4. Partial Orderings

4.1. Definition of a Partial Order.

DEFINITION 4.1.1.

(1) *A relation R on a set A is a* **partial order** *iff R is*
- *reflexive,*
- *antisymmetric, and*
- *transitive.*

(2) *(A, R) is called a* **partially ordered set** *or a* **poset**.

(3) *If, in addition, either aRb or bRa, for every $a, b \in A$, then R is called a* **total order** *or a* **linear order** *or a* **simple order**. *In this case (A, R) is called a* **chain**.

(4) *The notation $a \preceq b$ is used for aRb when R is a partial order.*

Discussion

The classic example of an order is the order relation on the set of real numbers: aRb iff $a \leq b$, which is, in fact, a total order. It is this relation that suggests the notation $a \preceq b$, but this notation is *not* used exclusively for total orders.

Notice that in a partial order on a set A it is *not required* that every pair of elements of A be related in one way or the other. That is why the word *partial* is used.

4.2. Examples.

EXAMPLE 4.2.1. *$(\mathbb{Z}, \leq)$ is a poset. Every pair of integers are related via $\leq$, so $\leq$ is a total order and $(\mathbb{Z}, \leq)$ is a chain.*

EXAMPLE 4.2.2. *If S is a set then $(P(S), \subseteq)$ is a poset.*

EXAMPLE 4.2.3. *$(\mathbb{Z}^+, |)$ is a poset. The relation $a|b$ means "a divides b."*

Discussion

Example 4.2.2 better illustrates the general nature of a partial order. This partial order is not necessarily a total order; that is, it is not always the case that either $A \subseteq B$ or $B \subseteq A$ for every pair of subsets of S. Can you think of the only occasions in which this would be a total order?

Example 4.2.3 is also only a partial order. There are many pairs of integers such that $a \not| b$ and $b \not| a$.

4.3. Pseudo-Orderings.

DEFINITION 4.3.1.

A relation $\prec$ on S is called a **pseudo-order** *if*

- *the relation is irreflexive and*
- *transitive.*

Some texts call this a **quasi-order**. *Rosen uses quasi-order to mean a different type of relation, though.*

THEOREM 4.3.1 (Theorems and Notation).

1. *Given a poset $(S, \preceq)$, we define a relation $\prec$ on S by $x \prec y$ if and only if $x \preceq y$ and $x \neq y$. The relation $\prec$ is a pseudo-order.*
2. *Given a set S and a pseudo-order $\prec$ on S, we define a relation $\preceq$ on S by $x \preceq y$ if and only if $x \prec y$ or $x = y$. The relation $\preceq$ is a partial order.*
3. *Given a poset $(S, \preceq)$, we define the relation $\succeq$ on S by $x \succeq y$ iff $y \preceq x$. $(S, \succeq)$ is a poset and $\succeq$ is the inverse of $\preceq$.*
4. *Given a set S and a pseudo-order $\prec$ on S, we define the relation $\succ$ on S by $x \succ y$ iff $y \prec x$. $\succ$ is a pseudo-order on S and $\succ$ is the inverse of $\prec$.*
5. *In any discussion of a partial order relation $\preceq$, we will use the notations $\prec$, $\succeq$, and $\succ$ to be the relations defined above, depending on $\preceq$. Similarly, if we are given a a pseudo-order, $\prec$, then $\preceq$ will be the partial order defined in part 2.*

Discussion

The notation above is analogous to the usual $\leq$, $\geq$, $<$, and $>$ notations used with real numbers. We do not require that the orders above be total orders, though. Another example you may keep in mind that uses similar notation is $\subseteq$, $\supseteq$, $\subset$, $\supset$ on sets. These are also partial and pseudo-orders.

