The Circular maximal function for Heisenberg radial functions

Jonathan Hickman

Postdoctoral Symposium : Madison

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David Beltran, BCAM

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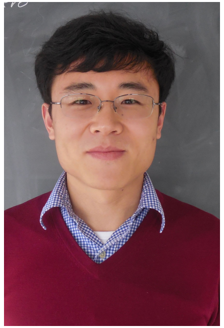

Shaoming Guo, UW Madison

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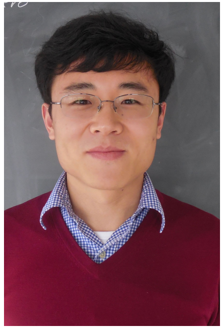

Andreas Seeger, UW Madison

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- Key features in the euclidean problem
  - Rotational and cinematic curvature
- New features in the Heisenberg setting.

1. Definitions and Results:

- The Euclidean Case

- Consider the circular means

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$$M f(x) := \sup_{t>0} A_t |f| (x).$$

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– Local smoothing for wave equation.

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- nth Heisenberg group

$\mathbb{H}^n := \mathbb{R}^{2n+1}$ with group operation

$$(x,u) * (y,v) := (x+y,\ u+v+\omega(x,y)).$$

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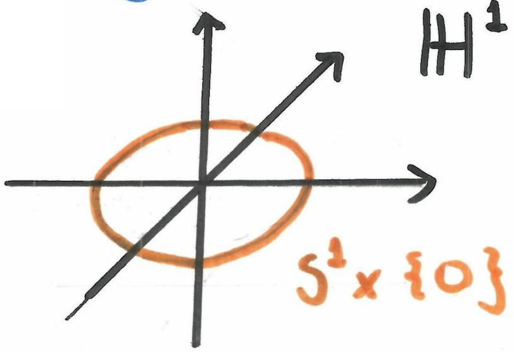

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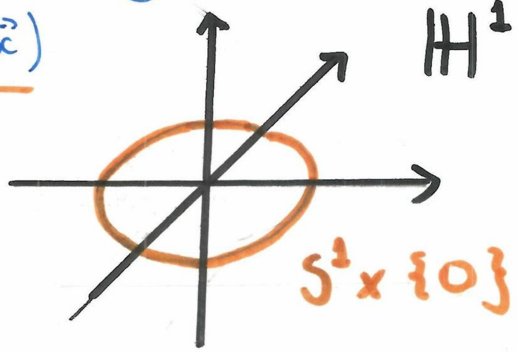

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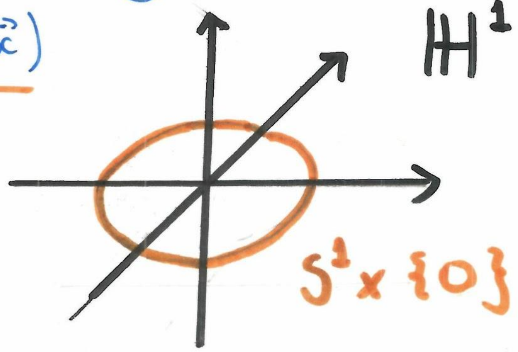

Remark :- There are alternative Heisenberg analogues – e.g. spheres with respect to Korányi norm (c.f. Cowling).

Explicitly,

$$A_t^{\mathbb{H}} f(x,u) = \int_{S^1} f\left(x - ty,\ u - \frac{t}{2}(x_2 y_1 - x_1 y_2)\right) d\sigma(y).$$

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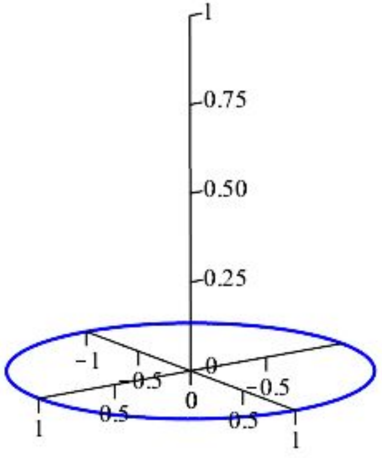

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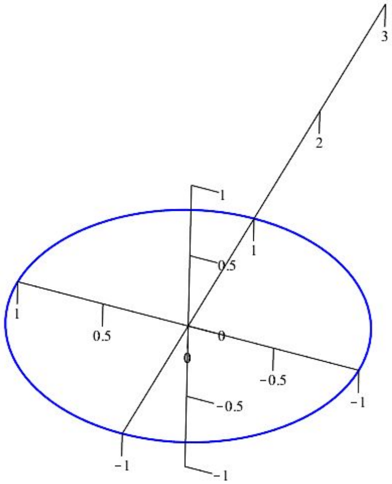

