CHAPTER 1

Relations

1. Relations and Their Properties

1.1. Definition of a Relation.

Definition 1.1.1. *A* **binary relation from a set** A **to a set** B *is a subset*

$$R \subseteq A \times B.$$

If $(a, b) \in R$ *we say* a *is* **Related** *to* b *by* R.

A *is the* **domain** *of* R*, and*

B *is the* **codomain** *of* R.

If $A = B$*,* R *is called a* **binary relation on the set** A.

Notation.

- If $(a, b) \in R$, then we write aRb.
- If $(a, b) \notin R$, then we write $a \not R b$.

Discussion

We recall the basic definitions and terminology associated with the concept of a relation from a set A to a set B. You should review the notes from MAD 2104 whenever you wish to see more examples or more discussion on any of the topics we review here.

A relation is a generalization of the concept of a function, so that much of the terminology will be the same. Specifically, if $f \colon A \to B$ is a function from a set A to a B, then the graph of f, $graph(f) = \{(x, f(x)) | x \in A\}$ is a relation from A to B. Recall, however, that a relation may differ from a function in two essential ways. If R is an arbitrary relation from A to B, then

- it is possible to have both $(a, b) \in R$ and $(a, b') \in R$, where $b' \neq b$; that is, an element a in A may be related to any number of elements of B; or

- it is possible to have some element a in A that is not related to any element in B at all.

EXAMPLE 1.1.1. *Suppose $R \subset \mathbb{Z} \times \mathbb{Z}^+$ is the relation defined by $(a, b) \in R$ if and only if $a|b$. (Recall that $\mathbb{Z}^+$ denotes the set of positive integers, and $a|b$, read "a divides b", means that $b = na$ for some integer n.) Then R fails to be a function on both counts listed above. Certainly, $-2|2$ and $-2|4$, so that -2 is related to more than one positive integer. (In fact, it is related to infinitely many integers.) On the other hand $0 \not| b$ for any positive integer b.*

REMARK 1.1.1. *It is often desirable in computer science to relax the requirement that a function be defined at* every *element in the domain. This has the beneficial effect of reducing the number of sets that must be named and stored. In order not to cause too much confusion with the usual mathematical definition of a function, a relation such as this is called a* **partial function**. *That is, a partial function $f : A \to B$ is a relation such that, for all $a \in A$, $f(a)$ is a uniquely determined element of B whenever it is defined. For example, the formula $f(x) = \frac{1}{1-x^2}$ defines a partial function $f : \mathbb{R} \to \mathbb{R}$ with domain $\mathbb{R}$ and codomain $\mathbb{R}$: f is not defined at $x = -1$ and $x = 1$, but, otherwise, $f(x)$ is uniquely determined by the formula.*

EXERCISE 1.1.1. *Let n be a positive integer. How many binary relations are there on a set A if $|A| = n$? [Hint: How many elements are there in $|A \times A|$?]*

1.2. Directed Graphs.

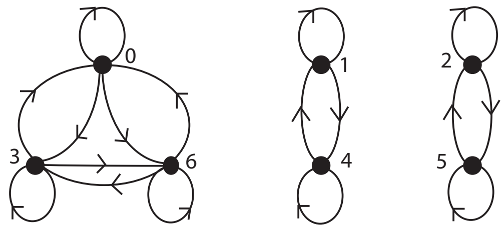
DEFINITIONS 1.2.1.

- *A* **directed graph** *or a* **digraph** *D from A to B is a collection of* **vertices** *$V \subseteq A \cup B$ and a collection of* **edges** *$R \subseteq A \times B$.*
- *If there is an ordered pair $e = (x, y)$ in R then there is an* **arc** *or* **edge** *from x to y in D.*
- *The elements x and y are called the* **initial** *and* **terminal** *vertices of the edge $e = (x, y)$, respectively.*

Discussion

A digraph can be a useful device for representing a relation, especially if the relation isn't "too large" or complicated. If R is a relation on a set A, we simplify the digraph G representing R by having only one vertex for each $a \in A$. This results, however, in the possibility of having **loops**, that is, edges from a vertex to itself, and having more than one edge joining distinct vertices (but with opposite orientations). A digraph for the relation R in Example 1.1.1 would be difficult to illustrate, and impossible to draw completely, since it would require infinitely many vertices and edges. We could draw a digraph for some finite subset of R.

