The Orbifold Notation for Surface Groups

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"Even quite ungainly objects, like chairs and tables, will become almost spherical if you wrap them in enough newspaper."

The symmetries of any finite object, such as a chair or a table, all fix a point, say the centre of gravity of the object, and so act on the surface of a sphere, for example any sphere centred on the centre of gravity.

The symmetries of a repeating pattern on a carpet or tiled floor, or on a wall, supposed continued to infinity, will probably constitute one of the 17 plane crystallographic groups.

Among the works of the Dutch draughtsman Maurits C. Escher, one can find examples of all these 17 groups, and also some even more interesting designs such as *Circle Limit I, II, ...*, whose symmetries are various discrete groups of isometries of the hyperbolic plane.

In this paper a **surface group** will be a discrete group of isometries of one of the following three surfaces:

1. the sphere
2. the Euclidean plane
3. the hyperbolic plane.

These are all the simply-connected surfaces of constant Gaussian curvature.

We shall present a simple and uniform notation that describes all three types of group. Since this notation is based on the concept of *orbifold* introduced by Bill Thurston, we shall call it **the orbifold notation**.

Roughly speaking, an orbifold is the quotient of a manifold by a discrete group acting on it. It therefore has one point for each orbit of the group on the manifold (*Orbifold = Orbit-manifold*).

Mirrors and mirror-boundaries

An orbifold may have boundary curves even though our three original surfaces do not. The boundary points arise from points lying on mirrors.

A **mirror** or **mirror-line** for a group is the line fixed by some reflection in the group. A point that lies on a mirror is called a **mirror-point**, an **ordinary mirror-point** if it lies on just one mirror, and an **m-fold mirror-point** if it lies on exactly m mirrors.

The points of an orbifold that correspond to mirror-points are boundary points. An **ordinary** boundary point is the image of an ordinary mirror-point, and a **type m corner-point** is the image of an m-fold mirror-point. At a type m corner, the boundary has an angle $\frac{\pi}{m}$.

We shall say that a boundary-curve has type $*ab\ldots c$ to mean that its corners have types $a, b, \ldots, c$, reading around the curve in some consistent direction.

Tables and chairs

A plain rectangular table has two planes of symmetry, which divide the sphere into four segments. We can consider any one of these segments as the orbifold– its boundary consists of two semicircles that intersect each other at the zenith and at the nadir, at angle $\frac{\pi}{2}$. So this boundary curve has type $*22$, and indeed $*22$ is the orbifold notation for the symmetry group of the table.

A plain square table has two further (diagonal) planes of symmetry, and the four symmetry planes divide the sphere into eight segments, the typical segment having two corners at angle $\frac{\pi}{4}$. This time the symmetry group is $*44$.

A chair has a single plane of symmetry, which cuts the sphere in a great circle, that is to say, a boundary curve without corners, type $*$. We might also write $1*$, so as to give the star something to hang on to– digits 1 have no significance in this notation, except as place-fillers.

Gyration-points and cone points

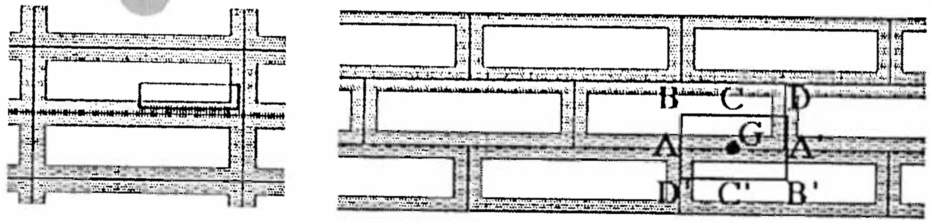
An orbifold may have some special points that do not lie on boundary curves. A **gyration** is a rotation in the group whose centre does *not* lie on any mirror. A point of the surface is called an **m-fold gyration point** if it is the centre of some gyration of order m, but not of any gyration of higher order. The image in the orbifold of an m-fold gyration point is called a **cone-point** of order m– it is a point around which the angle is $\frac{2\pi}{m}$ rather than 2π.

Some brick walls

The simple brick wall consisting of bricks laid directly above and directly to the side of each other in a rectangular array, has four types of mirror-line, namely

vertical in brick, horizontal in brick
vertical in cement, horizontal in cement.

