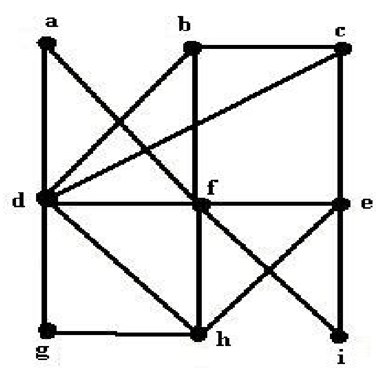
MAD 3105, Section 1 - Midterm 1 Solutions

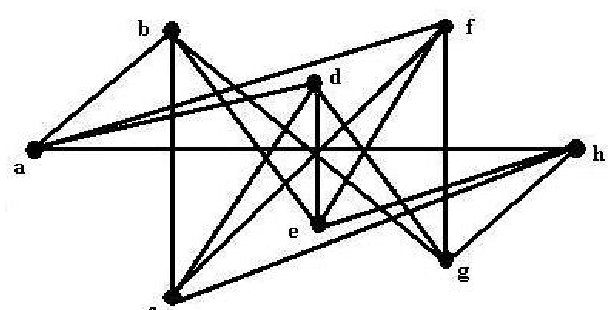
1. (20 points) Give precise definitions of the following:
   (a) A simple path in a graph: A path that repeats no edges.
   (b) The level number of a vertex in a rooted tree: The number of edges in the simple path from the root to the vertex.
   (c) A full m-ary tree: A rooted tree in which every non-leaf vertex has m children, and such that every leaf has the same level number.
   (d) A Hamilton graph: A graph that contains a Hamilton circuit.
   (e) The relation " $\leq$ " on a Boolean algebra: $x \leq y$ iff $x \vee y = y$.

2. (14 points) Does the graph given below have an Euler path? If so, then give a vertex sequence of one. If not, then explain why.



Solution: The given graph has exactly two vertices b and c of degree 3 and hence must have an Euler path. In order to construct an Euler path, add an extra edge between b and c so that every vertex will now have even degree in this new graph. Now we can apply the Euler circuit algorithm to the new graph to obtain an Euler circuit *bdghdfadcbfiehfecb*. Delete the extra edge bc that was added to obtain an Euler path in the original graph *bdghdfadcbfiehfec*.

3. For the graph given below, answer the following:



   (a) (10 points) Is the graph bipartite? If it is, then give the partitioning sets for the vertices. If it isn't, then explain why.
   Solution: The graph is bipartite with partitioning sets $V_1 = \{a, c, e, g\}$ and $V_1 = \{b, d, f, h\}$.

(b) (8 points) Give a Hamilton circuit or explain why none exists.

Solution: One possible Hamilton circuit is *abcdefgha*.

4. (a) (6 points) How many leaves does a full 7-ary tree of height 10 have? How many non-leaf vertices does this tree have?

   Solution: As seen in class, the number of leaves a full 7-ary tree of height 10 is 7^{10}, and the number of non-leaf vertices is $\frac{7^{10}-1}{7-1}$.

   (b) (8 points) Let T be a full m-ary tree. Let l be the number of leaves in T and let k be the number of non-leaf vertices in T. Show that $l = (m-1)k + 1$. (Note that this is independent of the height of the tree.)

   Solution: We note that the # of leaves l in a full m- ary tree and the # of non-leaf vertices k satisfy $k = \frac{l-1}{m-1}$. Solving for l yields $l = (m-1)k + 1$.

5. For all x, y and z in a Boolean algebra, the five axioms and properties 6a and 7a of a Boolean algebra are given below (their dual statements are not given. However, you are welcome to use them.):

   1a. $x \vee y = y \vee x$.
   2a. $(x \vee y) \vee z = x \vee (y \vee z)$.
   3a. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
   4a. $x \vee 0 = x$.
   5a. $x \vee x' = 1$.
   6a. $x \vee x = x$.
   7a. $x \vee 1 = 1$.

   (a) (10 points) Prove property 8a: $x \vee (x \wedge y) = x$. State which of the axioms or properties you have used in deducing your statements.

   Solution:

$$\begin{aligned} x \vee (x \wedge y) &= (x \wedge 1) \vee (x \wedge y) \\ &= x \wedge (1 \vee y) \\ &= x \wedge 1 \\ &= x. \end{aligned}$$

   (b) (10 points) Prove that if $x \vee y = y$, then $x \wedge y = x$. State which of the axioms or properties you have used. You can use property 8 if you want.

   Solution:

$$\begin{aligned} x \wedge y &= x \wedge (x \vee y) \\ &= x. \end{aligned}$$

6. (14 points) Write the function $f : \mathbb{B}^3 \to \mathbb{B}$ defined by $f(x, y, z) = xz' \vee xy' \vee x'yz$ as a join of atoms in BOOL(3).

   Solution: $f(x, y, z) = xyz' \vee xy'z \vee x'yz \vee xy'z'$. You should draw a table to see this.

   Bonus (6 points): How many Hamilton circuits does $K_{4,4}$ have if we count a Hamilton circuit as different if it has a different starting point or if it has a different vertex sequence?

   Solution: $K_{4,4}$ has $2(4!)^2$ different Hamilton circuits.