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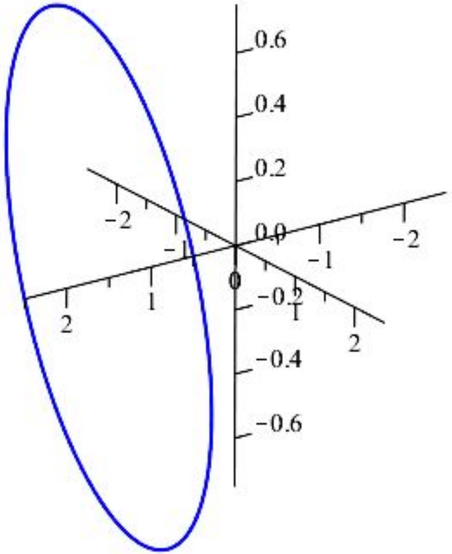

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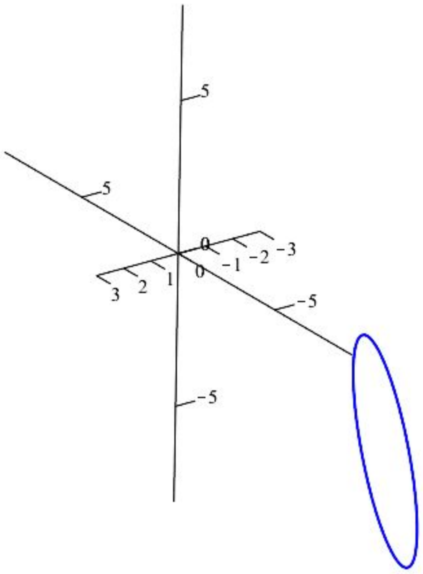
These actions can be combined...



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• More general groups:- Müller–Seeger.

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- Despite the radial assumption, the problem is significantly more involved than the euclidean circular maximal function.
- Equivalent to bounding a variable coefficient maximal function on $\mathbb{R}^2$.
- Fails rotational and cinematic curvature conditions. Non-smooth curve distribution.

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$\boxed{\text{where } \mathcal{A}_t \text{ is an averaging operator associated to a variable family of plane curves.}}$

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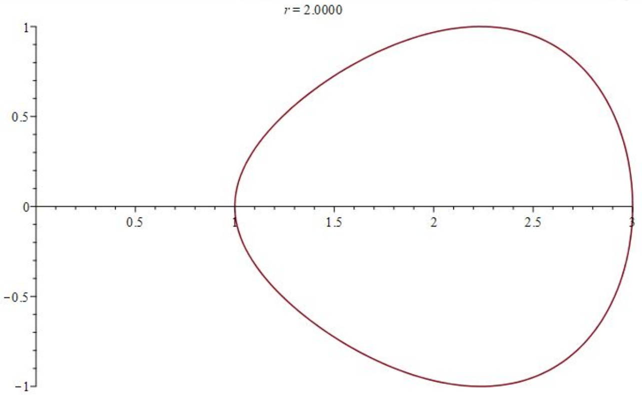

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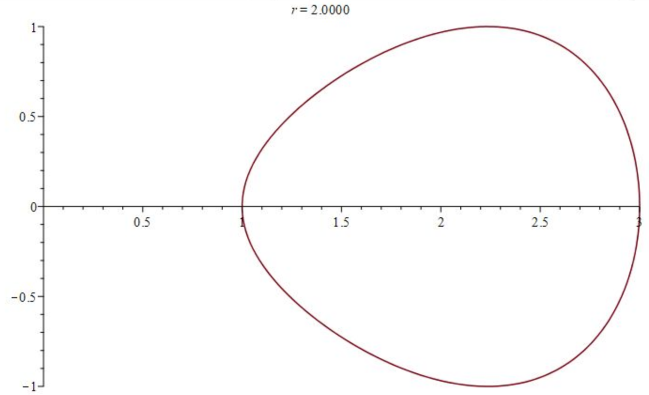

