CHAPTER 7

Introduction to Relations

1. Relations and Their Properties

1.1. Definition of a Relation. Definition: A **binary relation from a set** A **to a set** B is a subset

$$R \subseteq A \times B.$$

If $(a, b) \in R$ we say a is **related** to b by R.

A is the **domain** of R, and

B is the **codomain** of R.

If $A = B$, R is called a **binary relation on the set** A.

Notation:

- If $(a, b) \in R$, then we write aRb.
- If $(a, b) \notin R$, then we write $a \not R b$.

Discussion

Notice that a relation is simply a subset of $A \times B$. If $(a, b) \in R$, where R is some relation from A to B, we think of a as being assigned to b. In these senses students often associate relations with functions. In fact, a function is a special case of a relation as you will see in Example 1.2.4. Be warned, however, that a relation may differ from a function in two possible ways. If R is an arbitrary relation from A to B, then

- it is possible to have both $(a, b) \in R$ and $(a, b') \in R$, where $b' \neq b$; that is, an element in A could be related to any number of elements of B; or
- it is possible to have some element a in A that is not related to any element in B at all.

Often the relations in our examples do have special properties, but be careful not to assume that a given relation must have any of these properties.

1.2. Examples.

EXAMPLE 1.2.1. *Let* $A = \{a, b, c\}$ *and* $B = \{1, 2, 3, 4\}$, *and let* $R_1 = \{(a, 1), (a, 2), (c, 4)\}$.

EXAMPLE 1.2.2. *Let* $R_2 \subset \mathbb{N} \times \mathbb{N}$ *be defined by* $(m, n) \in R_2$ *if and only if* $m|n$.

EXAMPLE 1.2.3. *Let* A *be the set of all FSU students, and* B *the set of all courses offered at FSU. Define* R_3 *as a relation from* A *to* B *by* $(s, c) \in R_3$ *if and only if* s *is enrolled in* c *this term.*

Discussion

There are many different types of examples of relations. The previous examples give three very different types of examples. Let's look a little more closely at these examples.

Example 1.2.1. This is a completely abstract relation. There is no obvious reason for a to be related to 1 and 2. It just is. This kind of relation, while not having any obvious application, is often useful to demonstrate properties of relations.

Example 1.2.2. This relation is one you will see more frequently. The set R_2 is an infinite set, so it is impossible to list all the elements of R_2, but here are some elements of R_2:

$$(2, 6), (4, 8), (5, 5), (5, 0), (6, 0), (6, 18), (2, 18).$$

Equivalently, we could also write

$$2R_2 6, 4R_2 8, 5R_2 5, 5R_2 0, 6R_2 0, 6R_2 18, 2R_2 18.$$

Here are some elements of $\mathbb{N} \times \mathbb{N}$ that are **not** elements of R_2:

$$(6, 2), (8, 4), (2, 5), (0, 5), (0, 6), (18, 6), (6, 8), (8, 6).$$

Example 1.2.3. Here is an element of R_3: (you, MAD2104).

EXAMPLE 1.2.4. *Let* A *and* B *be sets and let* $f : A \to B$ *be a function. The graph of* f, *defined by* $graph(f) = \{(x, f(x))|x \in A\}$, *is a relation from* A *to* B.

Notice the previous example illustrates that any function has a relation that is associated with it. However, not all relations have functions associated with them.

EXERCISE 1.2.1. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is defined by* $f(x) = \lfloor x/2 \rfloor$.

(1) Find 5 elements of the relation $graph(f)$.

