x)]$
- *If any fox is sly, then it is not trustworthy:*
  $\forall x[(F(x) \wedge S(x) \to \neg T(x)] \Leftrightarrow \neg \exists x[F(x) \wedge S(x) \wedge T(x)]$

Discussion

Notice that in this example the last proposition may be written symbolically in the two ways given. Think about the how you could show they are the same using the logical equivalences in Module 2.2.

3.6. Additional Definitions.

- An assertion involving predicates is **valid** if it is true for every element in the universe of discourse.
- An assertion involving predicates is **satisfiable** if there is a universe and an interpretation for which the assertion is true. Otherwise it is **unsatisfiable**.
- The **scope** of a quantifier is the part of an assertion in which the variable is bound by the quantifier.

Discussion

You would not be asked to state the definitions of the terminology given, but you would be expected to know what is meant if you are asked a question like "Which of the following assertions are satisfiable?"

3.7. Examples.

EXAMPLE 3.7.1.

If the universe of discourse is $U = \{1, 2, 3\}$, *then*

(1) $\forall x P(x) \Leftrightarrow P(1) \wedge P(2) \wedge P(3)$
(2) $\exists x P(x) \Leftrightarrow P(1) \vee P(2) \vee P(3)$

Suppose the universe of discourse U *is the set of real numbers.*

(1) If $P(x)$ *is the predicate* $x^2 > 0$, *then* $\forall x P(x)$ *is false, since* $P(0)$ *is false.*
(2) If $P(x)$ *is the predicate* $x^2 - 3x - 4 = 0$, *then* $\exists x P(x)$ *is true, since* $P(-1)$ *is true.*
(3) If $P(x)$ *is the predicate* $x^2 + x + 1 = 0$, *then* $\exists x P(x)$ *is false, since there are no real solutions to the equation* $x^2 + x + 1 = 0$.
(4) If $P(x)$ *is the predicate "If* $x \neq 0$, *then* $x^2 \geq 1$*', then* $\forall x P(x)$ *is false, since* $P(0.5)$ *is false.*

EXERCISE 3.7.1. *In each of the cases above give the truth value for the statement if each of the* $\forall$ *and* $\exists$ *quantifiers are reversed.*

3.8. Multiple Quantifiers. Multiple quantifiers are read from left to right.

EXAMPLE 3.8.1. *Suppose $P(x, y)$ is "$xy = 1$", the universe of discourse for x is the set of positive integers, and the universe of discourse for y is the set of real numbers.*

(1) $\forall x \forall y P(x, y)$ may be read "For every positive integer x and for every real number y, $xy = 1$. This proposition is false.
(2) $\forall x \exists y P(x, y)$ may be read "For every positive integer x there is a real number y such that $xy = 1$. This proposition is true.
(3) $\exists y \forall x P(x, y)$ may be read "There exists a real number y such that, for every positive integer x, $xy = 1$. This proposition is false.

Discussion

Study the syntax used in these examples. It takes a little practice to make it come out right.

3.9. Ordering Quantifiers. The order of quantifiers is important; they may not commute.

For example,

(1) $\forall x \forall y P(x, y) \Leftrightarrow \forall y \forall x P(x, y)$, and
(2) $\exists x \exists y P(x, y) \Leftrightarrow \exists y \exists x P(x, y)$,
but
(3) $\forall x \exists y P(x, y) \not\Leftrightarrow \exists y \forall x P(x, y)$.