The orbifold can be identified with one quarter-brick (with a bit of cement adhering to it), whose boundary is a rectangle, type $*2222$, and this is the orbifold notation for the symmetry group of the wall.



*2222

2*22

Figure 1

A more common, because stronger, kind of wall, has the bricks staggered by half a length in adjacent layers.

This time we cannot directly identify the orbifold with a portion of the wall. In Figure 1, we have outlined a rectangle $ABCDA'B'C'D'$ bounded by mirrors, but the centre G of this rectangle does not lie on any mirror, and is a 2-fold gyration point, since the appearance of the wall would be undisturbed if it were rotated through half a revolution about G. The orbifold is obtained from this rectangle by identifying each point of the rectangle (including the interior points) with its image under the gyration about G. It is realisable as a conical surface with a cone point of order 2 at G, and a boundary curve that has just *two* right-angled corners, at $B = B'$ and $D = D'$. The orbifold notation for this group is 2*22, the initial 2 representing the cone-point, and the remainder of the symbol the boundary curve with its two corners.

The orbifold notation in general

The orbifold notation for any surface group consists of a number of digits $A, B, \ldots, C$, $a, b, \ldots, c$, $\alpha, \beta, \ldots, \gamma, \ldots$, together with some circles, stars and crosses:

$$\circ\circ\ldots\circ \quad AB\ldots C \quad *ab\ldots c \quad *\alpha\beta\ldots\gamma \quad \ldots \quad \times\times\ldots\times$$

Each initial circle represents a handle, and each final circle a crosscap. The digits $A, B, \ldots, C$ *not* preceded by stars are the orders of the distinct cone-points on the orbifold, and each star together with all the digits that immediately follow it indicates the type of a boundary curve.

We remind the reader that the classification theorem for 2-manifolds assures us that any connected compact 2-manifold can be obtained by adding handles and crosscaps to a sphere, and then punching a hole in it for each boundary curve. If we also take into account the various possibilities for local collapse, we can see that the above notation covers all possibilities for the orbifold of a surface group.

It can also be shown that every such notation does correspond to the orbifold of some group, with just the exceptions mn and $*mn$ ($m > n \geq 1$). (If $n = 1$, these exceptions appear as m and $*m$.)

The Euler characteristic of an orbifold

The Euler characteristic of an orbifold generalises the Euler characteristic ($V - E + F$) of a manifold in the natural way. It is a rational number which coincides with the usual integer-valued characteristic for an ordinary manifold without cone-points, and whose value gets divided by $|G|$ when we take the quotient by a discretely acting group G, of finite order $|G|$. From a 'map' it can be computed by the usual formula $V - E + F$, provided we make proper allowances for the divided nature of the vertices, edges, and faces, namely:-

a 'vertex'	at a cone-point of order m counts as $\frac{1}{m}$ of a vertex, while
a 'vertex'	at an ordinary boundary point is $\frac{1}{2}$ of a vertex, and
a 'vertex'	at a type m corner-point is $\frac{1}{2m}$ of a vertex. Also
an 'edge'	running along a boundary curve is $\frac{1}{2}$ of an edge, and finally
a 'face'	with just one internal cone-point, of order m, is $\frac{1}{m}$ of a face.

For example, for the orbifold from our second brick wall, we shall take the 'map' to consist of

$\frac{1}{4} + \frac{1}{4}$ vertices,	namely the corner-points $B = B'$ and $D = D'$
$\frac{1}{2} + \frac{1}{2}$ edges,	the images of $D'B$ and BD, and
$\frac{1}{2}$ of a face,	the image of the rectangle $ABCDA'B'C'D'$.

The Euler characteristic is therefore

$$(\frac{1}{4} + \frac{1}{4}) - (\frac{1}{2} + \frac{1}{2}) + \frac{1}{2} = \frac{1}{2} - 1 + \frac{1}{2} = 0.$$

There are other ways to compute the characteristic, namely from the Gauss-Bonnet formula as the integrated Gaussian curvature over the entire orbifold (in this example the curvature was everywhere 0), or directly from the orbifold notation, in the manner we shall now describe.

