3. Representing Graphs and Graph Isomorphism

3.1. Adjacency Matrix.

Definition 3.1.1. *The* **adjacency matrix**, *$A = [a_{ij}]$, for a simple graph $G = (V, E)$, where $V = \{v_1, v_2, ..., v_n\}$, is defined by*

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Discussion

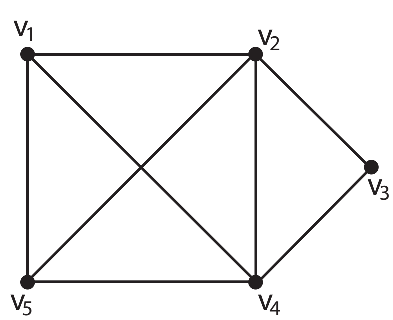
We introduce some alternate representations, which are extensions of connection matrices we have seen before, and learn to use them to help identify isomorphic graphs.

Remarks Here are some properties of the adjacency matrix of an undirected graph.

1. The adjacency matrix is always symmetric.
2. The vertices must be ordered: and the adjacency matrix depends on the order chosen.
3. An adjacency matrix can be defined for multigraphs by defining a_{ij} to be the **number** of edges between vertices i and j.
4. If there is a natural order on the set of vertices we will use that order unless otherwise indicated.

3.2. Example 3.2.1.

Example 3.2.1. *An adjacency matrix for the graph*



is the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Discussion

To find this matrix we may use a table as follows. First we set up a table labeling the rows and columns with the vertices.

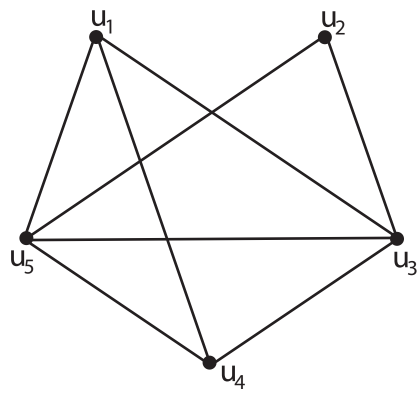
	v_1	v_2	v_3	v_4	v_5
v_1					
v_2					
v_3					
v_4					
v_5					

Since there are edges from v_1 to v_2, v_4, and v_5, but no edge between v_1 and itself or v_3, we fill in the first row and column as follows.

	v_1	v_2	v_3	v_4	v_5
v_1	0	1	0	1	1
v_2	1				
v_3	0				
v_4	1				
v_5	1				

We continue in this manner to fill the table with 0's and 1's. The matrix may then be read straight from the table.

EXAMPLE 3.2.2. *The adjacency matrix for the graph*



is the matrix

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

3.3. Incidence Matrices.

DEFINITION 3.3.1. *The* **incidence matrix**, $A = [a_{ij}]$, *for the undirected graph* $G = (V, E)$ *is defined by*

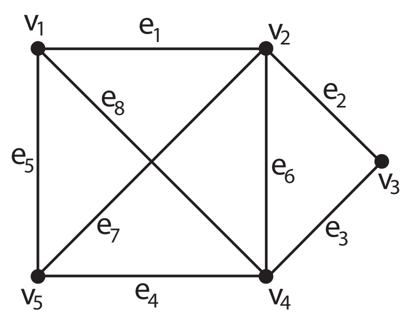
$$a_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is incident with vertex } i \\ 0 & \text{otherwise.} \end{cases}$$

Discussion

Remarks:

(1) This method requires the edges and vertices to be labeled and depends on the order in which they are written.
(2) Every column will have exactly two 1's.
(3) As with adjacency matrices, if there is a natural order for the vertices and edges that order will be used unless otherwise specified.

EXAMPLE 3.3.1. *The incidence matrix for the graph*



is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Again you can use a table to get the matrix. List all the vertices as the labels for the rows and all the edges for the labels of the columns.

3.4. Degree Sequence.

DEFINITION 3.4.1. *The* **degree sequence** *a graph G with n vertices is the sequence $(d_1, d_2, ..., d_n)$, where $d_1, d_2, ..., d_n$ are the degrees of the vertices of G and $d_1 \geq d_2 \geq \cdots \geq d_n$.*

Note that a graph could conceivably have infinitely many vertices. If the vertices are *countable* then the degree sequence would be an infinite sequence. If the vertices are not countable, then this degree sequence would not be defined.

