

2. Properties of Functions

2.1. Injections, Surjections, and Bijections.

DEFINITION 2.1.1. Given $f: A \rightarrow B$

1. f is **one-to-one** (short hand is 1 – 1) or **injective** if preimages are unique. In this case, $(a \neq b) \rightarrow (f(a) \neq f(b))$.
2. f is **onto** or **surjective** if every $y \in B$ has a preimage. In this case, the range of f is equal to the codomain.
3. f is **bijective** if it is surjective and injective (one-to-one and onto).

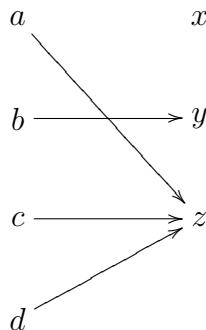
Discussion

We begin by discussing three very important properties functions defined above.

1. A function is *injective* or *one-to-one* if the preimages of elements of the range are unique. In other words, if every element in the range is assigned to exactly one element in the domain. For example, if a function is defined from a subset of the real numbers to the real numbers and is given by a formula $y = f(x)$, then the function is one-to-one if the equation $f(x) = b$ has *at most one solution* for every number b .
2. A function is *surjective* or *onto* if the range is equal to the codomain. In other words, if every element in the codomain is assigned to at least one value in the domain. For example, if, as above, a function is defined from a subset of the real numbers to the real numbers and is given by a formula $y = f(x)$, then the function is onto if the equation $f(x) = b$ has *at least one solution* for every number b .
3. A function is a *bijection* if it is both injective and surjective.

2.2. Examples.

EXAMPLE 2.2.1. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$. The function f is defined by the relation pictured below. This function is neither injective nor surjective.



EXAMPLE 2.2.2. $f: A \rightarrow B$ where $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$ defined by the relation below is a surjection, but not an injection.

$$a \longrightarrow x$$

$$b \longrightarrow y$$

$$c \longrightarrow z$$

$$d \longrightarrow z$$

EXAMPLE 2.2.3. $f: A \rightarrow B$ where $A = \{a, b, c, d\}$ and $B = \{v, w, x, y, z\}$ defined by the relation below is an injection, but not a surjection.

$$a \longrightarrow v$$

$$b \longrightarrow w$$

$$c \longrightarrow y$$

$$d \longrightarrow z$$

EXAMPLE 2.2.4. $f: A \rightarrow B$ where $A = \{a, b, c, d\}$ and $B = \{v, w, x, y\}$ defined by the relation below both a surjection and an injection, and therefore a bijection. Notice that for a function to be a bijection, the domain and codomain must have the same cardinality.

$$a \longrightarrow v$$

$$b \longrightarrow w$$

$$c \longrightarrow x$$

$$d \longrightarrow y$$

Discussion

The examples illustrate functions that are injective, surjective, and bijective. Here are further examples.

EXAMPLE 2.2.5. Let $f: [0, \infty) \rightarrow [0, \infty)$ be defined by $f(x) = \sqrt{x}$. This function is an injection and a surjection and so it is also a bijection.

EXAMPLE 2.2.6. Suppose $f(x) = x^2$. If the domain and codomain for this function is the set of real numbers, then this function would be neither a surjection nor an injection. It is not a surjection because the range is not equal to the codomain. For example, there is no number in the domain with image -1 which is an element of the codomain. It is not an injection since more than one distinct element in the domain is mapped to the same element in the codomain. For example, $f(-1) = f(1)$ but $-1 \neq 1$.

EXERCISE 2.2.1. What if we say the domain of the function in Example 2.2.6 is the set of all reals and the codomain is $[0, \infty)$. Which properties would the function have (injective and/or surjective)? Explain.

EXERCISE 2.2.2. Now, if we say the domain and the codomain are both $[0, \infty)$. What properties does the function in Example 2.2.6 have? Explain.

2.3. Example 2.3.1.

EXAMPLE 2.3.1. Prove that the function $f: \mathbb{N} \rightarrow \mathbb{N}$ be defined by $f(n) = n^2$ is injective.