Discussion

The lesson here is that you have to pay careful attention to the order of the quantifiers. The only cases in which commutativity holds are the cases in which both quantifiers are the same. In the one case in which equivalence does not hold,

$$\forall x \exists y P(x, y) \not\Leftrightarrow \exists y \forall x P(x, y),$$

there is an implication in one direction. Notice that if $\exists y \forall x P(x, y)$ is true, then there is an element c in the universe of discourse for y such that $P(x, c)$ is true for all x in the universe of discourse for x. Thus, for all x there exists a y, namely c, such that $P(x, y)$. That is, $\forall x \exists y P(x, y)$. Thus,

$$\exists y \forall x P(x, y) \Rightarrow \forall x \exists y P(x, y).$$

Notice predicates use function notation and recall that the variable in function notation is really a place holder. The statement $\forall x \exists y P(x, y)$ means the same as

$\forall s \exists t P(s,t)$. Now if this seems clear, go a step further and notice this will also mean the same as $\forall y \exists x P(y,x)$. When the domain of discourse for a variable is defined it is in fact defining the domain for the place that variable is holding at that time.

Here are some additional examples:

EXAMPLE 3.9.1. *$P(x,y)$ is "x is a citizen of y." $Q(x,y)$ is "x lives in y." The universe of discourse of x is the set of all people and the universe of discourse for y is the set of US states.*

(1) All people who live in Florida are citizens of Florida.

$$\forall x(Q(x, Florida) \to P(x, Florida))$$

(2) Every state has a citizen that does not live in that state.

$$\forall y \exists x(P(x,y) \wedge \neg Q(x,y))$$

EXAMPLE 3.9.2. *Suppose $R(x,y)$ is the predicate "x understands y," the universe of discourse for x is the set of students in your discrete class, and the universe of discourse for y is the set of examples in these lecture notes. Pay attention to the differences in the following propositions.*

(1) $\exists x \forall y R(x,y)$ is the proposition "There exists a student in this class who understands every example in these lecture notes."
(2) $\forall y \exists x R(x,y)$ is the proposition "For every example in these lecture notes there is a student in the class who understands that example."
(3) $\forall x \exists y R(x,y)$ is the proposition "Every student in this class understands at least one example in these notes."
(4) $\exists y \forall x R(x,y)$ is the proposition "There is an example in these notes that every student in this class understands."

EXERCISE 3.9.1. *Each of the propositions in Example 3.9.2 has a slightly different meaning. To illustrate this, set up the following diagrams: Write the five letters A, B, C, D, E on one side of a page, and put the numbers 1 through 6 on the other side. The letters represent students in the class and the numbers represent examples. For each of the propositions above draw the minimal number of lines connecting people to examples so as to construct a diagram representing a scenario in which the given proposition is true.*

Notice that for any chosen pair of propositions above, you can draw diagrams that would represent situations where the two propositions have opposite truth values.

EXERCISE 3.9.2. *Give a scenario where parts 1 and 2 in Example 3.9.2 have opposite truth values.*

EXERCISE 3.9.3. *Let $P(x, y)$ be the predicate $2x + y = xy$, where the domain of discourse for x is $\{u \in \mathbb{Z} | u \neq 1\}$ and for y is $\{u \in \mathbb{Z} | u \neq 2\}$. Determine the truth value of each statement. Show work or briefly explain.*

(1) $P(-1, 1)$
(2) $\exists x P(x, 0)$
(3) $\exists y P(4, y)$
(4) $\forall y P(2, y)$
(5) $\forall x \exists y P(x, y)$
(6) $\exists y \forall x P(x, y)$
(7) $\forall x \forall y [((P(x, y)) \wedge (x > 0)) \rightarrow (y > 1)]$

3.10. Unique Existential.

DEFINITION 3.10.1. *The* **unique existential quantification** *of $P(x)$ is the proposition "There exists a unique element x in the universe of discourse such that $P(x)$ is true."*

Notation: *"There exists unique x such that $P(x)$" or "There is exactly one x $P(x)$" is written*

$$\exists! x P(x).$$

Discussion

Continuing with Example 3.9.2, the proposition $\forall x \exists! y R(x, y)$ is the proposition "Every student in this class understands exactly one example in these notes (but not necessarily the same example for all students)."

