2. Representing Boolean Functions

2.1. Representing Boolean Functions.

DEFINITIONS 2.1.1.

1. *A* **literal** *is a Boolean variable or the complement of a Boolean variable.*
2. *A* **minterm** *is a product of literals. More specifically, if there are n variables, $x_1, x_2, \dots x_n$, a minterm is a product $y_1 y_2 \cdots y_n$ where y_i is x_i or $\overline{x}_i$.*
3. *A* **sum-of-products expansion** *or* **disjunctive normal form** *of a Boolean function is the function written as a sum of minterms.*

Discussion

Consider a particular element, say $(0, 0, 1)$, in the Cartesian product B^3. There is a unique Boolean product that uses each of the variables x, y, z or its complement (but not both) and has value 1 at $(0, 0, 1)$ and 0 at every other element of B^3. This product is $\overline{x}\,\overline{y}z$.

This expression is called a *minterm* and the factors, $\overline{x}$, $\overline{y}$, and z, are *literals*. This observation makes it clear that one can represent *any* Boolean function as a *sum-of-products* by taking Boolean sums of all minterms corresponding to the elements of B^n that are assigned the value 1 by the function. This sum-of-products expansion is analogous to the *disjunctive normal form* of a propositional expressions discussed in *Propositional Equivalences* in MAD 2104.

2.2. Example 2.2.1.

EXAMPLE 2.2.1. *Find the disjunctive normal form for the Boolean function F defined by the table*

x,	y	z	$F(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Solution: $F(x, y, z) = \overline{x}y\overline{z} + x\overline{y}\,\overline{z} + x\overline{y}z$

Discussion

The disjunctive normal form should have three minterms corresponding to the three triples for which F takes the value 1. Consider one of these: $F(0,1,0) = 1$. In order to have a product of literals that will equal 1, we need to multiply literals that have a value of 1. At the triple $(0,1,0)$ the literals we need are $\overline{x}$, y, and $\overline{z}$, since $\overline{x} = y = \overline{z} = 1$ when $x = 0$, $y = 1$, and $z = 0$. The corresponding minterm, $\overline{x}y\overline{z}$, will then have value 1 at $(0,1,0)$ and 0 at every other triple in B^3. The other two minterms come from considering $F(1,0,0) = 1$ and $F(1,0,1) = 1$. The sum of these three minterms will have value 1 at each of $(1,0,0)$, $(0,1,0)$, $(1,0,1)$ and 0 at all other triples in B^3.

2.3. Example 2.3.1.

EXAMPLE 2.3.1. *Simply the expression*

$$F(x,y,z) = \overline{x}y\overline{z} + x\overline{y}\,\overline{z} + x\overline{y}z$$

using properties of Boolean expressions.

<u>*Solution.*</u>

$$\begin{aligned}\overline{x}y\overline{z} + x\overline{y}\,\overline{z} + x\overline{y}z &= \overline{x}y\overline{z} + x\overline{y}(\overline{z} + z)\\ &= \overline{x}y\overline{z} + x\overline{y}\cdot 1\\ &= \overline{x}y\overline{z} + x\overline{y}\end{aligned}$$

Discussion

Example 2.3.1 shows how we might simplify the function we found in Example 2.2.1. Often sum-of-product expressions may be simplified, but any nontrivial simplification will produce an expression that is *not* in sum-of-product form. A sum-of-products form must be a sum of *minterms* and a minterm must have each variable or its compliment as a factor.

EXAMPLE 2.3.2. *The following are examples of "simplifying" that changes a sum-of-products to an expression that is not a sum-of-products:*

$$\begin{aligned}\textit{sum-of-product form:}\quad & x\overline{y}z + x\overline{y}\,\overline{z} + xyz\\ \textit{NOT sum-of-product form:}\quad & = x\overline{y} + xyz\\ \textit{NOT sum-of-product form:}\quad & = x(\overline{y} + yz)\end{aligned}$$

EXERCISE 2.3.1. *Find the disjunctive normal form for the Boolean function, G, of degree 4 such that $G(x_1,x_2,x_3,x_4) = 0$ if and only if at least 3 of the variables are 1.*

2.4. Functionally Complete.

DEFINITION 2.4.1. *A set of operations is called* **functionally complete** *if every Boolean function can be expressed using only the operations in the set.*

Discussion

Since every Boolean function can be expressed using the operations $\{+, \cdot, ^-\}$, the set $\{+, \cdot, ^-\}$ is *functionally complete.* The fact that every function may be written as a sum-of-products demonstrates that this set is functionally complete.

There are many other sets that are also functionally complete. If we can show each of the operations in $\{+, \cdot, ^-\}$ can be written in terms of the operations in another set, S, then the set S is functionally complete.

2.5. Example 2.5.1.

EXAMPLE 2.5.1. *Show that the set of operations* $\{\cdot, ^-\}$ *is functionally complete.*

PROOF. Since $\cdot$ and $^-$ are already members of the set, we only need to show that $+$ may be written in terms of $\cdot$ and $^-$.

We claim

$$x + y = \overline{\overline{x} \cdot \overline{y}}.$$

Proof of Claim Version 1

$\overline{\overline{x} \cdot \overline{y}}$	$= \overline{\overline{x}} + \overline{\overline{y}}$	De Morgan's Law
	$= x + y$	Law of Double Complement

Proof of Claim Version 2

x	$y,$	$\overline{x}$	$\overline{y}$	$\overline{x} \cdot \overline{y}$	$\overline{\overline{x} \cdot \overline{y}}$	$x + y$
1	1	0	0	0	1	1
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	1	0	0

□

Discussion

EXERCISE 2.5.1. *Show that* $\{+, ^-\}$ *is functionally complete.*

EXERCISE 2.5.2. *Prove that the set $\{+, \cdot\}$ is not functionally complete by showing that the function $F(x) = \overline{x}$ (of order 1) cannot be written using only x and addition and multiplication.*

2.6. NAND and NOR.

DEFINITIONS 2.6.1.

1. *The binary operation* **NAND**, *denoted* $|$, *is defined by the table*

x	y	$x\|y$
1	1	0
1	0	1
0	1	1
0	0	1

2. *The binary operation* **NOR**, *denoted* $\downarrow$, *is defined by the table*

x	y	$x \downarrow y$
1	1	0
1	0	0
0	1	0
0	0	1

Discussion

Notice the NAND operator may be thought of as “not and” while the NOR may be thought of as “not or.”

EXERCISE 2.6.1. *Show that $x|y = \overline{x \cdot y}$ for all x and y in $B = \{0, 1\}$.*

EXERCISE 2.6.2. *Show that $\{|\}$ is functionally complete.*

EXERCISE 2.6.3. *Show that $x \downarrow y = \overline{x + y}$ for all x and y in $B = \{0, 1\}$.*

EXERCISE 2.6.4. *Show that $\{\downarrow\}$ is functionally complete.*