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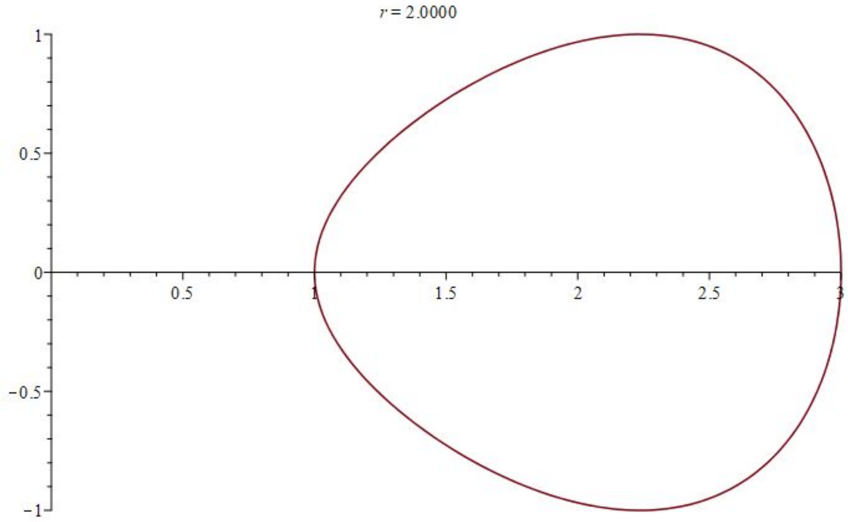
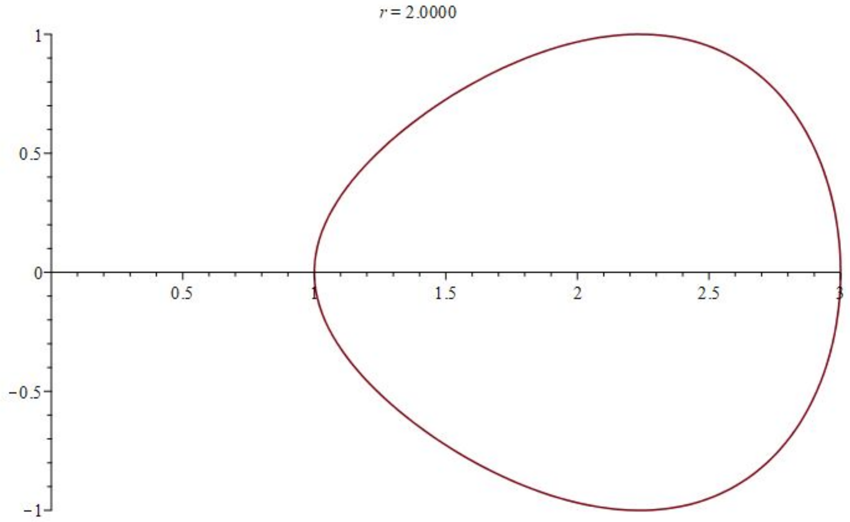
- Variable coefficient averaging operator.



Varying u translates $\Sigma_{r,u,1}$ in the vertical direction



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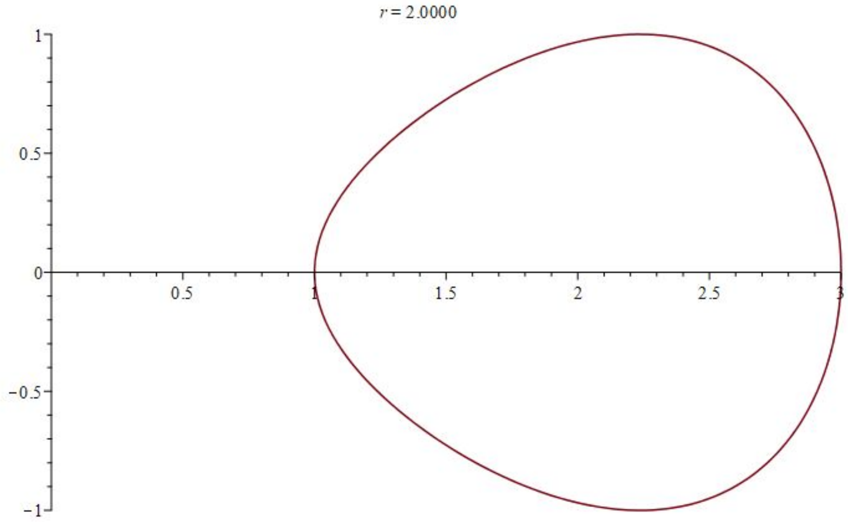

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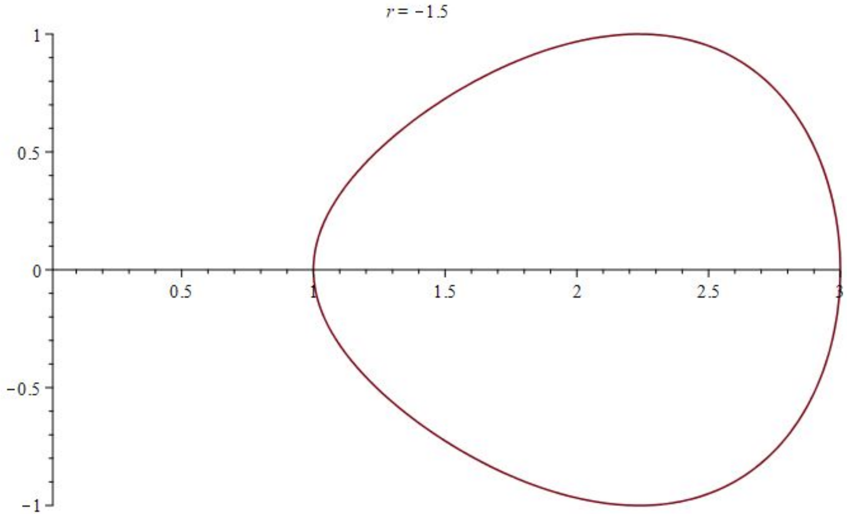

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$\Sigma_{r,0,1}$ for $\frac{1}{2} \leq r \leq 2$.