EXERCISE 4.3.1. *Prove a pseudo-order, $\prec$, is antisymmetric.*

EXERCISE 4.3.2. *Prove Theorem 4.3.1 part 1.*

EXERCISE 4.3.3. *Prove Theorem 4.3.1 part 2.*

4.4. Well-Ordered Relation.

DEFINITION 4.4.1. *Let R be a partial order on A and suppose $S \subseteq A$.*

1. *An element $s \in S$ is a* **least element** *of S iff sRb for every $b \in S$.*
2. *An element $s \in S$ is a* **greatest element** *of S iff bSs for every $b \in S$.*

(3) A chain (A, R) *is* **well-ordered** *iff every nonempty subset of* A *has a least element.*

Discussion

Notice that if s is a least (greatest) element of S, s must be an element of S and s must precede (be preceded by) all the other elements of S.

Confusion may arise when we define a partial order on a well-known set, such as the set $\mathbb{Z}^+$ of positive integers, that already has a natural ordering. One such ordering on $\mathbb{Z}^+$ is given in Example 4.2.3. As another example, one could perversely impose the relation $\preceq$ on $\mathbb{Z}^+$ by defining $a \preceq b$ iff $b \leq a$. With respect to the relation $\preceq$, $\mathbb{Z}^+$ would have no least element and it's "greatest" element would be 1! This confusion may be alleviated somewhat by reading $a \preceq b$ as "a precedes b" instead of "a is less than or equal to b", especially in those cases when the set in question already comes equipped with a natural order different from $\preceq$.

4.5. Examples.

EXAMPLE 4.5.1. $(\mathbb{Z}, \leq)$ *is a chain, but it is not well-ordered.*

EXAMPLE 4.5.2. $(\mathbb{N}, \leq)$ *is well-ordered.*

EXAMPLE 4.5.3. $(\mathbb{Z}^+, \geq)$ *is a chain, but is not well-ordered.*

Discussion

In Example 4.5.1, the set of integers does not have a least element. If we look at the set of positive integers, however, every nonempty subset (including $\mathbb{Z}^+$) has a least element. Notice that if we reverse the inequality, the "least element" is now actually the one that is larger than the others – look back at the discussion in Section 4.4 – and there is no "least element" of $\mathbb{Z}^+$.

Pay careful to the definition of what it means for a chain to be well-ordered. It requires every nonempty subset to have a *least* element, but it does not require that every nonempty subset have a greatest element.

EXERCISE 4.5.1. *Suppose* $(A, \preceq)$ *is a poset such that every nonempty subset of* A *has a least element. Prove that* $\preceq$ *is a total ordering on* A.

4.6. Lexicographic Order.

DEFINITION 4.6.1. *Given two posets* $(A_1, \preceq_1)$ *and* $(A_2, \preceq_2)$ *we construct an* **induced** *or* **lexicographic** *partial order* $\preceq_L$ *on* $A_1 \times A_2$ *by defining* $(x_1, y_1) \preceq_L (x_2, y_2)$ *iff*

- $x_1 \prec_1 x_2$ *or*
- $x_1 = x_2$ *and* $y_1 \preceq_2 y_2$.

This definition is extended recursively to Cartesian products of partially ordered sets $A_1 \times A_2 \times \cdots \times A_n$.

Discussion

EXERCISE 4.6.1. *Prove that if each of the posets* $(A_1, \preceq_1)$ *and* $(A_2, \preceq_2)$ *is a chain and* $\preceq_L$ *is the lexicographic order on* $A_1 \times A_2$*, then* $(A_1 \times A_2, \preceq_L)$ *is also a chain.*

EXERCISE 4.6.2. *Suppose for each positive integer* n*,* $(A_n, preceq_n)$ *is a poset. Give a recursive definition for the lexicographic order on* $A_1 \times A_2 \times \cdots \times A_n$ *for all positive integers* n.

4.7. Examples 4.7.1 and 4.7.2.

EXAMPLE 4.7.1. *Let* $A_1 = A_2 = \mathbb{Z}^+$ *and* $\preceq_1 = \preceq_2 = |$ *("divides"). Then*

- $(2, 4) \preceq_L (2, 8)$
- $(2, 4)$ *is not related under* $\preceq_L$ *to* $(2, 6)$.
- $(2, 4) \preceq_L (4, 5)$

EXAMPLE 4.7.2. *Let* $A_i = \mathbb{Z}^+$ *and* $\preceq_i = |$*, for* $i = 1, 2, 3, 4$*. Then*

- $(2, 3, 4, 5) \preceq_L (2, 3, 8, 2)$
- $(2, 3, 4, 5)$ *is not related under* $\preceq_L$ *to* $(3, 6, 8, 10)$.
- $(2, 3, 4, 5)$ *is not related under* $\preceq_L$ *to* $(2, 3, 5, 10)$.