3.5. Graph Invariants.

DEFINITION 3.5.1. *We say a property of graphs is a* **graph invariant** *(or, just invariant) if, whenever a graph G has the property, any graph isomorphic to G also has the property.*

THEOREM 3.5.1. *The following are invariants of a graph G:*

(1) G has r vertices.
(2) G has s edges.
(3) G has degree sequence $(d_1, d_2, ..., d_n)$.
(4) G is a bipartite graph.
(5) G contains r complete graphs K_n (as a subgraphs).
(6) G contains r complete bipartite graphs $K_{m,n}$.
(7) G contains r n-cycles.
(8) G contains r n-wheels.
(9) G contains r n-cubes.

Discussion

Recall from *Module 6.1 Introduction to Graphs* that two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection

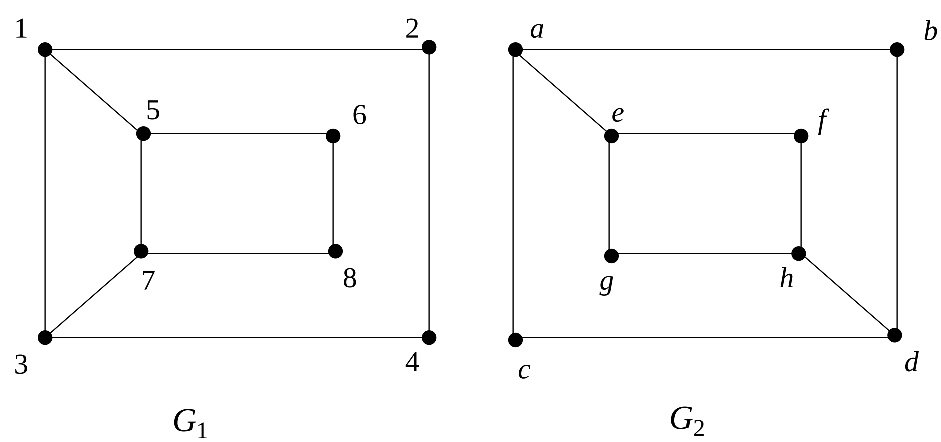
$$f : V_1 \to V_2$$

such that vertices u and v in V_1 are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. If there is such a function, we say f is an **isomorphism** and we write $G_1 \simeq G_2$.

It is often easier to determine when two graphs are *not* isomorphic. This is sometimes made possible by comparing *invariants* of the two graphs to see if they are different. The invariants in Theorem 3.5.1 may help us determine fairly quickly in some examples that two graphs are **not** isomorphic.

3.6. Example 3.6.1.

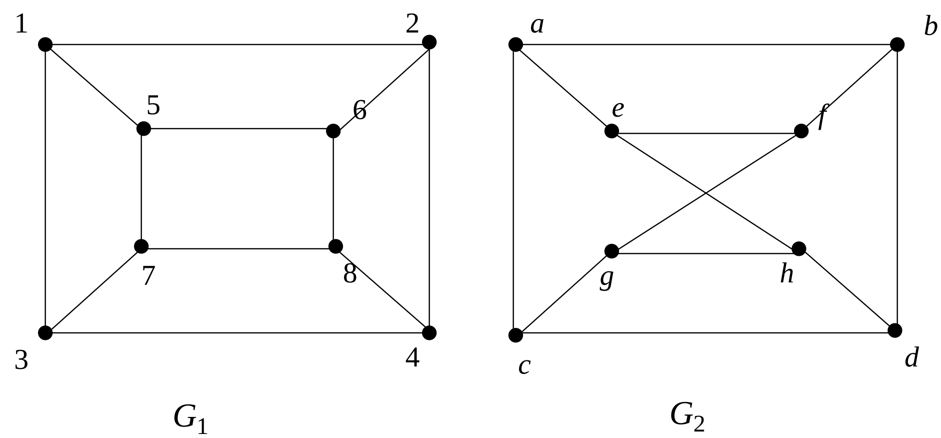
EXAMPLE 3.6.1. *Show that the following two graphs are not isomorphic.*



The two graphs have the same number of vertices, the same number of edges, and same degree sequences $(3, 3, 3, 3, 2, 2, 2, 2)$. Perhaps the easiest way to see that they *are not isomorphic is to observe that G_2 has only two 4-cycles, whereas G_1 has three 4-cycles. In fact, the four vertices of G_1 of degree 3 lie in a 4-cycle in G_1, but the four vertices of G_2 of degree 3 do not. Either of these two discrepancies is enough to show that the graphs are not isomorphic.*