EXAMPLE 1.2.1. *Suppose R is the relation on $\{0, 1, 2, 3, 4, 5, 6\}$ defined by mRn if and only if $m \equiv n(mod\ 3)$. The digraph that represents R is*



1.3. Representing Relations with Matrices.

DEFINITION 1.3.1. *Let R be a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$. An $m \times n$* **connection matrix** *$M_R = \{m_{ij}\}$ for R is defined by*

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{otherwise.} \end{cases}$$

1.4. Example 1.4.1.

EXAMPLE 1.4.1. *Let $A = \{a, b, c\}$, $B = \{e, f, g, h\}$, and $R = \{(a, e), (c, g)\}$.*

then the connection matrix

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Discussion

Recall the connection matrix for a finite relation is a method for representing a relation using a matrix.

REMARK 1.4.1. *The* **ordering** *of the elements in A and B is important. If the elements are rearranged the matrix would be different! If there is a natural order to the elements of the sets (like numerical or alphabetical) you would expect to use this order when creating connection matrices.*

To find this matrix we may use a table as follows. First we set up a table labeling the rows and columns with the vertices.

	e	f	g	h
a				
b				
c				

Since $(a, e) \in R$ we put a 1 in the row a and column e and since $(c, g) \in R$ we put a 1 in row c and column g.

	e	f	g	h
a	1			
b				
c			1	

Fill in the rest of the entries with 0's. The matrix may then be read straight from the table.

1.5. Inverse Relation.

DEFINITION 1.5.1. *Let R be a relation from A to B. Then $R^{-1} = \{(b, a)|(a, b) \in R\}$ is a relation from B to A.*

R^{-1} is called the **inverse of the relation** *R.*

Discussion

The inverse of a relation R is the relation obtained by simply reversing the ordered pairs of R. The inverse of a relation is also called the **converse** of a relation.

EXAMPLE 1.5.1. *Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$ and let $R = \{(a, 1), (a, 2), (c, 4)\}$. Then $R^{-1} = \{(1, a), (2, a), (4, c)\}$.*

EXERCISE 1.5.1. *Suppose A and B are sets and $f : A \to B$ is a function. The graph of f, $graph(f) = \{(x, f(x))|x \in A\}$ is a relation from A to B.*

(a) What is the inverse of this relation?
(b) Does f have to be invertible (as a function) for the inverse of this relation to exist?
(c) If f is invertible, find the inverse of the relation $graph(f)$ in terms of the inverse function f^{-1}.

1.6. Special Properties of Binary Relations. Definitions. Let A be a set, and let R be a binary relation on A.

(1) R is **reflexive** if
$\forall x[(x \in A) \to ((x, x) \in R)]$.
(2) R is **irreflexive** if
$\forall x[(x \in A) \to ((x, x) \notin R)]$.
(3) R is **symmetric** if
$\forall x \forall y[((x, y) \in R) \to ((y, x) \in R)]$.

(4) R is **antisymmetric** if
$\forall x \forall y[((x, y) \in R \wedge (y, x) \in R) \rightarrow (x = y)]$.
(5) R is **asymmetric** if
$\forall x \forall y[((x, y) \in R) \rightarrow ((y, x) \notin R)]$.
(6) R is **transitive** if
$\forall x \forall y \forall z[((x, y) \in R \wedge (y, z) \in R) \rightarrow ((x, z) \in R)]$.

Discussion

The definition above recalls six special properties that a relation may (or may not) satisfy. Notice that the definitions of reflexive and irreflexive relations are not complementary. That is, a relation on a set may be both reflexive and irreflexive or it may be neither. The same is true for the symmetric and antisymmetric properties, as well as the symmetric and asymmetric properties.