Discussion

Notice that $(2, 4)$ does not precede $(2, 6)$: although their first entries are equal, 4 does not divide 6. In fact, the pairs $(2, 4)$ and $(2, 6)$ are not related in any way. On the other hand, since $2|4$ (and $2 \neq 4$), we do not need to look any further than the first place to see that $(2, 4) \preceq_L (4, 5)$.

Notice also in Example 4.7.2, the first non-equal entries determine whether or not the relation holds.

4.8. Strings. We extend the lexigraphic ordering to strings of elements in a poset $(A, \preceq)$ as follows:

$$a_1a_2 \cdots a_m \preceq_L b_1b_2 \cdots b_n$$

iff

- $(a_1, a_2, \ldots, a_t) \preceq_L (b_1, b_2, \ldots, b_t)$ where $t = \min(m, n)$, or
- $(a_1, a_2, \ldots, a_m) = (b_1, b_2, \ldots, b_m)$ and $m < n$.

Discussion

The ordering defined on strings gives us the usual alphabetical ordering on words and the usual order on bit string.

EXERCISE 4.8.1. *Put the bit strings* $0110, 10, 01,$ *and* 010 *in increasing order using*

1. *numerical order by considering the strings as binary numbers,*
2. *lexicographic order using* $0 < 1$*,*

There are numerous relations one may impose on products. Lexicographical order is just one partial order.

EXERCISE 4.8.2. *Let* $(A_1, \preceq_1)$ *and* $(A_2, \preceq_2)$ *be posets. Define the relation* $\preceq$ *on* $A_1 \times A_2$ *by* $(a_1, a_2) \preceq (b_1, b_2)$ *if and only if* $a_1 \preceq_1 b_1$ *and* $a_2 \preceq_2 b_2$*. Prove* $\preceq$ *is a partial order. This partial order is called the* **product order***.*

4.9. Hasse or Poset Diagrams.

DEFINITION 4.9.1. *To construct a* **Hasse** *or* **poset diagram** *for a poset* (A, R)*:*

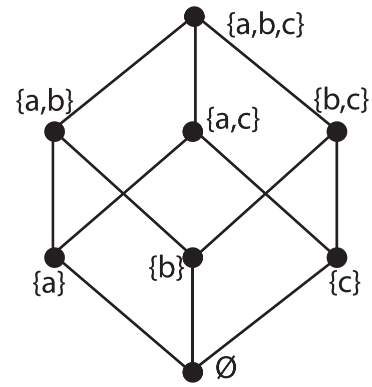
1. *Construct a digraph representation of the poset* (A, R) *so that all arcs point up (except the loops).*
2. *Eliminate all loops.*
3. *Eliminate all arcs that are redundant because of transitivity.*
4. *Eliminate the arrows on the arcs.*

Discussion

The Hasse diagram of a poset is a simpler version of the digraph representing the partial order relation. The properties of a partial order assure us that its digraph can be drawn in an oriented plane so that each element lies below all other elements it precedes in the order. Once this has been done, all redundant information can be removed from the digraph and the result is the Hasse diagram.