A visit to SymmetryLand

We suppose that every day of our holiday, we start with \$2 in our pocket (because the Euler characteristic of the sphere is 2), and go on a visit to SymmetryLand.

SYMMETRYLAND TICKET CHARGES

Ticket type	Symbol	Cost of ticket for Adult:	Child:
2-trip	2	$\frac{1}{2}$	$\frac{1}{4}$
3-trip	3	$\frac{2}{3}$	$\frac{1}{3}$
4-trip	4	$\frac{3}{4}$	$\frac{3}{8}$
5-trip	5	$\frac{4}{5}$	$\frac{2}{5}$
6-trip	6	$\frac{5}{6}$	$\frac{5}{12}$
n-trip	n	$\frac{(n-1)}{n}$	$\frac{(n-1)}{2n}$
TOP ticket	∘ or ×	2	1
Chaperone's	*	–	1

SYMMETRYLAND RULES:

- Children not in possession of TOP tickets must be chaperoned.
- A chaperone's ticket entitles the bearer to enter SymmetryLand alone, or in charge of any number of children, in which case the chaperone is responsible for keeping their behaviour within acceptable boundaries.
- SymmetryLand extends credit to regular visitors.

Every orbifold notation can be regarded as specifying a possible day's tickets at SymmetryLand, and its Euler characteristic is the amount of change we shall have in our pocket at the end of the day.

For example, 2*22 corresponds to a day when we purchase one adult's 2-trip ticket, one chaperone's ticket, and two child's 2-trip ones.

The total cost in dollars is $\frac{1}{2}+1+\frac{1}{4}+\frac{1}{4}=2$ so that the change from our \$2 is 0, agreeing with our earlier answer.

(It is fairly easy to justify this rule. It is well-known that the Euler characteristic of the sphere is 2, and that handles and crosscaps cause reductions by 2 and 1, respectively. Now punching a hole can be viewed as removing 1 face, so causing a decrease of 1, while changing an ordinary point to a conical point of order n can be viewed as replacing 1 whole vertex by $\frac{1}{n}$ of a vertex, causing a decrease of $\frac{(n-1)}{n}$ etc. The map can be chosen so that any point of particular interest is a vertex of it.)

The 17 plane crystallographic groups

It turns out that there are just 17 SymmetryLand outings from which we return flat broke, and these correspond to the 17 discrete groups acting on the flat(Euclidean) plane so that the orbifold has finite area, namely:-

Notation	English name	
*632	hexascope group	(hexatropic kaleidoscope group)
632	hexatrope group	(cheiral hexatropic group)
*442	tetrascope group	(tetratropic kaleidoscope group)
4*2	tetragyro group	(tetratropic gyrational group)
442	tetratrope group	(cheiral tetratropic group)
*333	triscope group	(tritropic kaleidoscope group)
3*3	trigyro group	(tritropic gyrational group)
333	tritrope group	(cheiral tritropic group)
*2222	discope group	(ditropic kaleidoscope group)
2*22	dirhomb group	(ditropic rhomboidal group)
22*	digyro group	(ditropic gyrational group)
22×	diglide group	(ditropic gliding group)
2222	ditrope group	(cheiral ditropic group)
** or 1**	monoscope group	(monotropic kaleidoscope group)
× or 1×	monorhomb group	(monotropic rhomboidal group)
×× or 1××	monoglide group	(monotropic gliding group)
∘ or ∘1	monotrope group	(cheiral monotropic group)

To verify the completeness of the enumeration, we consider applying the following modifications to a given group of characteristic 0:

1. Replace a group $AB\ldots C$ by $*AB\ldots C$ – this halves the characteristic
2. Replace an adult's TOP ticket (∘) by two child's ones
3. Replace a child's TOP ticket (×) by a chaperone's ticket (*)
4. Since a chaperone (*) is now present, replace an adult's n-trip ticket by two child's ones.

After these modifications we have at least one star (and at most two, since they cost \$1), and it is easy to see the group is one of

$$*632, *442, *333, *2222, **.$$

So the other groups are obtainable from these by reversing the above moves, namely:

replacing any two equal digits after the star by one before it;

replacing any final star by a final circle;

replacing two final crosses by one initial circle, or

deleting an initial star that was followed only by digits.

