CHAPTER 6

Introduction to Graph Theory

1. Introduction to Graphs

1.1. Simple Graphs.

DEFINITION 1.1.1. *A* **simple graph** (V, E) *consists of a nonempty set representing vertices,* V*, and a set of unordered pairs of elements of* V *representing edges,* E*. A simple graph has*

- *no arrows,*
- *no loops, and*
- *cannot have multiple edges joining vertices.*

Discussion

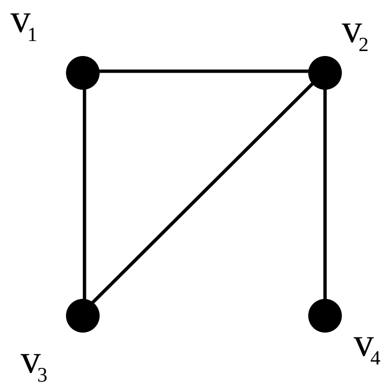
Graphs offer a convenient way to represent various kinds of mathematical objects. Essentially, any graph is made up of two sets, a set of vertices and a set of edges. Depending on the particular situation we are trying to represent, however, we may wish to impose restrictions on the type of edges we allow. For some problems we will want the edges to be *directed* from one vertex to another; whereas, in others the edges are undirected. We begin our discussion with **undirected graphs**.

The most basic graph is the **simple graph** as defined above. Since the edges of a simple graph are undirected, they are represented by **unordered pairs** of vertices rather than **ordered pairs**. For example, if $V = \{a, b, c\}$, then $\{a, b\} = \{b, a\}$ would represent the same edge.

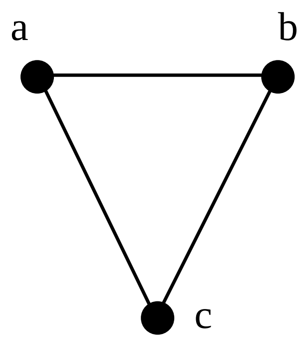
EXERCISE 1.1.1. *If a simple graph* G *has 5 vertices, what is the maximum number of edges that* G *can have?*

1.2. Examples.

EXAMPLE 1.2.1. $V = \{v_1, v_2, v_3, v_4\}$ $E = \{\{v_1v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}\}$



EXAMPLE 1.2.2. $V = \{a, b, c\}$, $E = \{\{a, b\}, \{b, c\}, \{a, c\}\}$



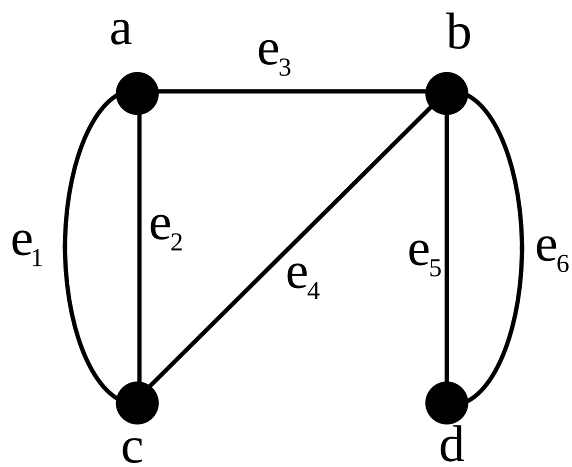
1.3. Multigraphs. Definition: A **multigraph** is a set of vertices, V, a set of edges, E, and a function

$$f : E \to \{\{u, v\} : u, v \in V \text{ and } u \neq v\}.$$

If $e_1, e_2 \in E$ are such that $f(e_1) = f(e_2)$, then we say e_1 and e_2 are **multiple** or **parallel edges**.

EXAMPLE 1.3.1. $V = \{a, b, c, d\}$, $E = \{e_1, e_2, \dots, e_6\}$, $f : E \to \{\{u, v\} : u, v \in V \text{ and } u \neq v\}$ *is defined as follows.*

e	e_1	e_2	e_3	e_4	e_5	e_6
$f(e)$	$\{a, c\}$	$\{a, c\}$	$\{a, b\}$	$\{c, b\}$	$\{b, d\}$	$\{b, d\}$



Discussion

In Example 1.3.1 e_1 and e_2 are parallel edges, but the edges e_2 and e_5 are not called parallel edges.

EXERCISE 1.3.1. *Find all the parallel edges Example 1.3.1.*

Notice that a multigraph allows for multiple edges between a pair of vertices, but does not allow for loops. In some applications it may be desirable to illustrate all the connections between the vertices. Say for example, in a network there may be multiple wires connecting the same units.