4.10. Example 4.10.1.

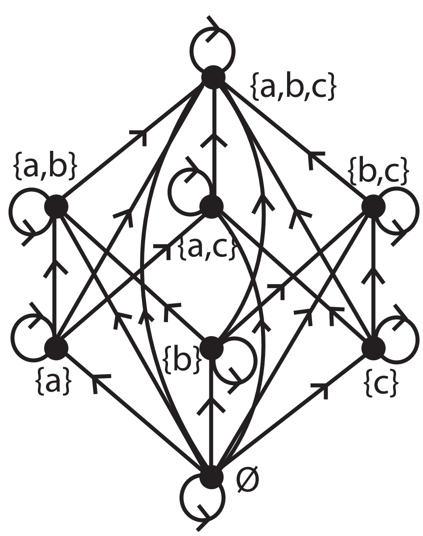
EXAMPLE 4.10.1. *The Hasse diagram for* $(P(\{a, b, c\}), \subseteq)$ *is*



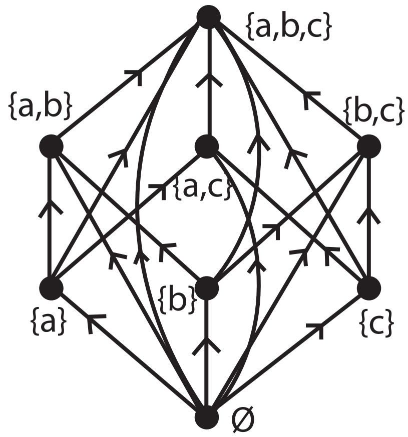
Discussion

The following steps could be used to get the Hasse diagram above.

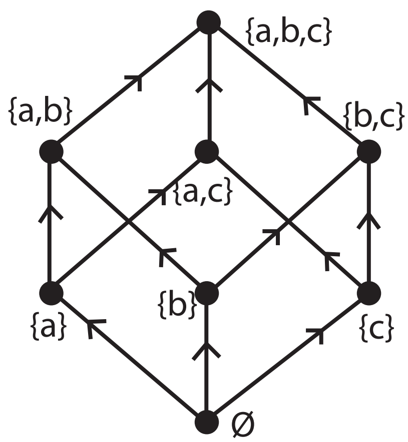
(1) You can see that even this relatively simple poset has a complicated digraph.



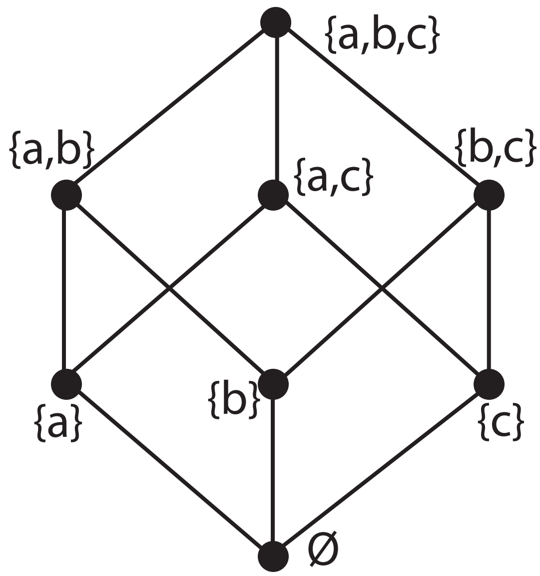
(2) Eliminate the loops.



(3) Now eliminate redundant arcs resulting from transitivity.



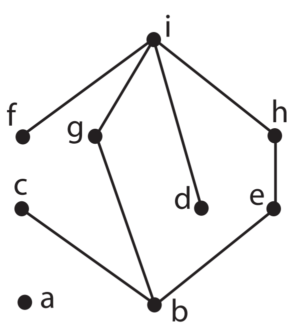
(4) Finally eliminate the arrows



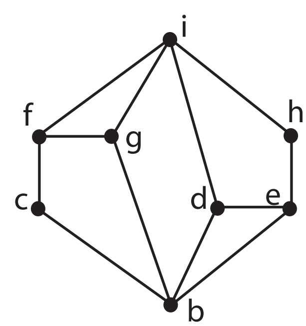
Exercise 4.10.1. *Construct the Hasse diagram for the poset* $(\{1, 2, 3, 4, 6, 9, 12\}, |)$, *where* $|$ *is the "divides" relation.*

Exercise 4.10.2. *Is the diagram the Hasse diagram for a partial order? If so, give the partial order and if not explain why.*

(a)



(b)



4.11. Maximal and Minimal Elements.

Definition 4.11.1. *Let* (A, R) *be a poset.*

(1) An element $a \in A$ *is a* **minimal element** *if there does not exist an element* $b \in A$, $b \neq a$, *such that* $b \preceq a$.
(2) An element $a \in A$ *is a* **maximal element** *if there does not exist an element* $b \in A$, $b \neq a$, *such that* $a \preceq b$.

