

5. Minimizing Circuits

5.1. Minimizing Circuits. A circuit is minimized if it is a sum-of-products that uses the least number of products of literals and each product contains the least number of literals possible to produce the desired output.

The laws of Boolean algebra can often be used to simplify a complicated Boolean expression, particularly the sum-of-products expressions that we have used to represent Boolean functions. Any such simplification also leads to a simplification of combinatorial circuits.

5.2. Example 5.2.1.

EXAMPLE 5.2.1. $F(x, y, z) = xyz + xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}$

Minimized Expression: $xy + \bar{y}\bar{z}$.

Discussion

We can simplify the expression in Example 5.2.1 as follows.

$$\begin{aligned} (xyz + xy\bar{z}) + (x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}) &= xy(z + \bar{z}) + \bar{y}\bar{z}(x + \bar{x}) \\ &= xy(1) + \bar{y}\bar{z}(1) = xy + \bar{y}\bar{z} \end{aligned}$$

There are various methods that do not require “ad-hoc” manipulation of variables for simplifying expressions and we cover two of the basic methods here: Karnaugh Maps and Quine McCluskey.

5.3. Karnaugh Maps.

- (1) For a given number of variables, n , form a table containing 1’s for all possible minterms arranged so the minterms that are adjacent (left to right or above and below) differ by only a single literal.
- (2) Circle blocks of adjacent 1’s. Start with the largest possible block with *number of rows and columns powers of 2*. Continue circling the largest block possible so that every 1 is within at least one block.
- (3) The simplified form is the sum of the products represented by each block.

5.4. Two Variables. The Table below shows how the Karnaugh map represents the minterms for two variables.

	y	\bar{y}
x	xy	$x\bar{y}$
\bar{x}	$\bar{x}y$	$\bar{x}\bar{y}$

EXAMPLE 5.4.1 (Example 1: Two Variables). *Simplify $xy + x\bar{y} + \bar{x}y$.*

Solution. *First create a table with 1's corresponding to each minterm.*

	y	\bar{y}
x	1	1
\bar{x}	1	

Next circle blocks of size 2×2 , 1×2 , and/or blocks of size 1×1

	y	\bar{y}
x	1	1
\bar{x}	1	

Last, write the sum of the terms represented by the circled blocks.

$$x + y$$

Discussion

When using the Karnaugh maps to simplify functions of two variable we try to find blocks of 1's. If all 4 squares have ones, we have a 2×2 block we circle. Otherwise look for 1×2 blocks of ones. Circle any block of this size possible. If there is a 1 not covered, see if it is in a 1×2 block. If it is circle the block, if not circle the one. Continue in this manner until all the ones are circled.

Now, suppose the blocks representing xy and $x\bar{y}$ are circled. The product that corresponds to this block is x since $xy + x\bar{y} = x(y + \bar{y}) = x$. A method of determining the product corresponding to a block is to look for the literal(s) that are the same in every entry in that block.

5.5. Three Variables. The Table below shows how the Karnaugh map represents the minterms for three variables.

	yz	$y\bar{z}$	$\bar{y}z$	$\bar{y}\bar{z}$
x	xyz	$xy\bar{z}$	$x\bar{y}z$	$x\bar{y}\bar{z}$
\bar{x}	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}z$	$\bar{x}\bar{y}\bar{z}$

The Karnaugh maps may be set up with the variables in different arrangements than the one in the above chart, but they *must* be set up so that adjacent squares differ by only one literal. Notice the entries in the left column differ from the ones in the right column by a single variable. We consider the top left adjacent to the top right and the bottom left adjacent to the bottom right. If you imagine rolling the chart and taping the red edges you have a tube which best represents this Karnaugh map.

EXAMPLE 5.5.1. *Example 2: Three Variables*

Simplify $xyz + x\bar{y}z + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$.

Solution: *First create a table with 1's corresponding to each minterm.*

	yz	$y\bar{z}$	$\bar{y}z$	$\bar{y}\bar{z}$
x	1			1
\bar{x}	1	1	1	1

Next circle blocks of size 2×4 , 1×4 , 2×2 , 1×2 and/or blocks of size 1×1

	yz	$y\bar{z}$	$\bar{y}z$	$\bar{y}\bar{z}$
x	1			1
\bar{x}	1	1	1	1

Last, write the sum of the terms represented by the circled blocks.

$$z + \bar{x}$$

Discussion

To simplify we check as follows

- (1) If all the squares have ones the function simplifies to 1.
- (2) Check for blocks of size 2×2 , 4×1 , 2×1 , or 1×1 . Circle the largest block you can find.
- (3) If there is a 1 not circled, find the largest possible block containing it and if possible try to use a block that contains other 1's that are not circled.
- (4) continue until all the 1's have been circled.
- (5) Write down the sum of the products represented by the circled blocks.

5.6. Four Variables. The Table below shows how the Karnaugh map represents the minterms for four variables.

	uv	$u\bar{v}$	$\bar{u}\bar{v}$	$\bar{u}v$
xy	$xyuv$	$xyu\bar{v}$	$xy\bar{u}\bar{v}$	$xy\bar{u}v$
$x\bar{y}$	$x\bar{y}uv$	$x\bar{y}u\bar{v}$	$x\bar{y}\bar{u}\bar{v}$	$x\bar{y}\bar{u}v$
$\bar{x}\bar{y}$	$\bar{x}\bar{y}uv$	$\bar{x}\bar{y}u\bar{v}$	$\bar{x}\bar{y}\bar{u}\bar{v}$	$\bar{x}\bar{y}\bar{u}v$
$\bar{x}y$	$\bar{x}yuv$	$\bar{x}y u\bar{v}$	$\bar{x}y\bar{u}\bar{v}$	$\bar{x}y\bar{u}v$

As with the map with three variables, we consider the squares on the left adjacent to the corresponding ones on the right. In addition we wish to consider the top

row entries adjacent to the corresponding entries in the bottom row. This may be visualized by rolling the chart so we tape the red edges together to create a tube. Now curve the tube around so the blue edges may be taped together to create a donut shape (also called a torus).