The terms reflexive, symmetric, and transitive is generally consistent from author to author. The rest are not as consistent in the literature

EXERCISE 1.6.1. *Before reading further, find a relation on the set* $\{a, b, c\}$ *that is neither*

(a) reflexive nor irreflexive.
(b) symmetric nor antisymmetric.
(c) symmetric nor asymmetric.

1.7. Examples of Relations and their Properties.

EXAMPLE 1.7.1. *Suppose* A *is the set of all residents of Florida and* R *is the relation given by* aRb *if* a *and* b *have the same last four digits of their Social Security Number. This relation is...*

- *reflexive*
- *not irreflexive*
- *symmetric*
- *not antisymmetric*
- *not asymmetric*
- *transitive*

EXAMPLE 1.7.2. *Suppose* T *is the relation on the set of integers given by* xTy *if* $2x - y = 1$. *This relation is...*

- *not reflexive*
- *not irreflexive*

- *not symmetric*
- *antisymmetric*
- *not asymmetric*
- *not transitive*

EXAMPLE 1.7.3. *Suppose $A = \{a, b, c, d\}$ and R is the relation $\{(a, a)\}$. This relation is. . .*

- *not reflexive*
- *not irreflexive*
- *symmetric*
- *antisymmetric*
- *not asymmetric*
- *transitive*

Discussion

When proving a relation, R, on a set A *has* a particular property, the property must be shown to hold for all appropriate combinations of members of the set. When proving a relation R does *not* have a property, however, it is enough to give a counterexample.

EXAMPLE 1.7.4. *Suppose T is the relation in Example 1.7.2 in Section 1.7. This relation is*

- *not reflexive*

  PROOF. 2 is an integer and $2 \cdot 2 - 2 = 2 \neq 1$. This shows that $\forall x[x \in \mathbb{Z} \to (x, x) \in T]$ is **not true**. $\square$

- *not irreflexive*

  PROOF. 1 is an integer and $2 \cdot 1 - 1 = 1$. This shows that $\forall x[x \in \mathbb{Z} \to (x, x) \notin T]$ is **not true**. $\square$

- *not symmetric*

  PROOF. Both 2 and 3 are integers, $2 \cdot 2 - 3 = 1$, and $2 \cdot 3 - 2 = 4 \neq 1$. This shows $2R3$, but $3 \not R 2$; that is, $\forall x \forall y[(x, y) \in \mathbf{Z} \to (y, x) \in T]$ is **not true**. $\square$

- *antisymmetric*

  PROOF. Let $m, n \in \mathbb{Z}$ be such that $(m, n) \in T$ and $(n, m) \in T$. By the definition of T, this implies both equations $2m - n = 1$ and $2n - m = 1$ must hold. We may use the first equation to solve for n, $n = 2m - 1$, and substitute this in for n in the second equation to get $2(2m - 1) - m = 1$. We may use this equation to solve for m and we find $m = 1$. Now solve for n and we get $n = 1$.

This shows that the only integers, m and n, such that both equations $2m - n = 1$ and $2n - m = 1$ hold are $m = n = 1$. This shows that $\forall m \forall n[((m, n) \in T \wedge (n, m) \in T) \rightarrow m = n]$. □

- *not asymmetric*

  PROOF. 1 is an integer such that $(1, 1) \in T$. Thus $\forall x \forall y[((x, y) \in T \rightarrow (b, a) \notin T]$ is **not true** (counterexample is $a = b = 1$). □

- *not transitive*

  PROOF. 2, 3, and 5 are integers such that $(2, 3) \in T$, $(3, 5) \in T$, but $(2, 5) \notin T$. This shows $\forall x \forall y \forall z[(x, y) \in T \wedge (y, z) \in T \rightarrow (x, z) \in T]$ is **not true**. □

EXERCISE 1.7.1. *Verify the assertions made about the relation in Example 1.7.1 in Section 1.7.*

EXERCISE 1.7.2. *Verify the assertions made about the relation in Example 1.7.3 in Section 1.7.*

EXERCISE 1.7.3. *Suppose $R \subset \mathbb{Z}^+ \times \mathbb{Z}^+$ is the relation defined by $(m, n) \in R$ if and only if $m|n$. Prove that R is*

(a) reflexive.
(b) not irreflexive.
(c) not symmetric.
(d) antisymmetric.
(e) not asymmetric.
(f) transitive.