PROOF. Let $a, b \in \mathbb{N}$ be such that $f(a) = f(b)$. This implies

$$a^2 = b^2 \text{ by the definition of } f.$$

Thus $a = b$ or $a = -b$. Since the domain of f is the set of natural numbers, both a and b must be nonnegative. Thus $a = b$.

This shows $\forall a \forall b [f(a) = f(b) \rightarrow a = b]$, which shows f is injective. \square

Discussion

In Example 2.3.1 we prove a function is injective, or one-to-one. Notice that to prove a function, $f: A \rightarrow B$ is one-to-one we must show the following:

$$(\forall x \in A)(\forall y \in A)[(x \neq y) \rightarrow (f(x) \neq f(y))].$$

This is equivalent to showing

$$(\forall x \in A)(\forall y \in A)[(f(x) = f(y)) \rightarrow (x = y)].$$

To prove this statement (which actually uses the contrapositive of the definition) we begin by choosing two arbitrary elements of the domain and assume the hypothesis

of the implication, i.e. we begin with “Let $x, y \in A$ and assume $f(x) = f(y)$.” We then use the rules of algebra to show that the conclusion must follow.

2.4. Example 2.4.1.

EXAMPLE 2.4.1. *Prove that the function $g: \mathbb{N} \rightarrow \mathbb{N}$, defined by $g(n) = \lfloor n/3 \rfloor$, is surjective.*

PROOF. Let $n \in \mathbb{N}$. Notice that $g(3n) = \lfloor (3n)/3 \rfloor = \lfloor (3n)/3 \rfloor = n$. Since $3n \in \mathbb{N}$, this shows n is in the range of g . Hence g is surjective. \square

Discussion

To prove a function, $f: A \rightarrow B$ is surjective, or onto, we must show $f(A) = B$. In other words, we must show the two sets, $f(A)$ and B , are equal. We already know that $f(A) \subseteq B$ if f is a well-defined function. While most functions encountered in a course using algebraic functions are well-defined, this should not be an automatic assumption in general. With that said, though, we will usually assume the functions given to us are well defined, so all that must be shown is that $B \subseteq f(A)$. To do this we may use the definition of a subset: show every element of B is also an element of $f(A)$. Thus we begin the proof by fixing an arbitrary element of B . We then use the tools at our disposal (definition of the function, algebra, any other known information) to show that this arbitrary element must in fact be the image of some element of A .

2.5. Example 2.5.1.

EXAMPLE 2.5.1. *Prove that the function $g: \mathbb{N} \rightarrow \mathbb{N}$, defined by $g(n) = \lfloor n/3 \rfloor$, is not injective.*

PROOF. The numbers 1 and 2 are in the domain of g and are not equal, but $g(1) = g(2) = 0$. Thus g is not injective. \square

Discussion

To show a function is *not* injective we must show

$$\neg[(\forall x \in A)(\forall y \in A)[(x \neq y) \rightarrow (f(x) \neq f(y))]].$$

This is equivalent to

$$(\exists x \in A)(\exists y \in A)[(x \neq y) \wedge (f(x) = f(y))].$$

Thus when we show a function is not injective it is enough to find an example of two different elements in the domain that have the same image.

2.6. Example 2.6.1.

EXAMPLE 2.6.1. *Prove that the function $f: \mathbb{N} \rightarrow \mathbb{N}$ be defined by $f(n) = n^2$, is not surjective.*

PROOF. The number 3 is an element of the codomain, \mathbb{N} . However, 3 is not the square of any integer. Therefore, there is no element of the domain that maps to the number 3, so f is not surjective. \square

Discussion

To show a function is not surjective we must show $f(A) \neq B$. Since a well-defined function must have $f(A) \subseteq B$, we should show $B \not\subseteq f(A)$. Thus to show a function is not surjective it is enough to find an element in the codomain that is not the image of any element of the domain. You may assume the familiar properties of numbers in this module as done in the previous examples.

2.7. Inverse Functions.

DEFINITION 2.7.1. *Suppose $f: A \rightarrow B$ is a bijection. Then the **inverse** of f , denoted*

$$f^{-1}: B \rightarrow A,$$

is the function defined by the rule

$$f^{-1}(y) = x \text{ if and only if } f(x) = y.$$

Discussion

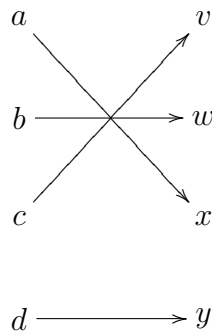
If a function f is a bijection, then it makes sense to define a new function that reverses the roles of the domain and the codomain, but uses the same rule that defines f . This function is called the inverse of the f . If the function is not a bijection, it does *not* have an inverse.

