4. Growth of Functions

4.1. Growth of Functions. Given functions f and g, we wish to show how to quantify the statement:

"g grows as fast as f".

The growth of functions is directly related to the complexity of algorithms. We are guided by the following principles.

- We only care about the behavior for "large" problems.
- We may ignore implementation details such as loop counter incrementation.

Discussion

When studying the complexity of an algorithm, we are concerned with the growth in the number of operations required by the algorithm as the size of the problem increases. In order to get a handle on its complexity, we first look for a function that gives the number of operations in terms of the size of the problem, usually measured by a positive integer n, to which the algorithm is applied. We then try to compare values of this function, for large n, to the values of some known function, such as a power function, exponential function, or logarithm function. Thus, the *growth of functions* refers to the relative size of the values of two functions for large values of the independent variable. This is one of the main areas in this course in which experience with the concept of a limit from calculus will be of great help.

Before we begin, one comment concerning notation for logarithm functions is in order. Most algebra and calculus texts use $\log x$ to denote $\log_{10} x$ (or, perhaps, $\log_e x$), but in computer science base 2 is used more prevalently. So we shall use $\log x$ to denote $\log_2 x$. As we shall see, in the context of this module it actually doesn't matter which base you use, since $\log_a x = \frac{\log_b x}{\log_b a}$ for any acceptable bases a and b.

EXERCISE 4.1.1. *Prove that* $\log_a x = \frac{\log_b x}{\log_b a}$ *for arbitrary positive real numbers* a *and* b *different from 1.*

4.2. The Big-O Notation.

DEFINITION 4.2.1. *Let* f *and* g *be functions from the natural numbers to the real numbers. Then* g **asymptotically dominates** f*, or*

f **is big-O of** g

if there are positive constants C *and* k *such that*

$$|f(x)| \leq C|g(x)| \quad \text{for } x \geq k.$$

If f is big-O of g, then we write

$$f(x) \text{ is } O(g(x))$$
$$\text{or}$$
$$f \in O(g).$$

THEOREM 4.2.1. *If* $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = L$, *where* $L \geq 0$, *then* $f \in O(g)$.

THEOREM 4.2.2. *If* $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \infty$, *then* f *is* **not** $O(g)$ $(f \notin O(g))$.

Discussion

The most basic concept concerning the growth of functions is *big-O*. The statement that f is big-O of g expresses the fact that for large enough x, f will be bounded above by some constant multiple of g. Theorem 4.2.1 gives a necessary condition for f to be big-O of g in terms of limits. The two notions aren't equivalent since there are examples where the definition holds, but the limit fails to exist. For the functions we will be dealing with, however, this will not happen.

When working the problems in the module you may find it helpful to use a graphing calculator or other graphing tool to graph the functions involved. For example, if you graph the functions $x^2 + 10$ and $3x^2$, then you will see that $x^2 + 10 \leq 3x^2$ when $x \geq 3$. (Actually, when $x \geq \sqrt{5}$.) This seems to imply that $f(x) = x^2 + 10$ is big-O of $g(x) = x^2$. This is NOT a proof, but it can give you some ideas as to what to look for. In particular, you wouldn't try to show that $f(x) \leq 3g(x)$ for $x \geq 2$. It isn't necessary that you find the best bound, k, for x, however, as long as you find one that works. Also, there is nothing unique about the choice of C.

EXAMPLE 4.2.1. *Show that* $x^2 + 10$ *is* $O(x^2)$.

Proof 1 (using Definition of Big-O). Let $C = 3$ and $k = 3$. Then, if $x \geq 3$,

$3x^2 = x^2 + 2x^2 \geq x^2 + 2 \cdot 3^2 \geq x^2 + 10.$ □

Proof 2 (using Definition of Big-O). Let $C = 2$ and $k = 4$. Then, if $x \geq 4$,

$2x^2 = x^2 + x^2 \geq x^2 + 4^2 \geq x^2 + 10.$ □

Proof 3 (using Theorem 4.2.1). $\lim_{x\to\infty} \frac{x^2 + 10}{x^2} = \lim_{x\to\infty} \left(1 + \frac{10}{x^2}\right) = 1 + 0 = 1.$
So, by Theorem 1, $x^2 + 10 \in O(x^2)$. □