1.8. Theorem 1.8.1: Connection Matrices v.s. Properties.

THEOREM 1.8.1. *Let R be a binary relation on a set A and let M_R be its connection matrix. Then*

- *R is reflexive iff $m_{ii} = 1$ for all i.*
- *R is irreflexive iff $m_{ii} = 0$ for all i.*
- *R is symmetric iff M is a symmetric matrix.*
- *R is antisymmetric iff for each $i \neq j$, $m_{ij} = 0$ or $m_{ji} = 0$.*
- *R is asymmetric iff for every i and j, if $m_{ij} = 1$, then $m_{ji} = 0$.*

Discussion

Connection matrices may be used to test if a finite relation has certain properties and may be used to determine the composition of two finite relations.

EXAMPLE 1.8.1. *Determine which of the properties: reflexive, irreflexive, symmetric, antisymmetric, asymmetric, the relation on* $\{a, b, c, d\}$ *represented by the following matrix has.*

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Solution. *The relation is irreflexive, asymmstric and antisymmetric only.*

EXERCISE 1.8.1. *Determine which of the properties: reflexive, irreflexive, symmetric, antisymmetric, asymmetric, the relation on* $\{a, b, c, d\}$ *represented by the following matrix has.*

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

1.9. Combining Relations. Suppose property P is one of the properties listed in Section 1.6, and suppose R and S are relations on a set A, each having property P. Then the following questions naturally arise.

(1) Does $\overline{R}$ (necessarily) have property P?
(2) Does $R \cup S$ have property P?
(3) Does $R \cap S$ have property P?
(4) Does $R - S$ have property P?

Discussion

Notice that when we combine two relations using one of the binary set operations we are combining *sets of ordered pairs*.

1.10. Example 1.10.1.

EXAMPLE 1.10.1. *Let* R *and* S *be transitive relations on a set* A. *Does it follow that* $R \cup S$ *is transitive?*

Solution. No. Here is a counterexample:

$$A = \{1, 2\}, \quad R = \{(1, 2)\}, \quad S = \{(2, 1)\}$$

$$\text{Therefore,} \qquad R \cup S = \{(1, 2), (2, 1)\}$$

Notice that R and S are both transitive (vacuously, since there are no two elements satisfying the hypothesis of the conditions of the property). However $R \cup S$ is not transitive. If it were it would have to have $(1, 1)$ and $(2, 2)$ in $R \cup S$.

Discussion

The solution to Example 1.10.1 gives a counterexample to show that the union of two transitive relations is not necessarily transitive. Note that you could find an example of two transitive relations whose union *is* transitive. The question, however, asks if the given property holds for two relations must it hold for the binary operation of the two relations. This is a general question and to give the answer "yes" we must know it is true for *every* possible pair of relations satisfying the property.

Here is another example:

EXAMPLE 1.10.2. *Suppose R and S are transitive relations on the set A. Is $R \cap S$ transitive?*

Solution. *Yes.*

PROOF. Assume R and S are both transitive and suppose $(a, b), (b, c) \in R \cap S$. Then $(a, b), (b, c) \in R$ and $(a, b), (b, c) \in S$. It is given that both R and S are transitive, so $(a, c) \in R$ and $(a, c) \in S$. Therefore $(a, c) \in R \cap S$. This shows that for arbitrary $(a, b), (b, c) \in R \cap S$ we have $(a, c) \in R \cap S$. Thus $R \cap S$ is transitive. □

As it turns out, the intersection of any two relations satisfying one of the properties in Section 1.6 also has that property. As the following exercise shows, sometimes even more can be proved.

EXERCISE 1.10.1. *Suppose R and S are relations on a set A.*

(a) Prove that if R and S are reflexive, then so is $R \cap S$.
(b) Prove that if R is irreflexive, then so is $R \cap S$.