Discussion

Another useful way to characterize minimal elements of a poset $(A, \preceq)$ is to say that a is a minimal element of A iff $b \preceq a$ implies $b = a$. A similar characterization holds for maximal elements. It is possible for a poset to have more than one maximal and minimal element. In the poset in Exercise 4.10.1, for example, 1 is the only minimal element, but both 9 and 12 are maximal elements. These facts are easily observable from the Hasse diagram.

4.12. Least and Greatest Elements.

Definition 4.12.1. *Let $(A, \preceq)$ be a poset.*

(1) An element $a \in A$ is the **least element** *of A if $a \preceq b$ for every element $b \in A$.*
(2) An element $a \in A$ is the **greatest element** *of A if $b \preceq a$ for every element $b \in A$.*

Theorem 4.12.1. *A poset $(A, \preceq)$ can have at most one least element and at most one greatest element. That is, least and greatest elements are unique, if they exist.*

Discussion

We revisit the definition of greatest and least elements, which were defined in Section 4.4. As with minimal and maximal elements, Hasse diagrams can be helpful in illustrating least and greatest elements. Although a poset may have many minimal (or maximal) elements, Theorem 4.12.1 guarantees that it may have no more than one least (or greatest) element. We ask you to explore the relationship among these concepts in the following exercise.

Exercise 4.12.1. *Let $(A, \preceq)$ be a poset.*

(a) Prove that if a is the least element of A, then a is a minimal element of A.
(b) Prove that if b is the greatest element of A, then b is a maximal element of A.
(c) Prove that if A has more than one minimal element, then A does not have a least element.
(d) Prove that if A has more than one maximal element, then A does not have a greatest element.

4.13. Upper and Lower Bounds.

Definition 4.13.1. *Let S be a subset of A in the poset $(A, \preceq)$.*

(1) If there exists an element $a \in A$ such that $s \preceq a$ for all $s \in S$, then a is called an **upper bound** *on S.*

(2) *If there exists an element $a \in A$ such that $a \preceq s$ for all $s \in S$, then a is called an* **lower bound** *on S.*

4.14. Least Upper and Greatest Lower Bounds.

Definition 4.14.1. *Suppose $(A, \preceq)$ is a poset, S a is subset of A, and $a \in A$.*

(1) *a is the* **least upper bound** *of S if*
   - *a is an upper bound of S and*
   - *if s is another upper bound of S, then $a \preceq s$.*

(2) *a is the* **greatest lower bound** *of S if*
   - *a is an lower bound of S and*
   - *if s is another lower bound of S, then $s \preceq a$.*

Discussion

In Section 4.13 we extend the concepts upper and lower bound as well as least upper bound and greatest lower bound to subsets of a poset. The difference between a least element of a subset of A and a lower bound for the subset of A is that the least element is required to be *in* the subset and the lower bound is not. Here are a few facts about lower bounds and minimal elements to keep in mind. You should rephrase each statement replacing "lower" with "upper", etc.

- For a to be a lower bound for a subset S, a need not be in S, but it must precede every element of S.
- A minimal element, a, of a subset S is a lower bound for the set of all elements in S preceded by a.
- A subset may have more than one lower bound, or it may have none.
- A subset may have lower bounds, but no greatest lower bound.