In the table, the groups obtained in these ways from one of the five starting groups above are listed immediately after that group.

The "English names" we give are new, but we recommend them since some thought has gone into their selection. The **tropicity** of a group is the maximal order of any rotation in it. A group is **cheiral** if it contains no reflections or glide-reflections, and **gyrational** if it contains *both* gyrations and reflections.

The finite spherical groups

A finite spherical group corresponds to a day in SymmetryLand from which we return with a positive amount of change. The cases are:-

Geometric name	Coxeter	Orbifold	Algebraic	Algebraic name
icosahedral reflection	$[3,5]$	$*532$	$\pm I$	diplo-icosahedral
icosahedral rotation	$[3,5]^+$	532	I	icosahedral
octahedral reflection	$[3,4]$	$*432$	$\pm O$	diplo-octahedral
octahedral rotation	$[3,4]^+$	432	O	octahedral
tetrahedral reflection	$[3,3]$	$*332$	TO	tetra-octahedral
gyro-octahedral	$[3^+,4]$	$3{*}2$	$\pm T$	diplo-tetrahedral
tetrahedral rotation	$[3,3]^+$	332	T	tetrahedral
prismatic	$[2,n]$	$*22n$ }	{ $\pm D_{2n}$	diplo-dihedral
antiprismatic	$[2^+,2n]$	$2{*}n$ }	{ DD_{4n}	dihedro-dihedral
dihedral	$[2,n]^+$	$22n$	D_{2n}	dihedral
pyramidal	$[n]$	$*nn$	CD_{2n}	cyclo-dihedral
gyro-prismatic	$[2,n^+]$	$n*$ }	{ $\pm C_n$	diplo-cyclic
gyro-antiprismatic	$[2^+,2n^+]$	$n\times$ }	{ CC_{2n}	cyclo-cyclic
cyclic	$[n]^+$	nn	C_n	cyclic

Here we have given no fewer than two systems of English names, and three notations. The reason is that the groups are classified differently according as we take the basic type of reflecting operation to be a **plain reflection**, or the **central inversion** (represented by the matrix -1). The first choice is usually appropriate for more geometrical purposes, and corresponds well to our orbifold notation, and to Coxeter's notation which regards the groups as subgroups of groups generated by reflections. The "geometric" names express the groups in terms of automorphisms of certain polyhedra.

However, for algebraic purposes, it is often useful to take the second approach, whereby the groups are obtained by modifying the pure rotation groups G by multiplying some of their elements by -1. We use the standard adjectives "cyclic", "dihedral", ...,"icosahedral" in the names of these groups.

If we straightforwardly adjoin -1 to G, we obtain the group $\pm G$, called "diplo-G", meaning "the double of G". Instead, we can multiply all those elements of G that lie outside H by -1, where H is a subgroup of index 2 in G, to obtain the group called HG. The "algebraic name" for this group is "adverbo-G", where the adverb describes the subgroup H.

As abstract groups, $\pm G$ is isomorphic to $C_2 \times G$, while HG is isomorphic to G, with rotation subgroup H. So, in the "adverbo–adjectival" name for HG, the adjective refers to the whole group G, and the adverb to its rotation subgroup H. (The rotation subgroup is the subgroup of elements of determinant +1.) The subscripts on our groups are **orders**.

So for example, CD_{24} is abstractly a dihedral group of order 24, whose rotation subgroup is a cyclic group of order 12, and we call it the cyclo-dihedral group of order 24.

The braces indicate that the correspondence between the geometric and algebraic systems is not always the same. One should interchange the two lines to the right of the brace whenever the parameter n is odd. Thus the symmetry group of a regular hexagonal prism, is the prismatic group $[2,6] = *226$, which is the *diplo*-dihedral group $\pm D_{12}$. However the symmetry group of the regular pentagonal prism, namely $[2,5] = *225$ is the *dihedro*-dihedral group DD_{20}. In general, we get the diplo-types from polyhedra with central symmetry, and otherwise the more subtle types.