EXERCISE 1.10.2. *Suppose R and S are relations on a set A.*

(a) Prove that if R and S are symmetric, then so is $R \cap S$.
(b) Prove that if R is antisymmetric, then so is $R \cap S$.
(c) Prove that if R is asymmetric, then so is $R \cap S$.

Exercise 1.10.3. *Suppose R and S are relations on a set A. Prove or disprove:*

(a) If R and S are reflexive, then so is $R \cup S$.
(b) If R and S are irreflexive, then so is $R \cup S$.

Exercise 1.10.4. *Suppose R and S are relations on a set A. Prove or disprove:*

(a) If R and S are symmetric, then so is $R \cup S$.
(b) If R and S are antisymmetric, then so is $R \cup S$.
(c) If R and S are asymmetric, then so is $R \cup S$.

1.11. Definition of Composition.

Definition 1.11.1.

(1) *Let*
- *R be a relation from A to B, and*
- *S be a relation from B to C.*

Then the **composition** *of R with S, denoted $S \circ R$, is the relation from A to C defined by the following property:*
$(x, z) \in S \circ R$ if and only if there is a $y \in B$ such that $(x, y) \in R$ and $(y, z) \in S$.

(2) *Let R be a binary relation on A. Then R^n is defined recursively as follows:*
Basis: $R^1 = R$
Recurrence: $R^{n+1} = R^n \circ R$, if $n \geq 1$.

Discussion

The composition of two relations can be thought of as a generalization of the composition of two functions, as the following exercise shows.

Exercise 1.11.1. *Prove: If $f: A \to B$ and $g: B \to C$ are functions, then $graph(g \circ f) = graph(g) \circ graph(f)$.*

Exercise 1.11.2. *Prove the composition of relations is an associative operation.*

Exercise 1.11.3. *Suppose R is a relation on A. Using the previous exercise and mathematical induction, prove that $R^n \circ R = R \circ R^n$.*

Exercise 1.11.4. *Prove an ordered pair $(x, y) \in R^n$ if and only if, in the digraph D of R, there is a directed path of length n from x to y.*

Notice that if there is no element of B such that $(a, b) \in R_1$ and $(b, c) \in R_2$ for some $a \in A$ and $c \in C$, then the composition is empty.

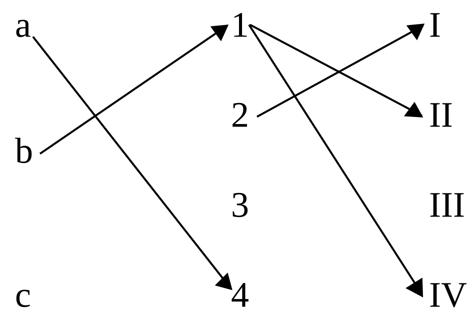
1.12. Example 1.12.1.

EXAMPLE 1.12.1. *Let* $A = \{a, b, c\}$*,* $B = \{1, 2, 3, 4\}$*, and* $C = \{I, II, III, IV\}$*.*

- *If* $R = \{(a, 4), (b, 1)\}$ *and*
- $S = \{(1, II), (1, IV), (2, I)\}$*, then*
- $S \circ R = \{(b, II), (b, IV)\}$*.*

Discussion

It can help to consider the following type of diagram when discussing composition of relations, such as the ones in Example 1.12.1 shown here.



EXAMPLE 1.12.2. *If* R *and* S *are transitive binary relations on* A*, is* $R \circ S$ *transitive?*

Solution. *No. Here is a counterexample: Let*

$$R = \{(1, 2), (3, 4)\}, \quad \text{and} \quad S = \{(2, 3), (4, 1)\}.$$

Both R *and* S *are transitive (vacuously), but*

$$R \circ S = \{(2, 4), (4, 2)\}$$

is not transitive. (Why?)