EXERCISE 4.2.1. *Let $a, b \in \mathbb{R}^+ - \{1\}$. Prove $\log_a x$ is $O(\log_b x)$. Hint: recall exercise 4.1.1.*

4.3. Proofs of Theorems 4.2.1 and 4.2.2.

Proof of Theorem 4.2.1. Suppose $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = L$, where L is a nonnegative real number. Then, by the definition of limit, we can make $\frac{|f(x)|}{|g(x)|}$ as close to L as we wish by choosing x large enough. In particular, we can ensure that $\frac{|f(x)|}{|g(x)|}$ is within a distance 1 of L by choosing $x \geq k$ for some positive number k. That is, there is a number $k \geq 0$ such that if $x \geq k$, then

$$\left| \frac{|f(x)|}{|g(x)|} - L \right| \leq 1.$$

In particular,

$$\frac{|f(x)|}{|g(x)|} - L \leq 1$$

$$\frac{|f(x)|}{|g(x)|} \leq L + 1$$

$$|f(x)| \leq (L+1)|g(x)|$$

So, we can choose $C = L + 1$. Thus $f \in O(g)$. □

Proof of Theorem 4.2.2. Suppose $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \infty$. This means that for every positive number C, there is a positive number N such that

$$\frac{|f(x)|}{|g(x)|} > C$$

if $x \geq N$. Thus, for all positive numbers C and k there is an $x \geq k$ (take x greater than the larger of k and N) such that

$$\frac{|f(x)|}{|g(x)|} > C$$

or

$$|f(x)| > C|g(x)|.$$

Thus $f \notin O(g)$. □

Discussion

How do you interpret the statement $f \notin O(g)$? That is, how do you negate the definition? Let's apply principles of logic from Module 2.3. The definition says:

> $f \in O(g)$ if and only if there exist constants C and k such that, for all x, if $x \geq k$, then $|f(x)| \leq C|g(x)|$.

The negation would then read:

> $f \notin O(g)$ if and only if for all constants C and k, there exist x such that $x \geq k$ and $|f(x)| > C|g(x)|$.

EXAMPLE 4.3.1. *Show that x^2 is not $O(x)$.*

Proof 1 (using the Definition of big-O). As we have just seen, the definition requires us to show that no matter how we choose positive constants C and k, there will be a number $x \geq k$ such that $x^2 > Cx$. So, suppose C and k are arbitrary positive constants. Choose x so that $x \geq k$ and $x > C$. Then $x^2 = x \cdot x > C \cdot x$. (We don't have to use the absolute value symbol, since $x > 0$.) □

Proof 2 (using Theorem 4.2.2). $\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty$. So, by Theorem 4.2.2, $x^2 \notin O(x)$. □

While it is true that most of the functions f and g that measure complexity have domain $\mathbb{N}$, they are often defined on the set of all positive real numbers, and, as we see, this is where the calculus can come in handy.

4.4. Example 4.4.1.

EXAMPLE 4.4.1. *Show that $2x^3 + x^2 - 3x + 2$ is $O(x^3)$.*

Proof 1 (using the Definition of big-O). By the triangle inequality,

$$\begin{aligned} |2x^3 + x^2 - 3x + 2| &\leq |2x^3| + |x^2| + |3x| + 2 \\ &= 2|x^3| + |x^2| + 3|x| + 2. \end{aligned}$$

Now, if $x \geq 2$, then $x^2 \leq x^3$, $x \leq x^3$, and $2 \leq x^3$.

Thus

$$|2x^3| + |x^2| + |3x| + 2 \leq 2|x^3| + |x^3| + 3|x^3| + |x^3| = 7|x^3|$$

Using these inequalities, $C = 7$, and $k = 2$, we see that f is $O(x^3)$. □

Proof 2 (using Theorem 4.2.2).

$$\lim_{x\to\infty} \frac{2x^3 + x^2 - 3x + 2}{x^3}$$

$$= \lim_{x\to\infty} \frac{2 + 1/x - 3/x^2 + 2/x^3}{1} = \frac{2}{1}$$

By Theorem 4.2.1, $2x^3 + x^2 - 3x + 2$ is $O(x^3)$.