EXERCISE 3.10.1. *Let $P(n, m)$ be the predicate $mn \geq 0$, where the domain for m and n is the set of integers. Which of the following statements are true?*

(1) $\exists! n \forall m P(n, m)$
(2) $\forall n \exists! m P(n, m)$
(3) $\exists! m P(2, m)$

EXERCISE 3.10.2. *Repeat Exercise 3.9.1 for the four propositions $\forall x \exists! y R(x, y)$, $\exists! y \forall x R(x, y)$, $\exists! x \forall y R(x, y)$, and $\forall y \exists! x R(x, y)$.*

Remember: A predicate is *not* a proposition until <u>all</u> variables have been bound either by quantification or by assignment of a value!

3.11. De Morgan's Laws for Quantifiers.

- $\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$
- $\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$

Discussion

The negation of a quantified statement are obtained from the De Morgan's Laws in Module 2.1.

So the negation of the proposition "Every fish in the sea has gills," is the proposition "there is at least one fish in the sea that does not have gills."

If there is more than one quantifier, then the negation operator should be passed from left to right across one quantifier at a time, using the appropriate De Morgan's Law at each step. Continuing further with Example 3.9.2, suppose we wish to negate the proposition "Every student in this class understands at least one example in these notes." Apply De Morgan's Laws to negate the symbolic form of the proposition:

$$\begin{aligned} \neg(\forall x \exists y R(x,y)) &\Leftrightarrow \exists x(\neg \exists y R(x,y)) \\ &\Leftrightarrow \exists x \forall y \neg R(x,y) \end{aligned}$$

The first proposition could be read "It is not the case that every student in this class understands at least one example in these notes." The goal, however, is to find an expression for the negation in which the verb in each predicate in the scope of the quantifiers is negated, and this is the intent in any exercise, quiz, or test problem that asks you to "negate the proposition ... ." Thus, a correct response to the instruction to negate the proposition "Every student in this class understands at least one example in these notes" is the proposition "There is at least one student in this class that does not understand any of the examples in these notes."

EXERCISE 3.11.1. *Negate the rest of the statements in Example 3.9.2.*

It is easy to see why each of these rules of negation is just another form of De Morgan's Law, if you assume that the universe of discourse is finite: $U = \{x_1, x_2, ..., x_n\}$. For example,

$$\forall x P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

so that

$$\begin{aligned}\neg\forall x P(x) &\Leftrightarrow \neg[P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)]\\ &\Leftrightarrow [\neg P(x_1) \vee \neg P(x_1) \vee \cdots \vee \neg P(x_1)]\\ &\Leftrightarrow \exists x \neg P(x)\end{aligned}$$

If U is an arbitrary universe of discouse, we must argue a little differently: Suppose $\neg\forall x P(x)$ is true. Then $\forall x P(x)$ is false. This is true if and only if there is some c in U such that $P(c)$ is false. This is true if and only if there is some c in U such that $\neg P(c)$ is true. But this is true if and only if $\exists x \neg P(x)$.

The argument for the other equivalence is similar.

EXERCISE 3.11.2. *Suppose $S(x, y)$ is the predicate "x saw y," $L(x, y)$ is the predicate "x liked y," and $C(y)$ is the predicate "y is a comedy." The universe of discourse of x is the set of people and the universe of discourse for y is the set of movies. Write the following in proper English. Do not use variables in your answers.*

(1) $\forall y \neg S(Margaret, y)$
(2) $\exists y \forall x L(x, y)$
(3) $\exists x \forall y [C(y) \rightarrow S(x, y)]$
(4) *Give the negation for part 3 in symbolic form with the negation symbol to the right of all quantifiers.*
(5) *state the negation of part 3 in English without using the phrase "it is not the case."*

EXERCISE 3.11.3. *Suppose the universe of discourse for x is the set of all FSU students, the universe of discourse for y is the set of courses offered at FSU, $A(y)$ is the predicate "y is an advanced course," $F(x)$ is "x is a freshman," $T(x, y)$ is "x is taking y," and $P(x, y)$ is "x passed y." Use quantifiers to express the statements*