- There are variable coefficient generalisations of Bourgain's theorem (c.f. Sogge 1988).

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- This operator falls well outside the scope of such results!

2. Key features in the euclidean problem.

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Question: What are the enemies?

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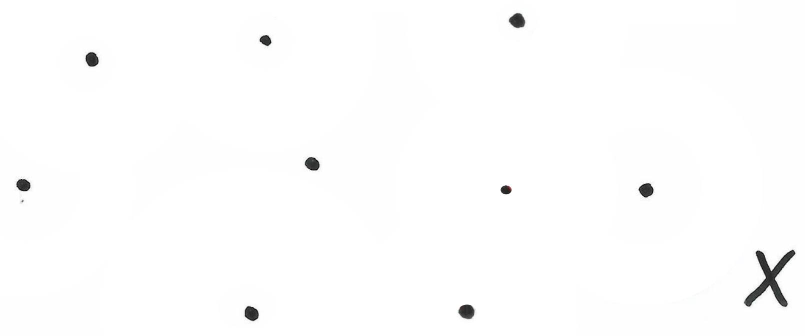

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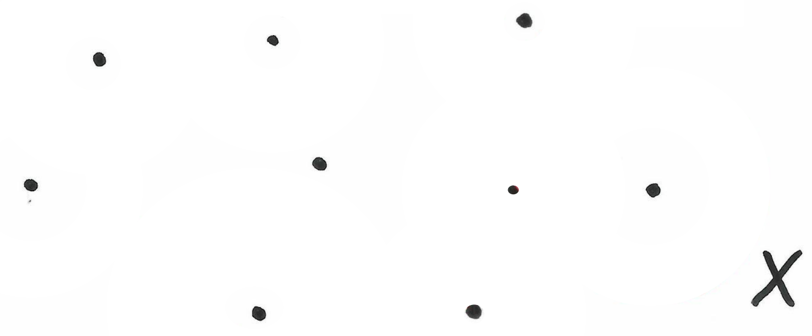

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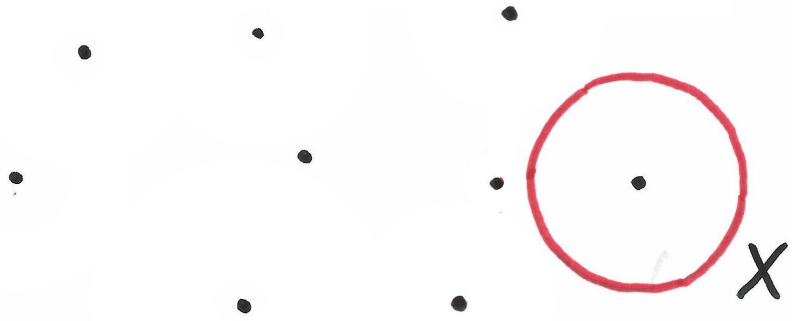

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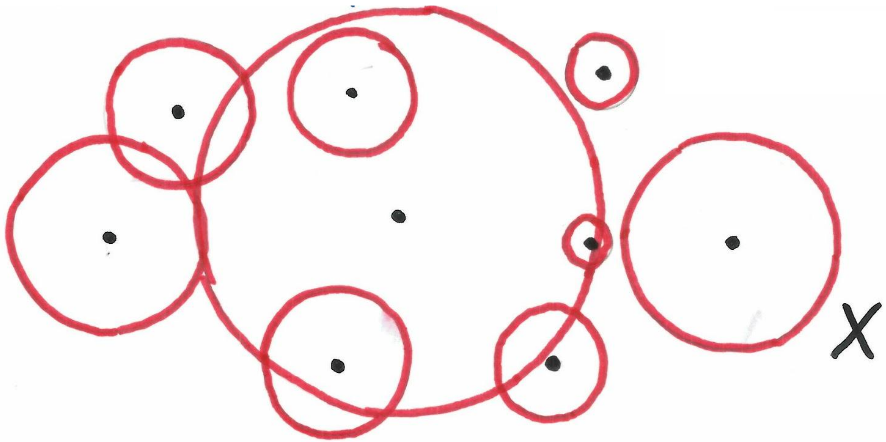

Corollary (Marstrand):- If $X \subseteq \mathbb{R}^2$ is a measurable set of centres and $C(X) \subseteq \mathbb{R}^2$ is a measurable union of circles formed around X, then

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Thus, $|X|^{1/p} \leq \|Mf\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)} = |C(X)|^{1/p}$. $\square$

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- How could this fail?

i.e. how could we make $C(X)$ small?