You have seen many functions in algebra and calculus that are defined as inverses of other functions. For example, the square-root function \sqrt{x} is defined by the rule $y = \sqrt{x}$ if and only if $y \geq 0$ and $y^2 = x$. That is, the square-root function is the inverse of the square function. Before we can invert the square function, however, its usual domain, the set of all real numbers, must be restricted to the set of nonnegative real numbers in order to have a bijection. That is, if $A = B = \{x \in \mathbb{R} : x \geq 0\}$,

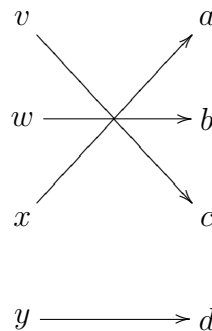
then the function $f: A \rightarrow B$ defined by $f(x) = x^2$ is a bijection, and its inverse $f^{-1}: B \rightarrow A$ is the square-root function, $f^{-1}(x) = \sqrt{x}$.

Another important example from algebra is the logarithm function. If a is a positive real number, different from 1, and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, the function $f: \mathbb{R} \rightarrow \mathbb{R}^+$ defined by $f(x) = a^x$ is a bijection. Its inverse, $f^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R}$, is the logarithm function with base a : $f^{-1}(x) = \log_a x$. In other words $y = \log_a x$ if and only if $a^y = x$.

EXAMPLE 2.7.1. Let $f: A \rightarrow B$, where $A = \{a, b, c, d\}$ and $B = \{v, w, x, y\}$, be defined as follows



Then the inverse function is $f^{-1}: B \rightarrow A$ defined as follows



EXAMPLE 2.7.2. Suppose $f: \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{1\}$ is defined by $f(x) = \frac{x}{x-2}$. Then the function $f^{-1}(x) = \frac{2x}{x-1}$ is the inverse of f . This is a good example of an algebraically defined function whose inverse has a nice formula specifying its rule. You may recall that the formula defining f^{-1} can be obtained by setting $y = f(x) = \frac{x}{x-2}$, interchanging x and y , and solving algebraically for y .

EXERCISE 2.7.1. Find the inverse of the function $f: \mathbb{R} - \{-2\} \rightarrow \mathbb{R} - \{1\}$ defined by $f(x) = \frac{x-1}{x+2}$.

EXERCISE 2.7.2. Find the inverse of the function $f: \mathbb{R} \rightarrow (-\infty, 1)$ defined by $f(x) = 1 - e^{-x}$.

THEOREM 2.7.1. If a function is a bijection, then its inverse is also a bijection.

PROOF. Let $f: A \rightarrow B$ be a bijection and let $f^{-1}: B \rightarrow A$ be its inverse. To show f^{-1} is a bijection we must show it is an injection and a surjection.

Let $x_1, x_2 \in B$ be such that $f^{-1}(x_1) = f^{-1}(x_2)$. Then by the definition of the inverse we have $x_1 = f(f^{-1}(x_1)) = f(f^{-1}(x_2)) = x_2$. This shows f^{-1} is injective.

We leave the proof that f^{-1} is surjective as an exercise for the reader. \square

EXERCISE 2.7.3. Finish the proof of Theorem 2.7.1.

2.8. Inverse Image.

DEFINITION 2.8.1. Let $f: A \rightarrow B$ and let S be a subset of B . Then the **inverse image** of S under f is the set

$$f^{-1}(S) = \{x \in A \mid f(x) \in S\}.$$

There is no requirement for f to be injective or surjective for this definition to hold.