(2) Find 5 elements of $\mathbb{R} \times \mathbb{R}$ *that are not in* $graph(f)$*.*

Exercise 1.2.2. *Find a relation from* $\mathbb{R}$ *to* $\mathbb{R}$ *that cannot be represented as the graph of a functions.*

Exercise 1.2.3. *Let* n *be a positive integer. How many binary relations are there on a set* A *if* $|A| = n$*? [Hint: How many elements are there in* $|A \times A|$*?]*

1.3. Directed Graphs.

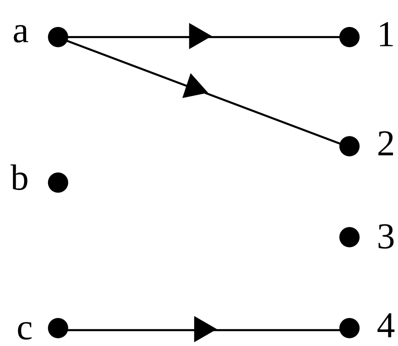
Definitions 1.3.1.

- *A* **directed graph** *or a* **digraph** D *from* A *to* B *is a collection of* **vertices** $V \subseteq A \cup B$ *and a collection of* **edges** $R \subseteq A \times B$*.*
- *If there is an ordered pair* $e = (x, y)$ *in* R *then there is an* **arc** *or* **edge** *from* x *to* y *in* D*.*
- *The elements* x *and* y *are called the* **initial** *and* **terminal** *vertices of the edge* $e = (x, y)$*, respectively.*

Discussion

A digraph can be a useful device for representing a relation, especially if the relation isn't "too large" or complicated.

The digraph that represents R_1 in Example 1.2.1 is:

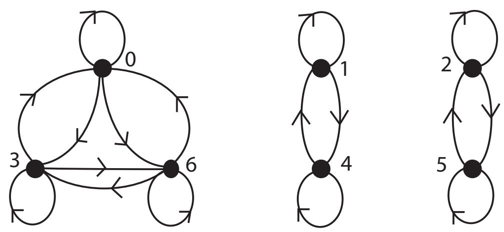


Discussion

If R is a relation on a set A, we simplify the digraph D representing R by having only one vertex for each $a \in A$. This results, however, in the possibility of having **loops**, that is, edges from a vertex to itself, and having more than one edge joining distinct vertices (but with opposite orientations).

A digraph for R_2 in Example 1.2.2 would be difficult to illustrate (and impossible to draw completely), since it would require infinitely many vertices and edges. We could draw a digraph for some finite subset of R_2. It is possible to indicate what the graph of some infinite relations might look like, but this one would be particularly difficult.

Example 1.3.1. *Let R_5 be the relation from $\{0, 1, 2, 3, 4, 5, 6\}$ defined by mR_5n if and only if $m \equiv n (mod\ 3)$. The digraph that represents R_5 is*



1.4. Inverse Relation.

Definition 1.4.1. *Let R be a relation from A to B. Then $R^{-1} = \{(b, a)|(a, b) \in R\}$ is a relation from B to A.*

R^{-1} is called the **inverse of the relation** *R.*

Discussion

The inverse of a relation R is simply the relation obtained by reversing the ordered pairs of R. The inverse relation is also called the **converse relation**.

Example 1.4.1. *Recall Example 1.2.1 $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$ and $R_1 = \{(a, 1), (a, 2), (c, 4)\}$. Then $R^{-1} = \{(1, a), (2, a), (4, c)\}$.*

Exercise 1.4.1. *Recall Example 1.2.4. A and B are sets and $f : A \to B$ is a function. The graph of f, $graph(f) = \{(x, f(x))|x \in A\}$ is a relation from A to B.*

1. *What is the inverse of this relation?*
2. *Does f have to be invertible for the inverse of this relation to exist?*
3. *If f is invertible, find the inverse of the relation $graph(f)$ in terms of the inverse function f^{-1}.*

1.5. Special Properties of Binary Relations.

DEFINITIONS 1.5.1. *Let A be a set, and let R be a binary relation on A.*

(1) *R is* **reflexive** *if*
$\forall x[(x \in A) \rightarrow ((x, x) \in R)]$.
(2) *R is* **irreflexive** *if*
$\forall x[(x \in A) \rightarrow ((x, x) \notin R)]$.
(3) *R is* **symmetric** *if*
$\forall x \forall y[((x, y) \in R) \rightarrow ((y, x) \in R)]$.
(4) *R is* **antisymmetric** *if*
$\forall x \forall y[([(x, y) \in R] \wedge [(y, x) \in R]) \rightarrow (x = y)]$.
(5) *R is* **asymmetric** *if*
$\forall x \forall y[((x, y) \in R) \rightarrow ((y, x) \notin R)]$.
(6) *R is* **transitive** *if*
$\forall x \forall y \forall z[([(x, y) \in R] \wedge [(y, z) \in R]) \rightarrow ((x, z) \in R)]$.