□

Discussion

In the first proof in Example 4.4.1 we used *the triangle inequality*, which is proved in the Appendix at the end of this module. We also need to use the fact $|ab| = |a||b|$.

Notice the strategy employed here. We did not try to decide what C and k were until after using the triangle inequality. The first constant we dealt with was k. After separating the function into the sum of absolute values we thought about what part of this function would be the biggest for large values of x and then thought about how large x needed to be in order for all the terms to be bounded by that largest term. This led to the choice of k. In general, the constant C depends on the choice of k and the two functions you are working with.

Exercise 4.4.1. *Use the definition to show that* $5x^3 - 3x^2 + 2x - 8 \in O(x^3)$.

Exercise 4.4.2. *Use Theorem 4.2.1 to show that* $10x^3 - 7x^2 + 5 \in O(x^3)$

Exercise 4.4.3. *Use Theorem 4.2.2 to show that* $x^5 \notin O(100x^4)$.

4.5. Calculus Definition.

Definition 4.5.1. *If f and g are such that*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

then we say f is **little-o** *of g, written*

$$f \in o(g).$$

As a corollary to Theorem 4.2.1, we have

Theorem 4.5.1. *If f is $o(g)$, then f is $O(g)$.*

Discussion

As Theorem 4.5.1 indicates, the *little-o* relation is stronger than big-O. Two of the most important examples of this relation are

(1) $\log_a x \in o(x)$, where a is a positive number different from 1, and
(2) $x^n \in o(a^x)$ if $a > 1$.

These are most easily seen using a version of l'Hôpital's rule from calculus:

l'Hôpital's Rule. If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, and if

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

(f' and g' denote the derivatives of f and g, respectively.)

Example 4.5.1. *Show that* $\log_a x \in o(x)$, *where* a *is a positive number different from 1.*

Proof. First observe that $\lim_{x\to\infty} \log_a x = \lim_{x\to\infty} x = \infty$. Recall that $\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$, where $\ln x = \log_e x$. By l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\log_a x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x\ln a}}{1} = 0.$$

□

Exercise 4.5.1. *Show that* $(\log_a x)^2 \in o(x)$.

Example 4.5.2. *Show that, if* $a > 1$, *then* $x \in o(a^x)$.

Proof. First observe that $\lim_{x\to\infty} x = \lim_{x\to\infty} a^x = \infty$. By l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x}{a^x} = \lim_{x\to\infty} \frac{1}{a^x \ln a} = 0,$$

since $a > 1$. □

Exercise 4.5.2. *Show that, if* $a > 1$, *then* $x^2 \in o(a^x)$.

EXERCISE 4.5.3. *Use mathematical induction to show that, if $a > 1$, then $x^n \in o(a^x)$ for every positive integer n.*

4.6. Basic Properties of Big-O. The following theorems and facts will be helpful in determining big-O.

THEOREM 4.6.1. *A polynomial of degree n is $O(x^n)$.*

Fact: Theorem 4.6.1. can be extended to functions with non-integral exponents (like $x^{1/2}$).

THEOREM 4.6.2. *If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then (f_1+f_2) is $O(max\{|g_1|, |g_2|\})$.*

COROLLARY 4.6.2.1. *If f_1 and f_2 are both $O(g)$, then $(f_1 + f_2)$ is $O(g)$.*

THEOREM 4.6.3. *If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $(f_1 f_2)$ is $O(g_1 g_2)$.*

THEOREM 4.6.4. *If f_1 is $O(f_2)$ and f_2 is $O(f_3)$, then f_1 is $O(f_3)$.*

THEOREM 4.6.5. *If f is $O(g)$, then (af) is $O(g)$ for any constant a.*

Discussion

Use these theorems when working the homework problems for this module.

