Cubic and face centered cubic lattice sphere packing densities

How do you compute the density of a lattice sphere packing? You would like to compute the ratio of volume covered by spheres to the volume of space. Since both of these are infinity, you have to cut space up into finite "tiles."

For the cubic (integer) lattice

$$\mathbf{Z}^3 = \{(x_1, x_2, x_3) \,|\, x_i \in \mathbf{Z}, i = 1, 2, 3\}$$

a sphere packing can be made by putting a sphere of radius $1/2$ and volume $\pi/6$ centered at every lattice point. Space can be divided into cubes of side 1 and volume 1, each containing one sphere. So the density is $\pi/6 \approx .52$.

(Recall that the volume of a sphere of radius r is $4\pi r^3/3$.)

The face centered cubic lattice is the set of all lattice points in $\mathbf{Z}^3$ such that $x_1+x_2+x_3$ is an even integer. A sphere packing can be made from this lattice by putting spheres of radius $\sqrt{2}/2$ and volume $2\pi/(3\sqrt{2})$ centered at every lattice point. Now space can be divided into cubes of side 2 and volume 8 each containing 1 sphere and 12 quarter spheres (see figure). Thus the density is $\pi/(3\sqrt{2}) \approx .74$.

So the FCC lattice packing is denser that the cubic lattice packing. In fact, it is the densest sphere packing in three dimensions.