The hyperbolic groups

We shall only briefly mention these here. They correspond to days at SymmetryLand when we overspend our \$2 allowance, and so return home in debt to the SymmetryLand owners. One such pattern is Escher's *Circle Limit IV*, which consists of alternating black devils and white angels. If one crawls along a mirror of this pattern until one hits another mirror and then turns right along this mirror and continues in the same way, one's path is a quadrilateral with four corners of $\frac{\pi}{3}$, whose centre appears as a 4-fold gyration point. This makes the orbifold notation $4{*}3$, since all four corners of the quadrilateral are identified by the gyration.

However, a more careful examination of the figure shows that every fourth figure– either angel or devil– is facing away from us rather than towards us, so that the indicated gyration is not actually a symmetry at this level of detail, and the group drops to $*3333$. Let us compute characteristics:

For $4{*}3$ we get $2 - \frac{3}{4} - 1 - \frac{1}{3} = -\frac{1}{12}$.

For $*3333$ we get $2 - 1 - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} = -\frac{1}{3}$.

The ratio between these numbers is indeed 4, indicating correctly that the second group $*3333$ has index 4 in $4{*}3$. The latter group has index 2 in the reflection group *642, which could be obtained by adjoining the reflexive symmetries of one of our quadrilaterals (which would interchange angels and devils).

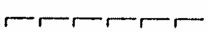
The work *Circle Limit III* is equally intriguing. It shows a pattern of fishes in four colours. There are some white lines that are misleading, since one's first guess is that they are hyperbolic straight lines, whereas they are in fact hypercycles ("circles of super-infinite radius"). These outline some "squares" and "triangles" which alternate rather like the faces of a cuboctahedron, except that at each vertex there are three triangles and three squares. The group is 433, there being gyration points of orders 4 and 3 at the centres of the squares and triangles, and another type of gyration point of order 3 at the vertices of the tessellation.

That, at least, is what we see if we are colourblind. If we take account of the colours, the group drops to 222222, since the square regions are now of six distinct types- each has two fish, each of two colours, and each of the six pairs of colours happens.

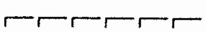
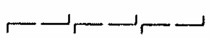
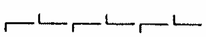
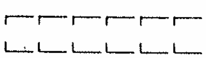
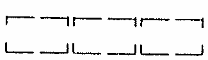
The characteristic of 433 is $2 - \frac{3}{4} - \frac{2}{3} - \frac{2}{3} = -\frac{1}{12}$, while that of 222222 is $2 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -1$. The index is 12, corresponding to the fact that every *even* permutation of the 4 colours is induced by some symmetry of the tessellation.

Stepping around the world, or across the plane

Please regard ┌— as a left footprint, and └— as a right one. Now hop around the equator, taking n hops to do so. Your footprints:



have the symmetry group nn. If you had walked, taking $2n$ full paces, the group would have been $n\times$. The table shows that the 7 infinite families of **axial** groups (the finite groups that fix an axis) correspond to the 7 ways to proceed around the world:

	hop	nn		spinning hop	$22n$
	step	$n\times$			
	jump	$n*$		spinning jump	$*22n$
	sidle	$*nn$		spinning sidle	$2*n$

But these figures could be used with more justice also for the trail left by a (doubly infinite) progression across the Euclidean plane. The resulting groups, which have been called the **frieze groups**, or the **2-dimensional line groups**, are naturally symbolised by replacing the digit n by ∞. They are

$$*22\infty,\ 2*\infty,\ 22\infty,\ *\infty\infty,\ \infty*,\ \infty\times,\ \infty\infty$$

The only other discrete groups acting on the Euclidean plane are the **2-dimensional point groups**. I call them (n) and $(*n)$. Here the parentheses may be considered as standing for the open-ended nature of the resulting orbifold, or as apologising for the slight illegitimacy of the notation. These groups can be thought of as obtained from nn and $*nn$ by letting the radius of the sphere tend to infinity.

The group (n) is generated by a rotation of order n, and the group $(*n)$ by two reflections in lines at angle $\frac{\pi}{n}$. So (4) is the group of the swastika, and (*5) the group of the regular pentagon.

Note

The philosophy that geometrical groups should be studied through their orbifolds is Bill Thurston's. I claim originality only for the simple and elegant notation introduced here. David Singerman tells me that Murray MacBeath has long described the orbifolds of surface groups in a less compact but essentially equivalent way.