EXAMPLE 1.12.3. *Suppose* R *is the relation on* $\mathbb{Z}^+$ *defined by* aRb *if and only if* $a|b$*. Then* $R^2 = R$*.*

EXERCISE 1.12.1. *Suppose* R *is the relation on the set of real numbers given by* xRy *if and only if* $\frac{x}{y} = 2$*.*

(a) Describe the relation R^2*.*
(b) Describe the relation R^n*,* $n \geq 1$*.*

EXERCISE 1.12.2. *Suppose* R *and* S *are relations on a set* A *that are reflexive. Prove or disprove the relation obtained by combining* R *and* S *in one of the following ways is reflexive. (Recall:* $R \oplus S = (R \cup S) - (R \cap S)$*.)*

(a) $R \oplus S$
(b) $R - S$
(c) $R \circ S$
(d) R^{-1}
(e) R^n, $n \geq 2$

EXERCISE 1.12.3. *Suppose R and S are relations on a set A that are symmetric. Prove or disprove the relation obtained by combining R and S in one of the following ways is symmetric.*

(a) $R \oplus S$
(b) $R - S$
(c) $R \circ S$
(d) R^{-1}
(e) R^n, $n \geq 2$

EXERCISE 1.12.4. *Suppose R and S are relations on a set A that are transitive. Prove or disprove the relation obtained by combining R and S in one of the following ways is transitive.*

(a) $R \oplus S$
(b) $R - S$
(c) $R \circ S$
(d) R^{-1}
(e) R^n, $n \geq 2$

1.13. Theorem 1.13.1: Characterization of Transitive Relations.

THEOREM 1.13.1. *Let R be a binary relation on a set A. R is transitive if and only if $R^n \subseteq R$, for $n \geq 1$.*

PROOF. To prove (R transitive) $\to$ ($R^n \subseteq R$) we assume R is transitive and prove $R^n \subseteq R$ for $n \geq 1$ by induction.

Basis Step, $n = 1$. $R^1 = R$, so the statement is vacuously true when $n = 1$, since $R^1 = R \subseteq R$ whether or not R is transitive.

Induction Step. Prove $R^n \subseteq R \to R^{n+1} \subseteq R$.

Assume $R^n \subseteq R$ for some $n \geq 1$. Suppose $(x, y) \in R^{n+1}$. By definition, $R^{n+1} = R^n \circ R$, so there must be some $a \in A$ such that $(x, a) \in R$ and $(a, y) \in R^n$. By the induction hypothesis, $R^n \subseteq R$, so $(a, y) \in R$. Since R is transitive, $(x, a), (a, y) \in R$ implies $(x, y) \in R$. Since (x, y) was an arbitrary element of R^{n+1}, we have shown $R^{n+1} \subseteq R$.

We now prove the reverse implication: $R^n \subseteq R$, for $n \geq 1$, implies R is transitive. We prove this directly using only the hypothesis for $n = 2$.

Assume $(x, y), (y, z) \in R$. The definition of composition implies $(x, z) \in R^2$. But $R^2 \subseteq R$, so $(x, z) \in R$. Thus R is transitive.

$\square$

Discussion

Theorem 1.13.1 gives an important characterization of the transitivity property. Notice that, since the statement of the theorem was a property that was to be proven for all positive integers, induction was a natural choice for the method of proof.

EXERCISE 1.13.1. *Prove that a relation R on a set A is transitive if and only if $R^2 \subseteq R$. [Hint: Examine not only the statement, but the proof of the Theorem 1.13.1.]*

1.14. Connection Matrices v.s. Composition. Recall: The **boolean product** of an $m \times k$ matrix $A = [a_{ij}]$ and a $k \times n$ matrix $B = [b_{ij}]$, denoted $A \odot B = [c_{ij}]$, is defined by

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

THEOREM 1.14.1. *Let X, Y, and Z be finite sets. Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. If M_1 is the connection matrix for R_1 and M_2 is the connection matrix for R_2, then $M_1 \odot M_2$ is the connection matrix for $R_2 \circ R_1$.*

We write $M_{R_2 \circ R_1} = M_{R_1} \odot M_{R_2}$.

COROLLARY 1.14.1.1. *$M_{R^n} = (M_R)^n$ (the boolean n^{th} power of M_R).*

EXERCISE 1.14.1. *Let the relation R on $\{a, b, c, d\}$ be represented by the matrix*

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Find the matrix that represents R^3.