Example 4.14.1. *Suppose we are given the poset $(A, |)$, where $A = \{1, 2, 3, 4, 6, 8, 9, 12\}$.*

(1) *The subset $\{2, 3, 4, 6\}$ has no greatest or least element.*
(2) *1 is the greatest lower bound for $\{2, 3, 4, 6\}$ and 12 is its least upper bound.*
(3) *The subset $\{1, 2, 3, 8\}$ has no upper bound in A.*
(4) *Every subset of A has a greatest lower bound.*

Exercise 4.14.1. *Consider the poset $(\mathcal{A}, \subseteq)$, where $\mathcal{A} = P(S)$ is the power set of a set S. Prove that every nonempty subset of $\mathcal{A}$ has a least upper bound and a greatest lower bound in $\mathcal{A}$.*

4.15. Lattices.

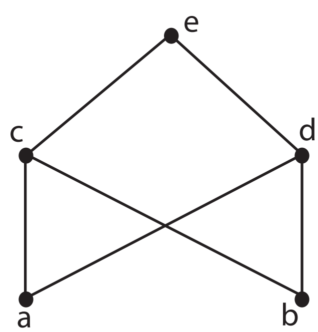
Definition 4.15.1. *A poset $(A, \preceq)$ is a* **lattice** *if every pair of elements has a least upper bound and a greatest lower bound in A.*

Discussion

To check if a poset is a lattice you must check every pair of elements to see if they each have a greatest lower bound and least upper bound. If you draw its Hasse diagram, you can check to see whether some pair of elements has more than one upper (or lower) bound on the same level. If so, then the poset is not a lattice.

4.16. Example 4.16.1.

EXAMPLE 4.16.1. *The poset given by the following Hasse diagram is not a lattice.*



Discussion

In Example 4.16.1 notice that $\{a, b\}$ has upper bounds c, d and e. Since e is larger than either c or d, it cannot be the least upper bound. But c and d are not related in any way. Thus there is no least upper bound for the subset $\{a, b\}$.

EXERCISE 4.16.1. *Prove that the poset* $(P(S), \subseteq)$ *is a lattice.*

4.17. Topological Sorting.

DEFINITION 4.17.1.

(1) A total ordering $\leq$ *on a set* A *is said to be* **compatible** *with a partial ordering* $\preceq$ *on* A, *if* $a \preceq b$ *implies* $a \leq b$ *for all* $a, b \in A$.

(2) A **topological sorting** *is a process of constructing a compatible total order for a given partial order.*

Discussion

A topological sorting is a process of creating a total order from a partial order. Topological sorting has a number of applications. For example:

- It can be useful in PERT charts to determine an ordering of tasks.
- It can be useful in graphics to render objects from back to front to expose hidden surfaces.
- A painter often uses a topological sort when applying paint to a canvas. He/she paints parts of the scene furthest from the view first.

There may be several total orders that are compatible with a given partial order.

4.18. Topological Sorting Algorithm. **procedure** *topological sort* $((A, \preceq)$: finite poset)
k:= 1
while $A \neq \emptyset$
begin
  a_k:= a minimal element of A
  A:= $A - \{a_k\}$
  k:= $k + 1$
end $\{a_1, a_2, ..., a_n$ is a compatible total ordering of $A\}$

Discussion

In order to justify the existence of a minimal element of S at each step in the topological sorting algorithm, we need to prove Theorem 4.19.1 in Section 4.19.

4.19. Existence of a Minimal Element.

THEOREM 4.19.1. *Suppose* $(A, \preceq)$ *is a finite nonempty poset. Then* A *has a minimal element.*

PROOF. Suppose $(A, \preceq)$ is a finite nonempty poset, where A has n elements, $n \geq 1$. We will prove that A has a minimal element by mathematical induction.

BASIS STEP. $n = 1$. Then $A = \{a\}$ and a is a minimal (least!) element of A.

INDUCTION STEP. Suppose that every poset having n elements has a minimal element. Suppose $(A, \preceq)$ is a poset having $n + 1$ elements. Let a be an arbitrary element of A, and let $S = A - \{a\}$. Then S, together with the partial order $\preceq$ restricted to S, is a poset with n elements. By the inductive hypothesis, S has a minimal element b. There are two possibilities.

(1) $a \preceq b$. Then a is a minimal element of A. Otherwise, there is an element c in A, different from a, such that $c \preceq a$. But then c is in S, c is different from b (why?), and $c \preceq b$, which contradicts the fact that b is a minimal element of S.