EXAMPLE 2.8.1. Let $f: A \rightarrow B$, where $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$ be defined as follows

$$a \longrightarrow x$$

$$b \longrightarrow y$$

$$c \longrightarrow z$$

$$d \longrightarrow z$$

- $f^{-1}(\{z\}) = \{c, d\}$
- $f^{-1}(\{x, y\}) = \{a, b\}$

Discussion

We revisit the definition of the inverse image of a set to emphasize the difference between the *inverse image of a subset of the codomain* and the *inverse of a function*. The inverse image of a set is the set of all elements that map into that set. The function does not have to be injective or surjective to find the inverse image of a set. For example, the function $f(n) = 1$ with domain and codomain all natural numbers

would have the following inverse images: $f^{-1}(\{1\}) = \mathbb{N}$ and $f^{-1}(\{5, 6, 7, 8, 9\}) = \emptyset$. This function does not have an inverse, however.

The context of the use of the notation f^{-1} usually indicates if the inverse image or the inverse function is intended. If A is a subset of the codomain we would always assume $f^{-1}(A)$ is the inverse image of A . When discussing a bijection the distinction between the inverse image and inverse function is often blurred.

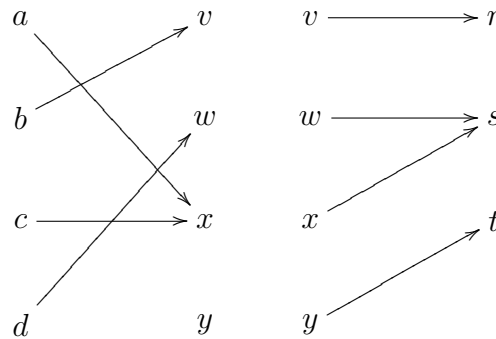
EXERCISE 2.8.1. Find an example of a function $f : A \rightarrow B$ and a set $S \subseteq A$ where $f(\overline{S}) \neq \overline{f(S)}$.

EXERCISE 2.8.2. Let $f : A \rightarrow B$ be a function and let $S \subseteq B$. Prove $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$

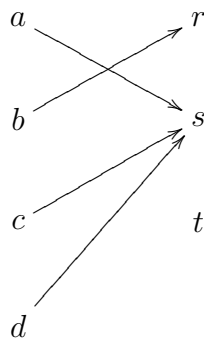
2.9. Composition.

DEFINITION 2.9.1. Let $f : A \rightarrow B$ and $g : B \rightarrow C$. The **composition** of g with f , denoted $g \circ f$, is the function from A to C defined by $(g \circ f)(x) = g(f(x))$.

EXAMPLE 2.9.1. Let $A = \{a, b, c, d\}$, $B = \{v, w, x, y\}$, and $C = \{r, s, t\}$; and let $f : A \rightarrow B$ and $g : B \rightarrow C$ by defined as follows



Then the composition $g \circ f$ is as follows



Discussion

The composition of two functions is defined by following one function by another. To define the composition $g \circ f$ we must have the range of f contained in the domain of g .

2.10. Example 2.10.1.

EXAMPLE 2.10.1. *Prove that the composition of two injective functions is injective.*

PROOF. Let A, B , and C be sets and let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two injections.

Suppose x and y are elements of A such that $(g \circ f)(x) = (g \circ f)(y)$. This means $g(f(x)) = g(f(y))$. Since the codomain of f is B , $f(x) \in B$ and $f(y) \in B$. Thus we have two elements of B , $f(x)$ and $f(y)$, such that $g(f(x)) = g(f(y))$. Since g is injective, we must have $f(x) = f(y)$. But now we may use the fact that f is injective to conclude $x = y$.

This shows that $g \circ f$ is an injection. □

Discussion

Sometimes we have to prove properties about functions without any specific formula for the functions. In Example 2.10.1 we prove that the composition of two injections is again an injection. We cannot use specific examples of function because that would not prove this more general statement. We have to use the tools given by the assumptions, namely, that we know the two functions that make up the composition are known to be injections.

EXERCISE 2.10.1. *Prove the composition of two surjections is a surjection.*

THEOREM 2.10.1 (Corollary to Example 2.10.1 and Exercise 2.10.1). *The composition of two bijections is a bijection.*

EXERCISE 2.10.2. *Prove or disprove: if the composition of two functions is an injection then the two original functions must be injections too.*

EXERCISE 2.10.3. *Prove or disprove: if the composition of two functions is a surjection then the two original functions must be surjections too.*