- The only way $C(X)$ can be small is if the circles overlap a lot:

Large overlap $\Leftrightarrow$ Small area.

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  Large overlap <=> Small area.

- <u>New problem</u>: understand configurations of circles with large overlap.

- **δ-discretize**: suppose X is a union of δ-balls and $r_x = r_y$ if $|x-y| < 2\delta$.

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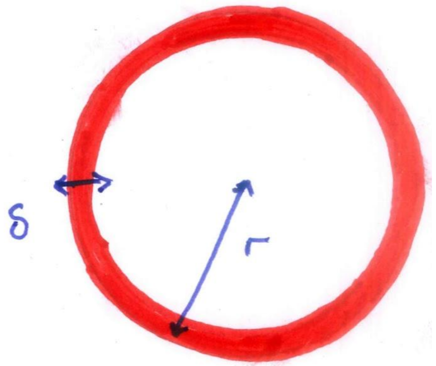

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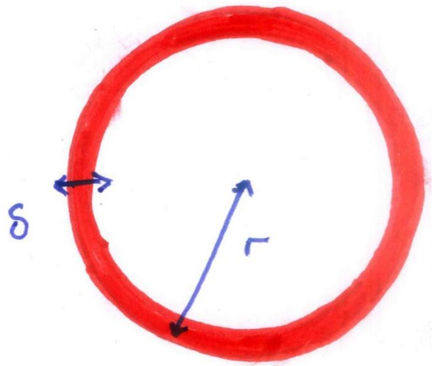

- Rather than considering infinite families of circles, we'll study finite families of annuli (we can then measure overlap using area).

- Curvature plays a key rôle.

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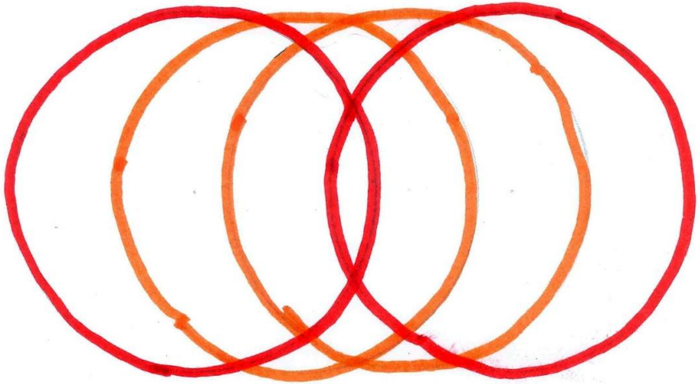
- If we fix $r > 0$, as soon as the centres are well-separated there is little overlap:



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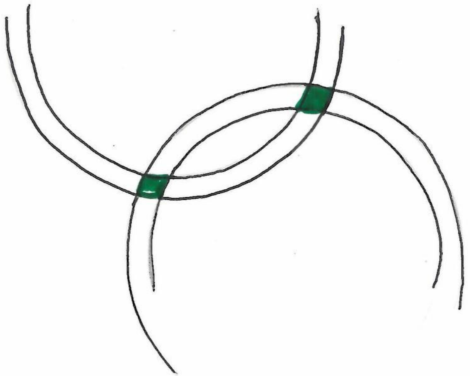

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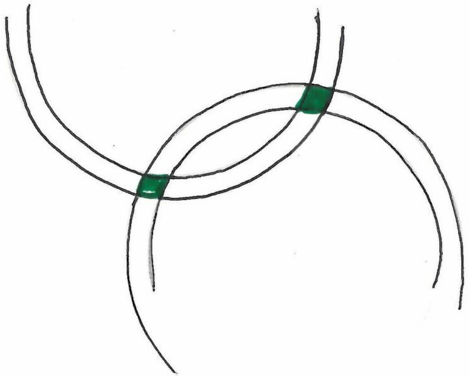

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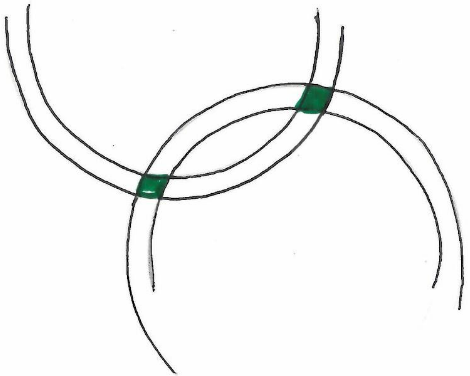

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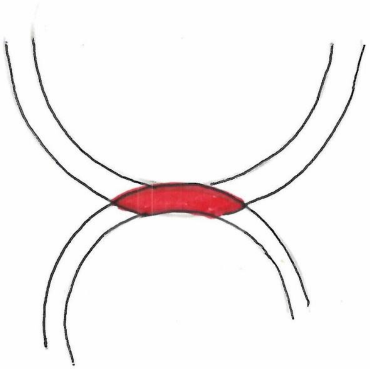
TANGENTIAL CASE:-