(2) $a \not\preceq b$. Suppose b is not a minimal element of A. Then there is a $c \in A$ with $c \preceq b$. Since $a \not\preceq b$ we have $c \neq a$. Thus $c \in S$ and we then conclude b is not minimal in S. This is a contradiction so there are no elements of A that preceed b. Hence b is a minimal element of A in this case.

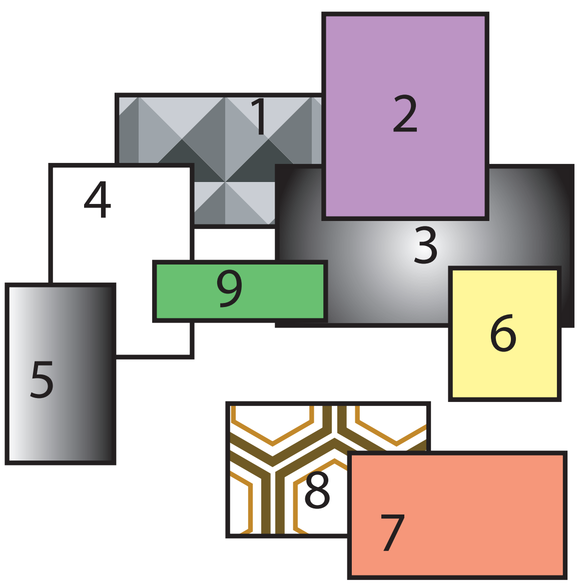
Thus, in any case A has a minimal element, and so by the principle of induction every finite poset has a minimal element. □

EXERCISE 4.19.1. *Let $(A, \preceq_A)$ be a poset and let $S \subseteq A$. Define $\preceq_S$ on S by*

$$\forall a, b \in S[a \preceq_S b \Leftrightarrow a \preceq_A b].$$

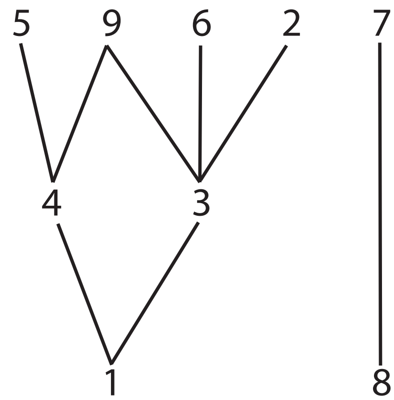
Prove $(S, \preceq_S)$ is a poset. The partial order $\preceq_S$ is the **Restriction** *of $\preceq_A$ to S and we usually use the same notation for both partial orders.*

EXAMPLE 4.19.1. *Consider the set of rectangles T and the relation R given by tRs if t is more distant than s from the viewer.*

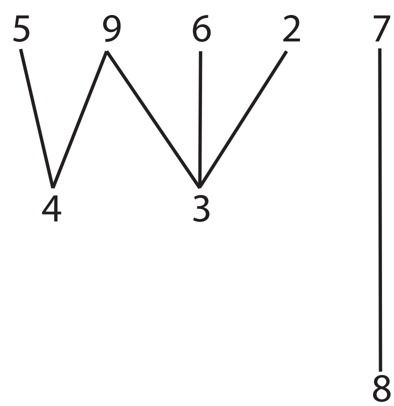


Here are some of the relations that we find from the figure: $1R2, 1R4, 1R3, 4R5, 3R2, 3R9, 3R6$.

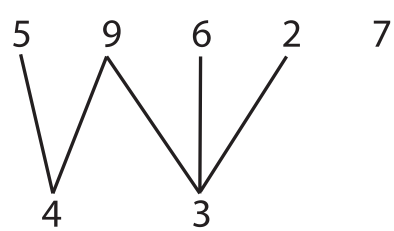
The Hasse diagram for R is



If we draw 1 (it would also be fine to use 8) out of the diagram and delete it we get



Then draw 8



and so on. By drawing minimal elements you may get the following total order (there are many other total orders compatible with the partial order):