EXAMPLE 5.6.1 (Example 3: Four Variables). *Simplify $xyuv + xy\bar{u}\bar{v} + x\bar{y}u\bar{v} + x\bar{y}\bar{u}\bar{z} + \bar{x}\bar{y}u\bar{v} + \bar{x}\bar{y}\bar{u}\bar{z} + \bar{x}yuv + \bar{x}y\bar{u}\bar{v}$.*

Solution. *First create a table with 1's corresponding to each minterm.*

	uv	$u\bar{v}$	$\bar{u}\bar{v}$	$\bar{u}v$
xy	1			1
$x\bar{y}$		1	1	
$\bar{x}\bar{y}$		1	1	
$\bar{x}y$	1			1

Next circle blocks of size 4×4 , 2×4 , 1×4 , 2×2 , 1×2 and/or blocks of size 1×1

	uv	$u\bar{v}$	$\bar{u}\bar{v}$	$\bar{u}v$
xy	1			1
$x\bar{y}$		1	1	
$\bar{x}\bar{y}$		1	1	
$\bar{x}y$	1			1

Last, write the sum of the terms represented by the circled blocks.

$$yv + \bar{y}\bar{v}$$

Discussion

This time we check for blocks of size 4×4 , 2×4 , 2×2 , 4×1 , 2×1 , or 1×1 .

The Karnaugh maps begin to lose their ease of use when we move to more variables. It is still possible to use Karnaugh maps for four variables, but the adjacent squares are more difficult to visualize.

EXERCISE 5.6.1. *Draw Karnaugh maps for five variables and identify which blocks are adjacent.*

5.7. Quine-McCluskey Method. The Quine-McCluskey method is a method used for simplifying the conjunctive normal form of a Boolean expression. This method is easier to program than the Karnaugh Maps.

EXAMPLE 5.7.1. *Simplify the expression*

$$xyuv + xy\bar{u}v + x\bar{y}u\bar{v} + x\bar{y}\bar{u}\bar{v} + \bar{x}y\bar{u}\bar{v} + \bar{x}\bar{y}\bar{u}\bar{v} + \bar{x}yuv$$

Solution.

- (1) Represent each minterm by a binary string. Use 1 for the variable and 0 for the complement:
- (2) Make a table listing the minterms, the binary strings, and the number of 1's in the strings. Enumerate the list.

	Minterm	String	no. of ones
1	$xyuv$	1111	4
2	$xy\bar{u}v$	1101	3
3	$\bar{x}yuv$	0111	3
4	$x\bar{y}u\bar{v}$	1010	2
5	$\bar{x}\bar{y}u\bar{v}$	0010	1
6	$x\bar{y}\bar{u}\bar{v}$	1000	1
7	$\bar{x}\bar{y}\bar{u}\bar{v}$	0000	0

- (3) Find the strings that differ by only one bit. Replace the different bit with a dash and remove the variable from the product that corresponds to that bit.
- (4) Create a new table by listing the combined product, the new product, and the binary string.

	Product	String
(1, 2)	xyv	11 - 1
(1, 3)	yuv	-111
(4, 5)	$\bar{y}u\bar{v}$	-010
(4, 6)	$x\bar{y}\bar{v}$	10 - 0
(5, 7)	$\bar{x}\bar{y}\bar{v}$	00 - 0
(6, 7)	$\bar{y}\bar{u}\bar{v}$	-000

- (5) Use the previous table to continue to reduce the products by finding strings that differ by one bit.

	Product	String
$((4, 5), (6, 7))$	$\bar{y}\bar{v}$	$-0 - 0$

- (6) Since there are no bit strings in the last table that differ by a single bit we are ready to find the simplified expression.
- (7) From the last table we have the string $\bar{y}\bar{v}$. This simplified the minterms numbered 4 through 7. From the table before we look for strings that simplify the minterms numbered 1 through 3. We use xyv and yuv to “cover” the minterms 1, 2, and 3. The simplified form is

$$\bar{y}\bar{v} + xyv + yuv$$

Discussion

Notice that a string with a certain number of 1's will have exactly one more or one less 1 if one of the bits are changed. Thus when we are looking for bit strings that differ by only one bit we only need to look at the bits that have exactly one more or one less 1. This is the only reason we add a column with the number of 1's in it. We could also add this column in the later tables.

Combining (4, 5) and (6, 7) gives us the same string as we get by combining (4, 6) and (5, 7), so we only use one of these combinations. Any time the numbers are the same we will obtain the same string.

We continue creating tables until we get to a table where none of the bit strings in that table differ by a single bit (including the dashes). Once we have all the tables created we need to “cover” all the minterms. In this example, we start with the last table and use the strings given in that table to cover the minterms numbered 4, 5, 6, 7. Since this was the only product in the last table we now move to the second to the last table and look one or more products that cover the remaining minterms, 1, 2, and 3. Since there is a product that covers 1 and 2 we will use it. Now we have to cover 3. Any product that covers 3 will do, but there is only one choice in this table so we use it. If there had not been a product in the second to the last table that covered 3, we would move up to the previous table.

It is possible to an expression to have more than one minimal expression. It is also possible that there are several choices of strings from a given table that could be used to cover the minterms. We chose the combinations that use the fewest possible strings to cover the most minterms.