Discussion

Study the definitions of the definitions of the properties given above. You must know these properties, be able to recognize whether or not a relation has a particular property, and be able to prove that a relation has or does not have a particular property. Notice that the definitions of reflexive and irreflexive relations are not complementary. In fact, a relation on a set may be neither reflexive nor irreflexive. The same is true for the symmetric and antisymmetric properties, as well as the symmetric and asymmetric properties. Some texts use the term antireflexive for irreflexive.

EXERCISE 1.5.1. *Before reading further, find a relation on the set $\{a, b, c\}$ that is neither*

(a) reflexive nor irreflexive.
(b) symmetric nor antisymmetric.
(c) symmetric nor asymmetric.

1.6. Examples of Relations and Their Properties.

EXAMPLE 1.6.1. *Suppose A is the set of FSU students and R is the relation given by aRb if students a and b have the same last name. This relation is...*

- *reflexive*
- *not irreflexive*
- *symmetric*
- *not antisymmetric*

- *not asymmetric*
- *transitive*

EXAMPLE 1.6.2. *Suppose T is the relation on the set of integers given by xTy if $2x - y = 1$. This relation is...*

- *not reflexive*
- *not irreflexive*
- *not symmetric*
- *antisymmetric*
- *not asymmetric*
- *not transitive*

EXAMPLE 1.6.3. *Suppose $A = \{a, b, c, d\}$ and R is the relation $\{(a, a)\}$. This relation is...*

- *not reflexive*
- *not irreflexive*
- *symmetric*
- *antisymmetric*
- *not asymmetric*
- *transitive*

Discussion

The examples above illustrate three rather different relations. Some of the relations have many of the properties defined on Section 1.5, whereas one has only one of the property. It is entirely possible to create a relation with none of the properties given in Section 1.5.

EXERCISE 1.6.1. *Give an example of a relation that does not satisfy any property given in Section 1.5.*

1.7. Proving or Disproving Relations have a Property.

EXAMPLE 1.7.1. *Suppose T is the relation on the set of integers given by xTy if $2x - y = 1$. This relation is*

- *not reflexive*

  PROOF. 2 is an integer and $2 \cdot 2 - 2 = 2 \neq 1$. This shows that $\forall x[x \in \mathbb{Z} \to (x, x) \in T]$ is **not true**. □

- *not irreflexive*

  PROOF. 1 is an integer and $2 \cdot 1 - 1 = 1$. This shows that $\forall x[x \in \mathbb{Z} \to (x, x) \notin T]$ is **not true**. □

- *not symmetric*

  Proof. Both 2 and 3 are integers, $2 \cdot 2 - 3 = 1$, and $2 \cdot 3 - 2 = 4 \neq 1$. This shows $2R3$, but $3 \not R 2$; that is, $\forall x \forall y[(x,y) \in \mathbf{Z} \to (y,x) \in T]$ is **not true**. □

- *antisymmetric*

  Proof. Let $m, n \in \mathbb{Z}$ be such that $(m,n) \in T$ and $(n,m) \in T$. By the definition of T, this implies both equations $2m - n = 1$ and $2n - m = 1$ must hold. We may use the first equation to solve for n, $n = 2m - 1$, and substitute this in for n in the second equation to get $2(2m-1) - m = 1$. We may use this equation to solve for m and we find $m = 1$. Now solve for n and we get $n = 1$.