(1) *No student is taking every advanced course.*
(2) *Every freshman passed calculus.*
(3) *Some advanced course(s) is(are) being taken by no students.*
(4) *Some freshmen are only taking advanced courses.*
(5) *No freshman has taken and passed linear algebra.*

Here is a formidable example from the calculus. Suppose a and L are fixed real numbers, and f is a real-valued function of the real variable x. Recall the rigorous definition of what it means to say "the limit of $f(x)$ as x tends to a is L":

$$\lim_{x \to a} f(x) = L \Leftrightarrow$$

for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every x,

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Here, the universe of discourse for the variables ϵ, δ, and x is understood to be the set of all real numbers.

What does it mean to say that $\lim_{x \to a} f(x) \neq L$? In order to figure this out, it is useful to convert this proposition into a symbolic proposition. So, let $P(\epsilon, \delta, x)$ be the predicate "$0 < |x - a| < \delta$" and let $Q(\epsilon, \delta, x)$ be the predicate "$|f(x) - L| < \epsilon$." (It is perfectly OK to list a variable in the argument of a predicate even though it doesn't actually appear!) We can simplify the proposition somewhat by restricting the universe of discourse for the variables ϵ and δ to be the set of *positive* real numbers. The definition then becomes

$$\forall \epsilon \exists \delta \forall x [P(\epsilon, \delta, x) \to Q(\epsilon, \delta, x)].$$

Use De Morgan's Law to negate:

$$\neg[\forall \epsilon \exists \delta \forall x [P(\epsilon, \delta, x) \to Q(\epsilon, \delta, x)]] \Leftrightarrow \exists \epsilon \forall \delta \exists x [P(\epsilon, \delta, x) \wedge \neg Q(\epsilon, \delta, x)],$$

and convert back into words:

There exists $\epsilon > 0$ such that, for every $\delta > 0$ there exists x such that, $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.

3.12. Distributing Quantifiers over Operators.

(1) $\forall x [P(x) \wedge Q(x)] \Leftrightarrow \forall x P(x) \wedge \forall x Q(x)$, but
(2) $\forall x [P(x) \vee Q(x)] \not\Leftrightarrow \forall x P(x) \vee \forall x Q(x)$.
(3) $\exists x [P(x) \vee Q(x)] \Leftrightarrow \exists x P(x) \vee \exists x Q(x)$, but
(4) $\exists x [P(x) \wedge Q(x)] \not\Leftrightarrow \exists x P(x) \wedge \exists x Q(x)$.

Discussion

Here we see that in only half of the four basic cases does a quantifier distribute over an operator, in the sense that doing so produces an equivalent proposition.

EXERCISE 3.12.1. *In each of the two cases in which the statements are not equivalent, there is an implication in one direction. Which direction? In order to help you analyze these two cases, consider the predicates* $P(x) = [x \geq 0]$ *and* $Q(x) = [x < 0]$*, where the universe of discourse is the set of all real numbers.*

EXERCISE 3.12.2. *Write using predicates and quantifiers.*

(1) *For every* $m, n \in \mathbb{N}$ *there exists* $p \in \mathbb{N}$ *such that* $m < p$ *and* $p < n$.
(2) *For all nonnegative real numbers a, b, and c, if* $a^2 + b^2 = c^2$, *then* $a + b \geq c$.

(3) *There does not exist a positive real number a such that* $a + \frac{1}{a} < 2$.
(4) *Every student in this class likes mathematics.*
(5) *No student in this class likes mathematics.*
(6) *All students in this class that are CS majors are going to take a 4000 level math course.*

EXERCISE 3.12.3. *Give the negation of each statement in example 3.12.2 using predicates and quantifiers with the negation to the right of all quantifiers.*

EXERCISE 3.12.4. *Give the negation of each statement in example 3.12.2 using an English sentence.*