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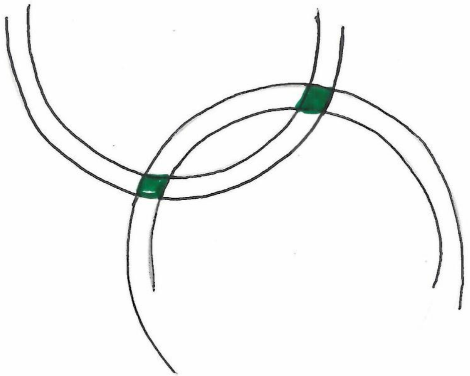

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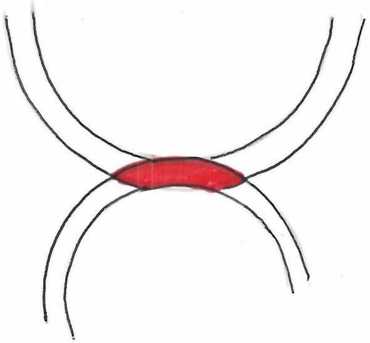

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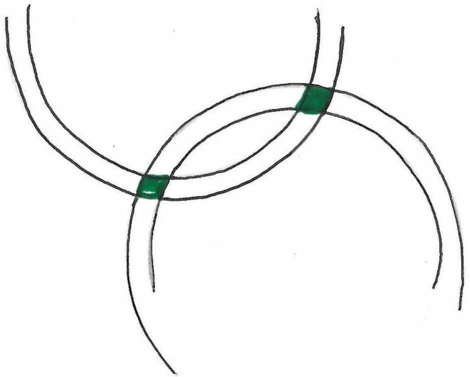

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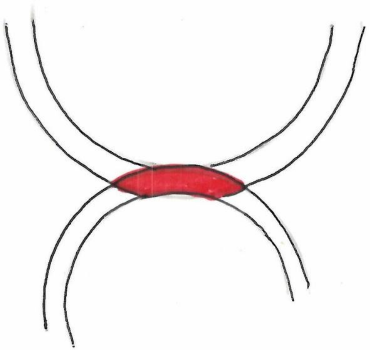

Overlap on a big, long "rectangle".

Tangent case is the bad guy!

KNOWING OUR ENEMY: CIRCLE TANGENCIES.

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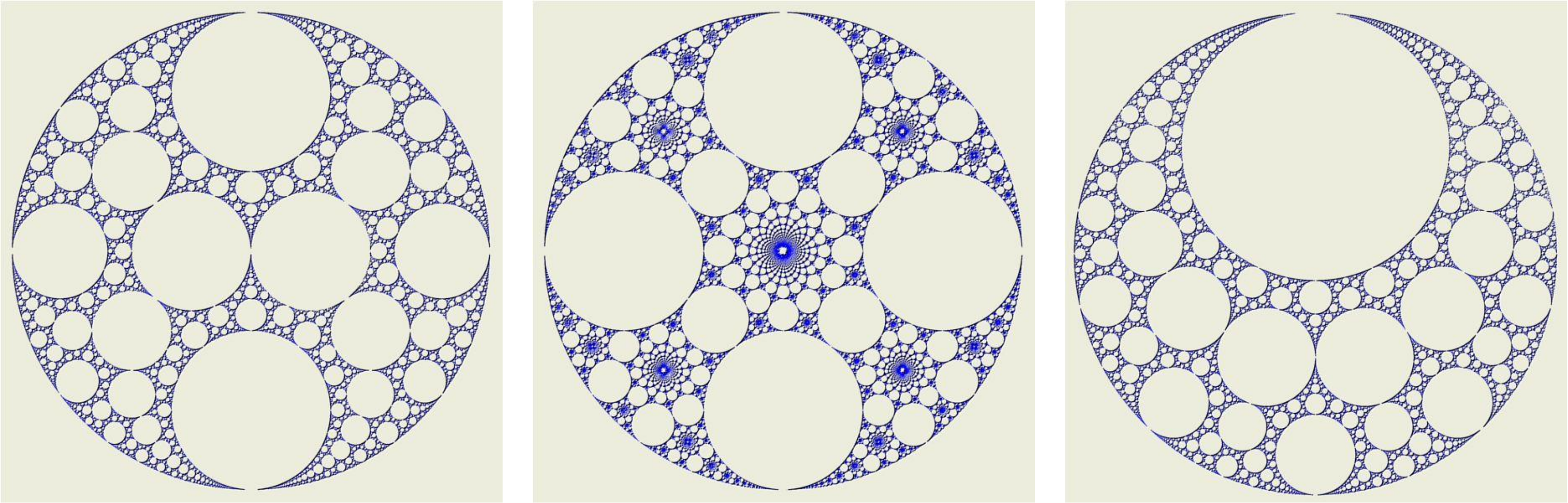
"Bad" configurations of circles will have many tangencies:-



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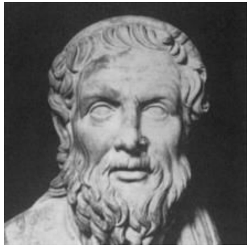
THEOREM (APOLLONIUS OF PERGA): Generically, any three fixed circles have 8 different circles tangent to them.