  This shows that the only integers, m and n, such that both equations $2m - n = 1$ and $2n - m = 1$ hold are $m = n = 1$. This shows that $\forall m \forall n[((m,n) \in T \wedge (n,m) \in T) \to m = n]$. □

- *not asymmetric*

  Proof. 1 is an integer such that $(1,1) \in T$. Thus $\forall x \forall y[((x,y) \in T \to (b,a) \notin T]$ is **not true** (counterexample is $a = b = 1$). □

- *not transitive*

  Proof. 2, 3, and 5 are integers such that $(2,3) \in T$, $(3,5) \in T$, but $(2,5) \notin T$. This shows $\forall x \forall y \forall z[(x,y) \in T \wedge (y,z) \in T \to (x,z) \in T]$ is **not true**. □

Example 1.7.2. *Recall Example 1.2.2: $R_2 \subset \mathbb{N} \times \mathbb{N}$ was defined by $(m,n) \in R_2$ if and only if $m|n$.*

- *reflexive*

  Proof. Since $n|n$ for all integers, n, we have nR_2n for every integer. This shows R_2 is reflexive. □

- *not irreflexive*

  Proof. 1 is an integer and clearly $1R_21$. This shows R_2 is not irreflexive. (you could use any natural number to show R_2 is not irreflexive). □

- *not symmetric*

  Proof. 2 and 4 are natural numbers with $2|4$, but $4 \not| 2$, so $2R_24$, but $4 \not R_2 2$. This shows R_2 is not reflexive. □

- *antisymmetric*

  Proof. Let $n, m \in \mathbb{N}$ be such that nR_2m and mR_2n. By the definition of R_2 this implies $n|m$ and $m|n$. Hence we must have $m = n$. This shows R_2 is antisymmetric. □

- *not asymmetric*

  PROOF. Let $m = n$ be any natural number. Then nR_2m and mR_2n, which shows R_2 is not asymmetric. (You may use a particular number to show R_2 is not asymmetric. □

- *transitive*

  PROOF. Let $p, q, r \in \mathbb{N}$ and assume pR_2q and qR_2r. By the definition of R_2 this means $p|q$ and $q|r$. We have proven in *Integers and Division* that this implies $p|r$, thus pR_2r. This shows R_2 is transitive. □

Discussion

When proving a relation, R, on a set A **has** a particular property, the property must be shown to hold for all possible members of the set. For example, if you wish to prove that a given relation, R, on A is reflexive, you must take an arbitrary element x from A and show that xRx. Some properties, such as the symmetric property, are defined using implications. For example, if you are asked to show that a relation, R, on A is symmetric, you would suppose that x and y are arbitrary elements of A such that xRy, and then try to prove that yRx. It is possible that a property defined by an implication holds **vacuously** or **trivially**.

EXERCISE 1.7.1. *Let R be the relation on the set of real numbers given by xRy if and only if $x < y$. Prove R is antisymmetric.*

When proving R does **not** have a property, it is enough to give a counterexample. Recall $\neg[\forall x \forall y P(x, y)] \Leftrightarrow \exists x \exists y \neg P(x, y)$.

EXERCISE 1.7.2. *Prove whether or not each of the properties in Section 1.5 holds for the relation in Example 1.6.1.*

EXERCISE 1.7.3. *Prove whether or not each of the properties in Section 1.5 holds for the relation in Example 1.6.3.*

1.8. Combining Relations. Important Question: Suppose property P is one of the properties listed in Section 1.5, and suppose R and S are relations on a set A, each having property P. Then the following questions naturally arise.

(1) Does $\overline{R}$ (necessarily) have property P?
(2) Does $R \cup S$ have property P?
(3) Does $R \cap S$ have property P?
(4) Does $R - S$ have property P?