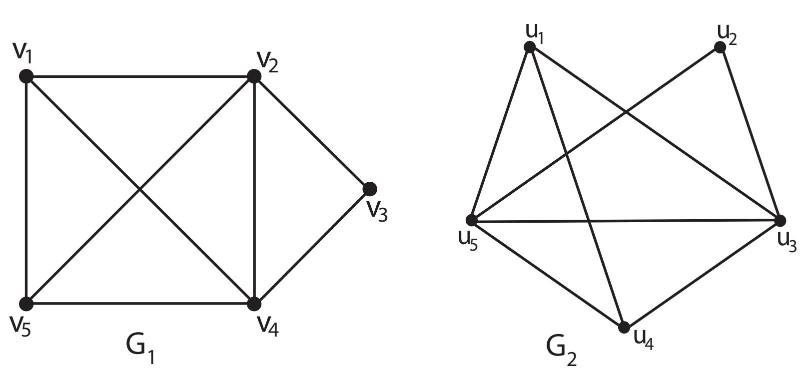
Another way we could recognize the graphs above are not isomorphic is to consider the adjacency relationships. Notice in G_1 all the vertices of degree 3 are adjacent to 2 vertices of degree 3 and 1 of degree 2. However, in graph G_2 all of the vertices of degree 3 are adjacent to 1 vertex of degree 3 and 2 vertices of degree 3. This discrepancy indicates the two graphs cannot be isomorphic.

EXERCISE 3.6.1. *Show that the following two graphs are not isomorphic.*



3.7. Example .

EXAMPLE 3.7.1. *Determine whether the graphs G_1 and G_2 are isomorphic.*



Solution: We go through the following checklist that might tell us immediately if the two are not isomorphic.

- *They have the same number of vertices, 5.*
- *They have the same number of edges, 8.*
- *They have the same degree sequence* $(4, 4, 3, 3, 2)$.

Since there is no obvious reason to think they are not isomorphic, we try to construct an isomorphism, f*. (Note that the above does not tell us there is an isomorphism, only that there might be one.)*

The only vertex on each that have degree 2 are v_3 *and* u_2*, so we must have* $f(v_3) = u_2$.

Now, since $deg(v_1) = deg(v_5) = deg(u_1) = deg(u_4)$*, we must have either*

- $f(v_1) = u_1$ *and* $f(v_5) = u_4$*, or*
- $f(v_1) = u_4$ *and* $f(v_5) = u_1$.

It is possible only one choice would work or both choices may work (or neither choice may work, which would tell us there is no isomorphism).

We try $f(v_1) = u_1$ *and* $f(v_5) = u_4$.

Similarly we have two choices with the remaining vertices and try $f(v_2) = u_3$ *and* $f(v_4) = u_5$*. This defines a bijection from the vertices of* G_1 *to the vertices of* G_2*. We still need to check that adjacent vertices in* G_1 *are mapped to adjacent vertices in* G_2*. To check this we will look at the adjacency matrices.*

The adjacency matrix for G_1 *(when we list the vertices of* G_1 *by* v_1, v_2, v_3, v_4, v_5*) is*

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We create an adjacency matrix for G_2*, using the bijection* f *as follows: since* $f(v_1) = u_1$*,* $f(v_2) = u_3$*,* $f(v_3) = u_2$*,* $f(v_4) = u_5$*, and* $f(v_5) = u_4$*, we rearrange the order of the vertices of* G_2 *to* u_1, u_3, u_2, u_5, u_4*. With this ordering, the adjacency* *matrix for G_2 is*