1.4. Pseudograph.

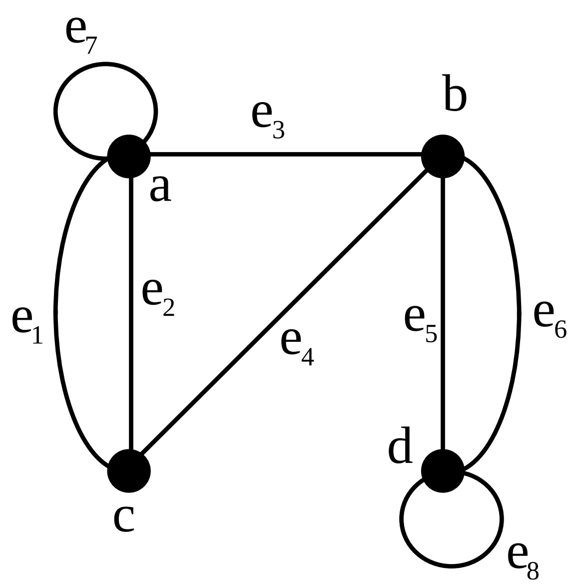
DEFINITION 1.4.1. *A* **pseudograph** *is a set of vertices, V, a set of edges, E, and a function $f : E \to \{\{u, v\} : u, v \in V\}$. If $e \in E$ is such that $f(e) = \{u, u\} = \{u\}$, then we say e is a* **loop**.

EXAMPLE 1.4.1. $V = \{a, b, c, d\}$, $E = \{e_1, e_2, \ldots, e_8\}$,

$$f : E \to \{\{u, v\} : u, v \in V\}$$

is defined as follows.

e	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
$f(e)$	$\{a, c\}$	$\{a, c\}$	$\{a, b\}$	$\{c, b\}$	$\{b, d\}$	$\{b, d\}$	$\{a\}$	$\{d\}$



Discussion

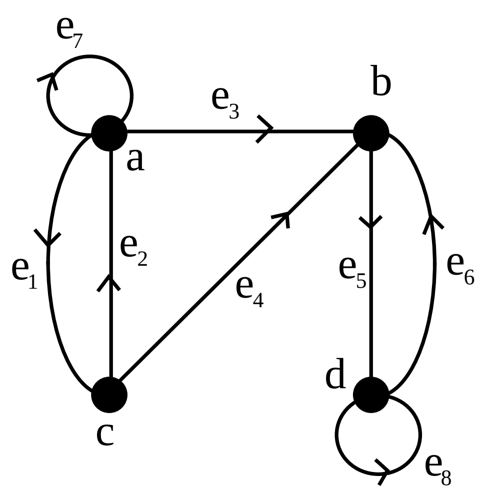
The pseudograph adds the possibility of loops. For example, a diagnostic line may be used in a network, which is a line connecting a computer to itself.

1.5. Directed Graph.

Definition 1.5.1. *A* **directed graph** (V, E) *consists of a set of vertices, V, and a set of directed edges, E. The elements of E are* **ordered pairs** *of vertices.*

Example 1.5.1. $V = \{a, b, c, d\}$,

$E = \{(a, c), (c, a), (a, b), (c, b), (b, d), (d, b), (a, a), (d, d)\}$,



Discussion

A directed graph, or **digraph**, allows loops and allows two edges joining the same vertex, but going in the opposite direction. More than one edge going in the same direction between vertices, however, is not allowed. A directed edge is defined by an **ordered pair** rather than an unordered pair. That is, the ordered pair (a, b) is different from the ordered pair (b, a), while the unordered pair $\{a, b\} = \{b, a\}$. Be careful of the notation you use when writing an edge.

EXERCISE 1.5.1. *If a directed graph G has 5 vertices, what is the maximum number of (directed) edges of G?*

1.6. Directed Multigraph.

DEFINITION 1.6.1. *A* **directed multigraph** (V, E) *consists of vertices, V, and edges, E, and a function*

$$f : E \to V \times V = \{(u, v) | u, v \in V\}.$$

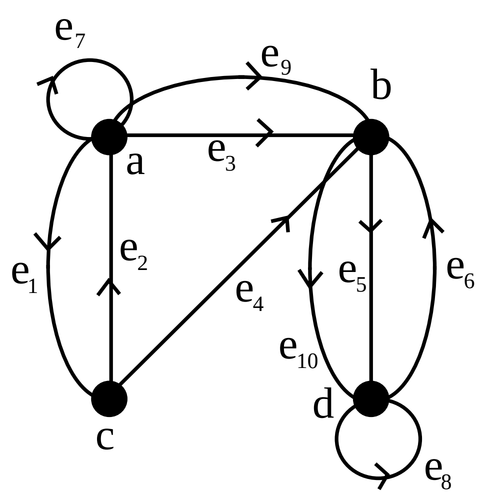
The edges e_1 and e_2 are **multiple edges** *if $f(e_1) = f(e_2)$*