1.9. Example of Combining Relations.

EXAMPLE 1.9.1. *Let R_1 and R_2 be transitive relations on a set A. Does it follow that $R_1 \cup R_2$ is transitive?*

Solution: *No. Here is a counterexample:*

$$A = \{1, 2\}, \quad R_1 = \{(1, 2)\}, \quad R_2 = \{(2, 1)\}$$

$$\textit{Therefore,} \qquad R_1 \cup R_2 = \{(1, 2), (2, 1)\}$$

Notice that R_1 and R_2 are both transitive (vacuously, since there are no two elements satisfying the conditions of the property). However $R_1 \cup R_2$ is not transitive. If it were it would have to have $(1, 1)$ and $(2, 2)$ in $R_1 \cup R_2$.

Discussion

Example 1.9.1 gives a counterexample to show that the union of two transitive relations is not necessarily transitive. Note that you could find an example of two transitive relations whose union *is* transitive. However, the question asks if the given property holds for two relations must it hold for the binary operation of the two relations. This is a general question and to give the answer "yes" we must know it is true for *every* possible pair of relations satisfying the property.

Here is another example:

EXAMPLE 1.9.2. *Suppose R and S are transitive relations on the set A. Is $R \cap S$ transitive?*

Solution: *Yes.*

PROOF. Assume R and S are both transitive and let $(a, b), (b, c) \in R \cap S$. Then $(a, b), (b, c) \in R$ and $(a, b), (b, c) \in S$. It is given that both R and S are transitive, so $(a, c) \in R$ and $(a, c) \in S$. Therefore $(a, c) \in R \cap S$. This shows that for arbitrary $(a, b), (b, c) \in R \cap S$ we have $(a, c) \in R \cap S$. Thus $R \cap S$ is transitive. □

1.10. Composition.

DEFINITIONS 1.10.1.

(1) Let
- *R_1 be a relation from A to B, and*
- *R_2 be a relation from B to C.*

Then the **composition** *of* R_1 *with* R_2*, denoted* $R_2 \circ R_1$*, is the relation from* A *to* C *defined by the following property:*

$(x, z) \in R_2 \circ R_1$ *if and only if there is a* $y \in B$ *such that* $(x, y) \in R_1$ *and* $(y, z) \in R_2$.

(2) Let R *be a binary relation on* A*. Then* R^n *is defined recursively as follows:*
Basis: $R^1 = R$
Recurrence: $R^{n+1} = R^n \circ R$, *if* $n \geq 1$.

Discussion

The composition of two relations can be thought of as a generalization of the composition of two functions, as the following exercise shows.

EXERCISE 1.10.1. *Prove: If* $f : A \to B$ *and* $g : B \to C$ *are functions, then* $graph(g \circ f) = graph(g) \circ graph(f)$.

EXERCISE 1.10.2. *Prove the composition of relations is an associative operation.*

EXERCISE 1.10.3. *Let* R *be a relation on* A*. Prove* $R^n \circ R = R \circ R^n$ *using the previous exercise and induction.*

EXERCISE 1.10.4. *Prove an ordered pair* $(x, y) \in R^n$ *if and only if, in the digraph* D *of* R*, there is a directed path of length* n *from* x *to* y.

Notice that if there is no element of B such that $(a, b) \in R_1$ and $(b, c) \in R_2$ for some $a \in A$ and $c \in C$, then the composition is empty.

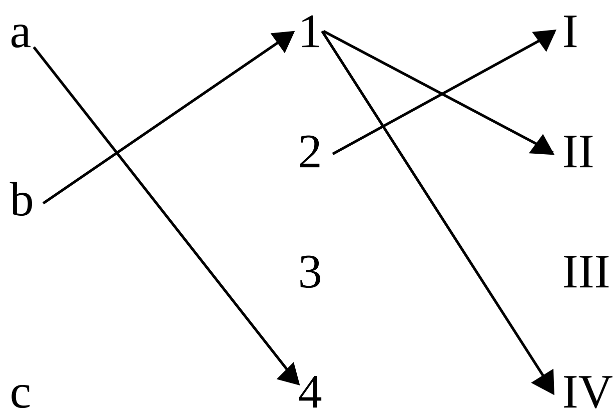
1.11. Example of Composition.