EXAMPLE 4.6.1. *Find the least integer n such that $(x^4+5\log x)/(x^3+1)$ is $O(x^n)$*

Solution: First we consider $\frac{x^4}{x^3+1}$. If you think back to calculus and consider which part of this function "takes over" when x gets large, that provides the clue that this function should be $O(x)$. To see this, we take the following limit;

$$\lim_{x\to\infty} \frac{(x^4)/(x^3+1)}{x} = \lim_{x\to\infty} \frac{x^3}{x^3+1} = 1.$$

Since that limit is 1, we have verified $\frac{x^4}{x^3+1}$ is $O(x)$. Theorem 4.2.2 can be used to show that $\frac{x^4}{x^3+1}$ is not $O(x^0) = O(1)$:

$$\lim_{x\to\infty} \frac{(x^4)/(x^3+1)}{1} = \lim_{x\to\infty} \frac{x}{1+1/x^3} = \infty.$$

Now consider $\frac{5\log x}{x^3+1}$. Since $\log x$ is $O(x)$, $\frac{5\log x}{x^3+1}$ is $O(\frac{5x}{x^3+1})$, and, by taking a limit as above, $\frac{5x}{x^3+1}$ is $o(x)$, hence, $O(x)$.

Since the original function is the sum of the two functions, each of which is $O(x)$, the sum $(x^4 + 5\log x)/(x^3 + 1)$ is $O(x)$, by Corollary 4.6.2.1.

4.7. Proof of Theorem 4.6.3.

PROOF OF THEOREM 4.6.3. Suppose f_1, f_2, g_1, g_2 are all functions with domain and codomain $\mathbb{R}$ such that f_1 is $O(g_1)$ and f_2 is $O(g_2)$.

Then by definition of big-O, there are positive constants C_1, k_1, C_2, k_2 such that

$$\forall x \geq k_1[|f_1(x)| \leq C_1|g_1(x)|] \text{ and } \forall x \geq k_2[|f_2(x)| \leq C_2|g_2(x)|].$$

Let $k = max\{k_1, k_2\}$ and $C = C_1C_2$. Then if $x \geq k$ we have

$$\begin{aligned} |(f_1f_2)(x)| &= |f_1(x)| \cdot |f_2(x)| \\ &\leq C_1|g_1(x) \cdot C_2|g_2(x)| \\ &= C_1C_2|(g_1g_2)(x)| \\ &= C|(g_1g_2)(x)| \end{aligned}$$

This shows f_1f_2 is $O(g_1g_2)$. □

4.8. Example 4.8.1.

EXAMPLE 4.8.1. *Suppose there are two computer algorithms such that*

- *Algorithm 1 has complexity* $n^2 - n + 1$, *and*
- *Algorithm 2 has complexity* $n^2/2 + 3n + 2$.

Then both are $O(n^2)$, *but to indicate Algorithm 2 has a smaller leading coefficient, and hence would be faster, we write*

- *Algorithm 1 has complexity* $n^2 + O(n)$, *and*
- *Algorithm 2 has complexity* $n^2/2 + O(n)$.

Discussion

Example 4.8.1 illustrates the way in which the big-O notation may be used to discuss complexity of algorithms.

4.9. Big-Omega.

DEFINITION 4.9.1. *f is* **big-Omega** *of g, written $f \in \Omega(g)$, if there are positive constants C and k such that*

$$|f(x)| \geq C|g(x)| \qquad \text{for } x > k.$$

Big-Omega is very similar to big-O. Big-Ω notation is used to indicate a lower bound on functions for large values of the independent variable. Notice that f is $\Omega(g)$ if and only if g is $O(f)$. Using this fact we see the properties for big-O give similar properties for big-Ω.

