

### 3. Euler and Hamilton Paths

#### 3.1. Euler and Hamilton Paths.

DEFINITIONS 3.1.1.

- (1) An **Euler Circuit** in a graph  $G$  is a path in  $G$  that uses every edge exactly once and begins and ends at the same vertex.
- (2) An **Euler path** in  $G$  is a path in  $G$  that uses every edge exactly once, but does not necessarily begin and end at the same vertex.

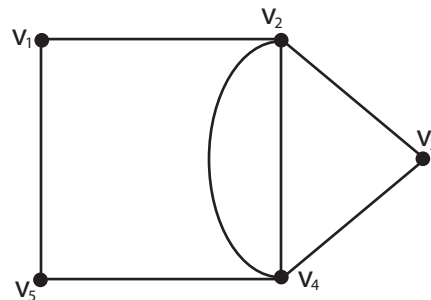
#### Discussion

There are several special types of paths in graphs that we will study in this section.

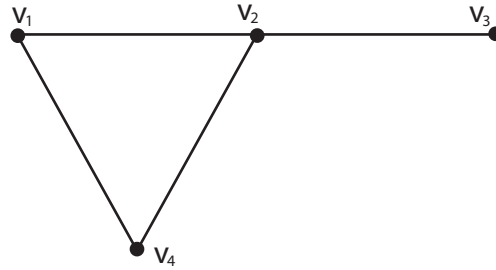
An Euler path or circuit should use every single *edge* exactly one time. The difference between an Euler path and Euler circuit is simply whether or not the path begins and ends at the same vertex. Remember a circuit begins and ends at the same vertex. If the graph is a directed graph then the path must use the edges in the direction given.

#### 3.2. Examples.

EXAMPLE 3.2.1. *This graph has the Euler circuit (and hence Euler path)  $v_1, v_2, v_3, v_4, v_2, v_4, v_5, v_1$ .*



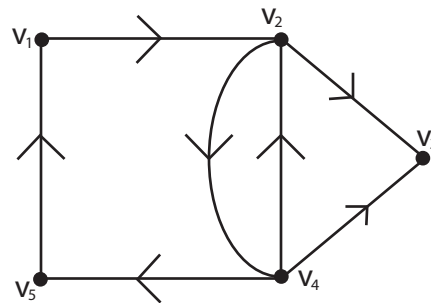
EXAMPLE 3.2.2. *This graph does not have an Euler circuit, but has the Euler path  $v_2, v_4, v_1, v_2, v_3$ .*



## Discussion

Not all graphs have Euler circuits or Euler paths. See page 634, Example 1  $G_2$ , in the text for an example of an undirected graph that has no Euler circuit nor Euler path.

In a directed graph it will be less likely to have an Euler path or circuit because you must travel in the correct direction. Consider, for example,



This graph has neither an Euler circuit nor an Euler path. It is impossible to cover both of the edges that travel to  $v_3$ .

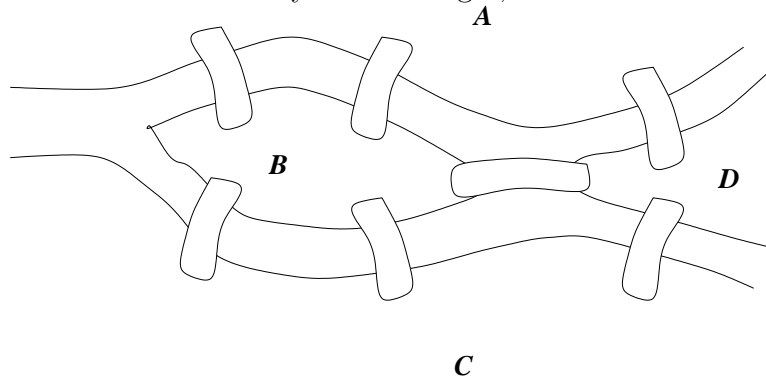
### 3.3. Necessary and Sufficient Conditions for an Euler Circuit.

**THEOREM 3.3.1.** *A connected, undirected multigraph has an Euler circuit if and only if each of its vertices has even degree.*

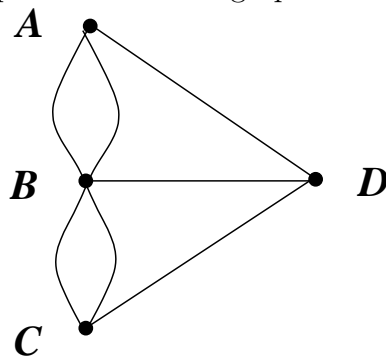
## Discussion

This is a wonderful theorem which tells us an easy way to check if an undirected, connected graph has an Euler circuit or not. There is an extension for Euler paths which we will soon see.

This theorem allows us to solve the famous Königsberg problem: The town, once called Königsberg, Prussia, now Kaliningrad, Russia), was divided by the river Pregel into parts, which were connected by seven bridges, as illustrated below.



When the people of Königsberg would walk through the town, they wondered whether they could plan their walk so that they would cross each bridge exactly once and end up at their starting point. Leonhard Euler, a Swiss mathematician, solved the problem in the negative, by discovering and proving a theorem, which is essentially Theorem 3.3.1. The problem can be modelled by the multigraph below, and the solution depends upon whether the graph has an Euler circuit.