EXAMPLE 1.11.1. *Let* $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, *and* $C = \{I, II, III, IV\}$.

- $R_1 = \{(a, 4), (b, 1)\}$
- $R_2 = \{(1, II), (1, IV), (2, I)\}$
- *Then* $R_2 \circ R_1 = \{(b, II), (b, IV)\}$

Discussion

It can help to consider the following type of diagram when discussing composition of relations, such as the ones in Example 1.11.1 as shown here.



Example 1.11.2. *If R and S are transitive binary relations on A, is $R \circ S$ transitive?*

Solution: No. Here is a counterexample: Let

$$R = \{(1,2),(3,4)\}, \quad \text{and} \quad S = \{(2,3),(4,1)\}.$$

Then both R and S are transitive (vacuously). However,

$$R \circ S = \{(2,4),(4,2)\}$$

is not transitive. (Why?)

Example 1.11.3. *Suppose R is the relation on $\mathbf{Z}$ defined by aRb if and only if $a|b$. Then $R^2 = R$.*

Exercise 1.11.1. *Let R be the relation on the set of real numbers given by xRy if and only if $\frac{x}{y} = 2$.*

(1) Describe the relation R^2.
(2) Describe the relation R^n.

Exercise 1.11.2. *Let P be a property given below and let R and S be relations on A satisfying property P. When does the relation obtained by combining R and S using the operation given satisfy property P?*

(1) P is the reflexive property.
  (a) $R \cup S$
  (b) $R \cap S$
  (c) $R \oplus S$
  (d) $R - S$
  (e) $R \circ S$
  (f) R^{-1}
  (g) R^n
(2) P is the symmetric property.

(a) $R \cup S$
(b) $R \cap S$
(c) $R \oplus S$
(d) $R - S$
(e) $R \circ S$
(f) R^{-1}
(g) R^n

(3) P is the transitive property.

(a) $R \cup S$
(b) $R \cap S$
(c) $R \oplus S$
(d) $R - S$
(e) $R \circ S$
(f) R^{-1}
(g) R^n

1.12. Characterization of Transitive Relations.

THEOREM 1.12.1. *Let R be a binary relation on a set A. R is transitive if and only if $R^n \subseteq R$, for $n \geq 1$.*

PROOF. To prove (R transitive) $\rightarrow$ ($R^n \subseteq R$) we assume R is transitive and prove $R^n \subseteq R$ for $n \geq 1$ by induction.

Basis Step, $n = 1$. $R^1 = R$, so this is obviously true.

Induction Step. Prove $R^n \subseteq R \rightarrow R^{n+1} \subseteq R$.

Assume $R^n \subseteq R$ for some $n \geq 1$. Suppose $(x, y) \in R^{n+1}$. By definition $R^{n+1} = R^n \circ R$, so there must be some $a \in A$ such that $(x, a) \in R$ and $(a, y) \in R^n$. However, by the induction hypothesis $R^n \subseteq R$, so $(a, y) \in R$. R is transitive, though, so $(x, a), (a, y) \in R$ implies $(x, y) \in R$. Since (x, y) was an arbitrary element of R^{n+1}, this shows $R^{n+1} \subseteq R$.

Now we must show the other direction : $R^n \subseteq R$, for $n \geq 1$, implies R is transitive. We prove this directly.

Assume $(x, y), (y, z) \in R$. But by the definition of composition, this implies $(x, z) \in R^2$. But $R^2 \subseteq R$, so $(x, z) \in R$. This shows R is transitive.

$\square$

Discussion

Theorem 1.12.1 gives an important theorem characterizing the transitivity relation. Notice that, since the statement of the theorem was a property that was to be proven for all positive integers, induction was a natural choice for the proof.

EXERCISE 1.12.1. *Prove that a relation R on a set A is transitive if and only if $R^2 \subseteq R$. [Hint: Examine not only the statement, but the proof of Theorem 1.12.1.]*