EXAMPLE 4.9.1. *x is $\Omega(\log x)$.*

EXAMPLE 4.9.2. *$2x^3 + x^2 - 3x + 2$ is $\Omega(x^3)$.*

PROOF USING THE DEFINITION OF BIG-Ω: Let $x \geq 3$. Then $x^2 - 3x \geq 0$ and so $x^2 - 3x + 2 \geq 0$ as well. Thus

$$|2x^3 + x^2 - 3x + 2| = 2x^3 + x^2 - 3x + 2 \geq 2x^3.$$

By choosing $C = 2$ and $k = 3$ in the definition of big-Ω the above work shows $2x^3 + x^2 - 3x + 2$ is $\Omega(x^3)$. ☐

EXERCISE 4.9.1. *Let $a, b \in \mathbb{R}^+ - \{1\}$. Prove $\log_a x$ is $\Omega(\log_b x)$.*

4.10. Big-Theta.

DEFINITION 4.10.1. *f is* **big-Theta** *of g, written $f \in \Theta(g)$, if f is both $O(g)$ and $\Omega(g)$.*

Discussion

The definition given for big-Θ is equivalent to the following:

THEOREM 4.10.1. *f is $\Theta(g)$ if and only if f is $O(g)$ and g is $O(f)$.*

EXERCISE 4.10.1. *Prove Theorem 4.10.1.*

EXAMPLE 4.10.1. *$(2x^2-3)/(3x^4+x^3-2x^2-1)$ is $\Theta(x^{-2})$.*

$$\begin{aligned}
\frac{(2x^2-3)/(3x^4+x^3-2x^2-1)}{x^{-2}} &= \frac{2x^2-3}{3x^4+x^3-2x^2-1}\cdot x^2 \\
&= \frac{2x^4-3x^2}{3x^4+x^3-2x^2-1} \\
&= \frac{2-3/x^2}{3+1/x-2/x^2-1/x^4} \\
\lim_{x\to\infty}\frac{(2x^2-3)/(3x^4+x^3-2x^2-1)}{x^{-2}} &= \frac{2}{3}
\end{aligned}$$

You now will show through the following exercise that any two logarithm functions have the same growth rate; hence, it doesn't matter what (acceptable) base is used.

EXERCISE 4.10.2. *If a and b are positive real numbers different from 1, show that $\log_a x \in \Theta(\log_b x)$.*

4.11. Summary. Suppose f and g are functions such that $\lim_{x\to\infty}\frac{|f(x)|}{|g(x)|} = L$, where $0 \le L \le \infty$.

1. If $L = 0$, then f is $o(g)$ (hence, $O(g)$), and g is $\Omega(f)$ (hence, not $O(f)$).
2. If $L = \infty$, then f is $\Omega(g)$ (hence, not $O(g)$), and g is $o(f)$ (hence, $O(f)$).
3. If $0 < L < \infty$, then f is $\Theta(g)$ (hence, $O(g)$), and g is $\Theta(f)$ (hence, $O(f)$).

4.12. Appendix. Proof of the Triangle Inequality. Recall the triangle inequality: for all real numbers a and b,

$$|a+b| \leq |a| + |b|.$$

PROOF. Recall from Module 1.2 that the absolute value function $f(x) = |x|$ is defined by

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

We first observe that for any real numbers x and y, if $y \geq 0$, then $|x| \leq y$ if and only if $-y \leq x \leq y$. To see this, look at two cases:

Case 1. $x \geq 0$. Then $|x| = x$, and so $|x| \leq y$ if and only if $-y \leq 0 \leq x \leq y$, or $-y \leq x \leq y$.

Case 2. $x < 0$. Then $|x| = -x$, and so $|x| \leq y$ if and only if $-y \leq 0 \leq -x \leq y$. Multiplying through by -1 and reversing the inequalities, we get $y \geq x \geq -y$, or $-y \leq x \leq y$.

We now prove the triangle inequality. For arbitrary real numbers a and b, apply the above to $x = a$ and $y = |a|$, and then to $x = b$ and $y = |b|$, to get inequalities

$$\begin{aligned} -|a| &\leq a \leq |a| \\ -|b| &\leq b \leq |b|. \end{aligned}$$

Then

$$-|a| - |b| \leq a + b \leq |a| + |b|$$

or

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Now apply the assertion above to $x = a + b$ and $y = |a| + |b|$ to get:

$$|a+b| \leq |a| + |b|.$$

□