**PROOF OF THEOREM 3.3.1.** Assume  $G$  is a connected, undirected multigraph with an Euler circuit. The degree of any given vertex may be counted by considering this circuit, since the circuit traverses every edge exactly once. While traveling the circuit we move into a vertex by one edge and leave by another edge, so there must be an even number of edges adjacent to each vertex.

Conversely, if the graph  $G$  is such that every edge has an even degree, then we can build an Euler circuit by the following algorithm: We begin at some arbitrary vertex and travel an edge out from that vertex to another. Then travel to another

vertex using an unused edge. Since each vertex has even degree there will always be an unused edge to travel out if we have traveled into the vertex until we reach the beginning vertex and have used all the edges.

□

Try your hand at the following exercise before you read further.

**EXERCISE 3.3.1.** *Is it possible for the people in Königsberg to plan a walk that crosses each bridge exactly once and ends up in a part of town different from where they started? (That is, is there an Euler path?) Either show that this is possible or explain why it is not.*

There is another algorithm one may use to find an Euler circuit given a graph with all vertices of even degree. The algorithm is written in pseudocode in the text, but the general idea is to start with some arbitrary circuit which we consider the "main" circuit. Now find a circuit that uses edges not used in the main circuit but begins and ends at some vertex in the main circuit. Insert this circuit into the main circuit. Repeat until all edges are used.

### 3.4. Necessary and Sufficient Conditions for an Euler Path.

**THEOREM 3.4.1.** *A connected, undirected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.*

#### Discussion

Now you can determine precisely when a graph has an Euler path. If the graph has an Euler circuit, then it has an Euler path (why?). If it does not have an Euler circuit, then we check if there are exactly two vertices of odd degree.

**PROOF OF THEOREM 3.4.1.** Suppose  $G$  is a connected multigraph that does not have an Euler circuit. If  $G$  has an Euler path, we can make a new graph by adding on one edge that joins the endpoints of the Euler path. If we add this edge to the Euler path we get an Euler circuit. Thus there is an Euler circuit for our new graph. By the previous theorem, this implies every vertex in the new graph has even degree. However, this graph was obtained from  $G$  by adding the one edge between distinct vertices. This edge added one to the degrees of these two vertices. Thus in  $G$  these vertices must have odd degree and are the only vertices in  $G$  with odd degree.

Conversely, suppose  $G$  has exactly two vertices with odd degree. Again, add an edge joining the vertices with odd degree. The previous theorem tells us there is an Euler circuit. Since it is a circuit, we could consider the circuit as one which begins and ends at one of these vertices where the degree is odd in  $G$ . Now, remove the edge

we added earlier and we get  $G$  back and an Euler path in  $G$ .

□

### 3.5. Hamilton Circuits.

DEFINITIONS 3.5.1.

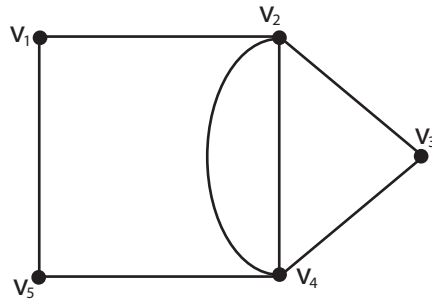
- (1) A **Hamilton path** is a path in a graph  $G$  that passes through every vertex exactly once.
- (2) A **Hamilton circuit** is a Hamilton path that is also a circuit.

#### Discussion

The difference between a Hamilton path and an Euler path is the Hamilton path must pass through each *vertex* exactly once and we do not worry about the edges, while an Euler path must pass through every *edge* exactly once and we do not worry about the vertices.

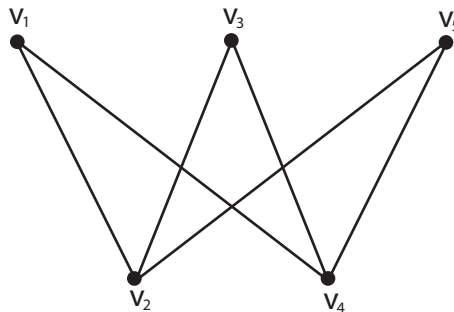
### 3.6. Examples.

EXAMPLE 3.6.1. The circuit  $v_1, v_2, v_3, v_4, v_5, v_1$  is a Hamilton circuit (and so a path



too).

EXAMPLE 3.6.2. This graph has no Hamilton circuit, but  $v_1, v_2, v_3, v_4, v_5$  is a Hamilton path.



**3.7. Sufficient Condition for a Hamilton Circuit.**

**THEOREM 3.7.1.** *Let  $G$  be a connected, simple graph with  $N$  vertices, where  $N \geq 3$ . If the degree of each vertex is at least  $n/2$ , then  $G$  has a Hamilton circuit.*

## Discussion

Unfortunately, there are no necessary and sufficient conditions to determine if a graph has a Hamilton circuit and/or path. Fortunately, there are theorems that give sufficient conditions for the existence of a Hamilton circuit. Theorem 3.7.1 above is just one example.