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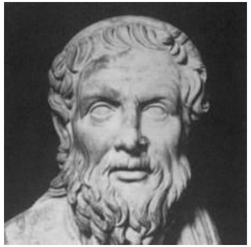

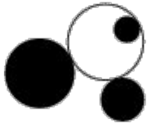

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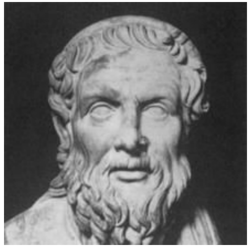

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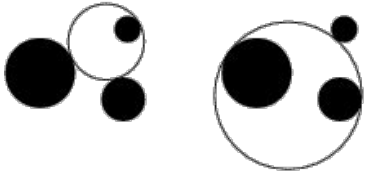

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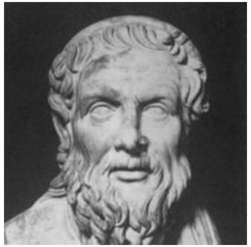

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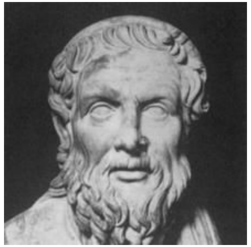

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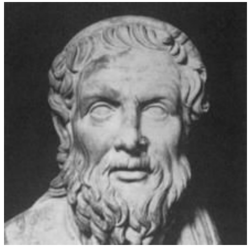

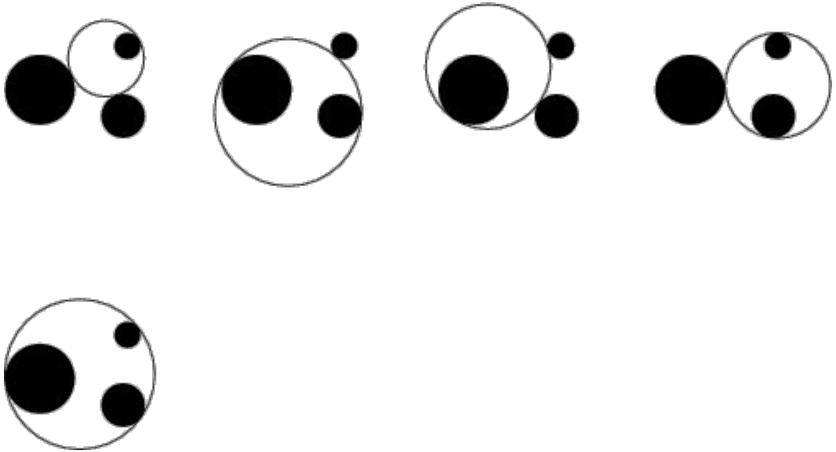

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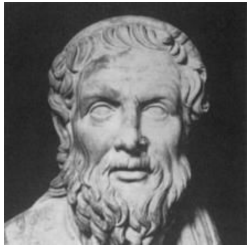

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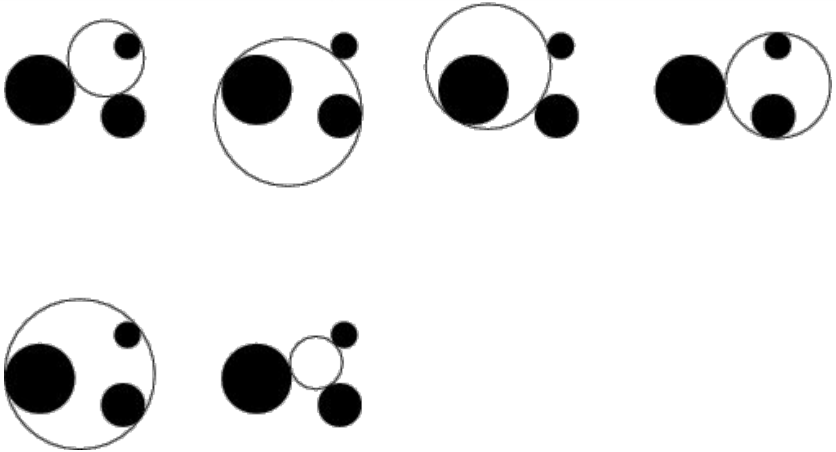