EXAMPLE 1.6.1. $V = \{a, b, c, d\}$, $E = \{e_1, e_2, \ldots, e_{10}\}$,

$$f : E \to \{(u, v) : u, v \in V\}$$

is defined as follows.

e	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
$f(e)$	(a, c)	(c, a)	(a, b)	(c, b)	(b, d)	(d, b)	(a, a)	(d, d)	(a, b)	(b, d)



Discussion

Notice the difference between a directed graph and a directed multigraph: a directed graph allows more than one edge to connect the same two vertices as long as they have opposite directions; whereas, no such restriction is placed on the edges of a directed multigraph.

EXERCISE 1.6.1. *Give all the multiple edges in Example 1.6.1.*

1.7. Graph Isomorphism.

DEFINITION 1.7.1. *Let $G_1 = (V, E)$ and $G_2 = (U, F)$ be simple graphs. The graphs G_1 and G_2 are* **isomorphic** *if there exists a bijection*

$$f\colon V \to U$$

such that for all $v_1, v_2 \in V$, v_1 and v_2 are adjacent in G_1 if and only if $f(v_1)$ and $f(v_2)$ are adjacent in G_2.

DEFINITION 1.7.2. *If f is a bijection as described above, then f is called an* **isomorphism** *between G_1 and G_2, and we often write*

$$f\colon G_1 \to G_2.$$

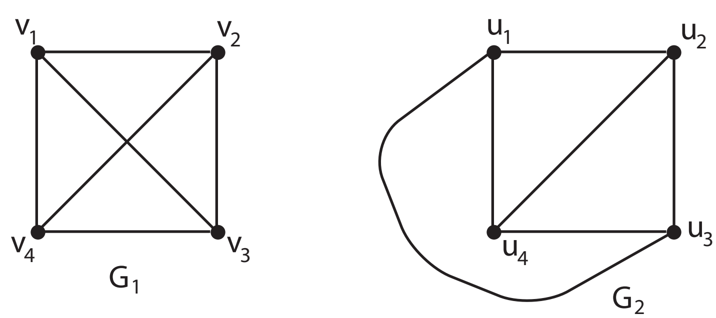
Discussion

There are several notations that are used to represent an isomorphism. We will use a common notation $G_1 \simeq G_2$ to mean that G_1 is isomorphic to G_2.

Trying to construct an isomorphism between graphs can be a very difficult problem in general. If simple graphs G_1 and G_2 are isomorphic, then, clearly, they must have the same number of vertices. As the next exercise shows, G_1 and G_2 must also have the same number of edges. **Having the same number of vertices and edges, however, is in no way sufficient for graphs G_1 and G_2 to be isomorphic.** Often to prove existence of an isomorphism between two graphs one must actually construct the isomorphism.

EXERCISE 1.7.1. *Prove that if simple graphs G_1 and G_2 are isomorphic, then G_1 and G_2 have the same number of edges.*

EXAMPLE 1.7.1. *The graphs G_1 and G_2 below are isomorphic. The bijection is defined by $f(v_i) = u_i$.*



Example 1.7.1 illustrates a situation in which it is very easy to construct an isomorphism. The graph G_2 is merely an alteration of G_1 obtained by moving one of the edges so it goes around rather than crossing over another edge and relabeling its vertices.

One way to visualize when two graphs are isomorphic is to imagine that all the vertices are beads and each edge is represented by a sting with each end tied to the beads that represents its endpoints. If you pick one or more beads up and place it in another location without untying the strings, you obtain a graph that is isomorphic to the original. In fact, if you can move the vertices to different positions keeping the edges attached to go from one graph to another, the two graphs are isomorphic. Edges are allowed to "pass through" each other, so a straight edge and a knotted edge would be considered the same edge.

When two graphs are isomorphic, they share many of the important properties of graphs. In many instances we do not differentiate between two graphs that are isomorphic. Until we study isomorphism in detail, we will not differentiate between two isomorphic graphs. We will discuss graph isomorphisms further in Module 6.3.

Exercise 1.7.2. *Construct a definition for "isomorphism" between*

(a) two multigraphs.
(b) two pseudographs.
(c) two directed graphs.
(d) two directed multigraphs.