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

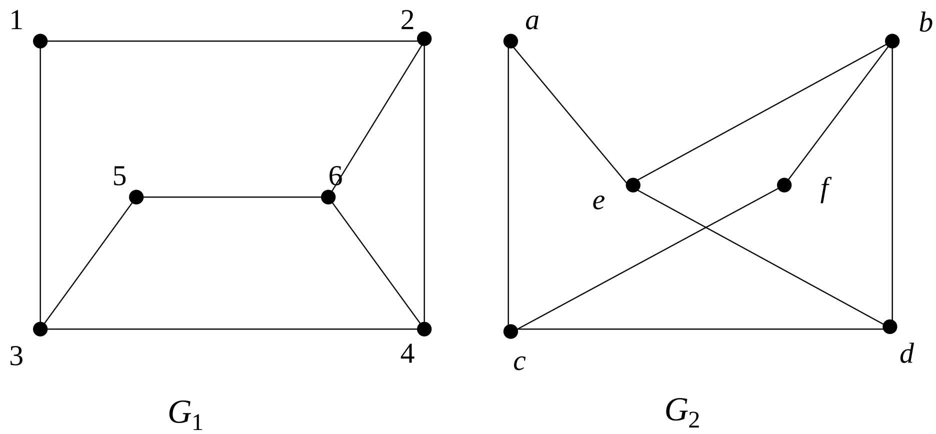
Since $A = B$, adjacency is preserved under this bijection. Hence the graphs are isomorphic.

Discussion

Notice that, trying to establish that the two graphs are isomorphic, it is *not* enough to show that they have the same number of vertices, edges, and degree sequence. In fact, if we knew they were isomorphic and we were asked to prove it, we would proceed to trying to find a bijection that preserves adjacency. That is, the check list is not necessary if you already know they are isomorphic. On the other hand, having found a bijection between two graphs that doesn't preserve adjacency doesn't tell us the graphs are not isomorphic, because some other bijection that would work. If we go down this path, we would have to show that *every* bijection fails to preserve adjacency.

The advantage of the checklist is that it will give you a quick and easy way to show two graphs are *not* isomorphic if some invariant of the graphs turn out to be different. If you examine the logic, however, you will see that if two graphs have all of the same invariants we have listed so far, we still wouldn't have a proof that they are isomorphic. Indeed, there is no known list of invariants that can be efficiently checked to determine when two graphs are isomorphic. The best algorithms known to date for determining graph isomorphism have exponential complexity (in the number n of vertices).

Exercise 3.7.1. *Determine whether the following two graphs are isomorphic.*



Exercise 3.7.2. *How many different isomorphism (that is, bijections that preserve adjacencies) are possible between the graphs in Example 3.7.1?*

Exercise 3.7.3. *There are 14 nonisomorphic pseudographs with 3 vertices and 3 edges. Draw all of them.*

Exercise 3.7.4. *How many nonisomorphic simple graphs with 6 vertices, 5 edges, and no cycles are there. In other words, how many different simple graphs satisfying the criteria that it have 6 vertices, 5 edges, and no cycles can be drawn so that no two of the graphs are isomorphic?*

3.8. Proof of Theorem 3.5.1 Part 3 for Finite Simple Graphs.

Proof. Let G_1 and G_2 be isomorphic finite simple graphs having degree sequences. By part 1 of Theorem 3.5.1 the degree sequences of G_1 and G_2 have the same number of elements. Let $f : V(G_1) \to V(G_2)$ be an isomorphism and let $v \in V(G_1)$. We claim $deg_{G_1}(v) = deg_{G_2}(f(v))$. If we show this, then f defines a bijection between the vertices of G_1 and G_2 that maps vertices to vertices of the same degree. This will imply the degree sequences are the same.

Proof of claim: Suppose $deg_{G_1}(v) = k$. Then there are k vertices in G_1 adjacent to v, say $u_1, u_2, \ldots, u_k$. The isomorphism maps each of the vertices to k distinct vertices adjacent to $f(u)$ in G_2 since the isomorphism is a bijection and preserves adjacency. Thus $deg_{G_2}(f(u)) \geq k$. Suppose $deg_{G_2}(f(u)) > k$. Then there would be a vertex, $w_{k+1} \in V(G_2)$, not equal to any of the vertices $f(u_1), \ldots, f(u_k)$, and adjacent to $f(u)$. Since f is a bijection there is a vertex u_{k+1} in G_1 that is not equal to any of $u_1, \ldots, u_k$ such that $f(u_{k+1}) = w_{k+1}$. Since f preserves adjacency we would have u_{k+1} and v are adjacent. But this contradicts that $deg_{G_1}(v) = k$. Thus $deg_{G_2}(f(u)) = k = deg_{G_1}(u)$ □

Exercise 3.8.1. *Prove the first 2 properties listed in Theorem 3.5.1 for finite simple graphs using only the properties listed before each and the definition of isomorphism.*