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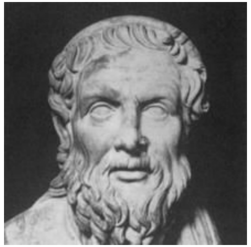

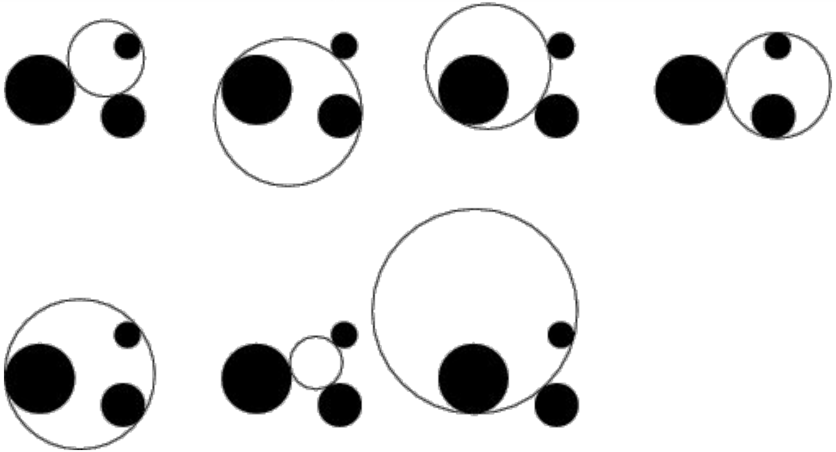

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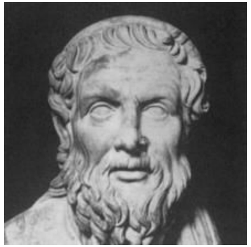

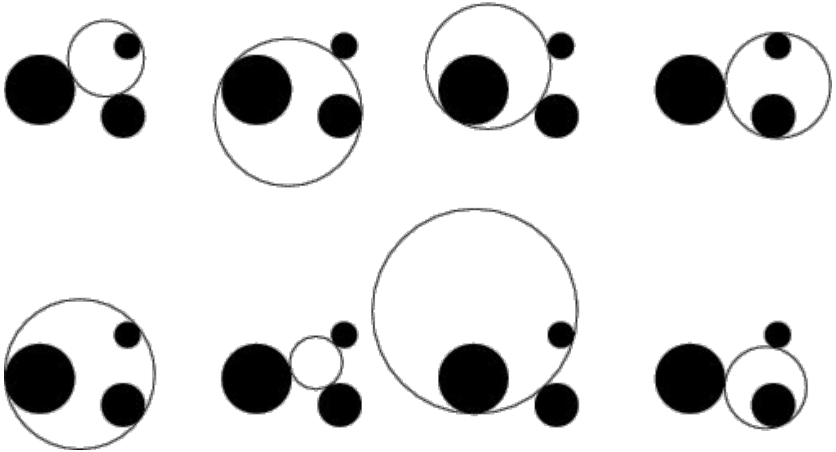

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- Can be used to control circle tangencies and rule out the existence of a 'bad' example – c.f. Schlag.
- Not only true for Circles – holds for families of curves satisfying Sogge's "cinematic curvature condition"!

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All these features manifest themselves in the proof of Bourgain's theorem.

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- Consider :-

$$A_t f(x) := \int_{\Sigma_{x,t}} f(y)\, d\sigma_{x,t}(y)$$

$$Mf(x) := \sup_{1 \le t \le 2} A_t |f|(x).$$

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Uncertainty principle :

δ - discretization <=> δ^{-1} Fourier localisation

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Goal: $\| \sup_{1 \leq t \leq 2} |A_t^j f| \|_{L^p(\mathbb{R}^2)} \lesssim 2^{-\varepsilon(p) j} \|f\|_{L^p(\mathbb{R}^2)}$.

Step 2 : L^2- bounds.

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$$\int_1^2 \|A_t^j f\|_2^2 dt \quad , \quad \int_1^2 \left\|\frac{\partial}{\partial t} A_t^j f\right\|_2^2 dt .$$

Def$^{\underline{n}}$: Φ_t satisfies the rotational curvature condition if $\det \begin{pmatrix} \Phi_t & \partial_y \Phi_t^T \\ \partial_x \Phi_t & \partial^2_{xy} \Phi_t \end{pmatrix}(x,y) \neq 0$ whenever $\Phi_t(x,y) = 0$.

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Lemma: Under this hypothesis,

- $\| A_t f \|_{L^2(\mathbb{R}^2)} \lesssim 2^{-j/2} \|f\|_{L^2(\mathbb{R}^2)}$
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Thus, $\|\sup_{1\le t\le 2} |A_t^j f|\|_2 \lesssim \left(\int_1^2 \|A_t^j f\|_2^2\, dt\right)^{1/2} + \left(\int_1^2 \|A_t^j f\|_2^2\, dt\right)^{1/4}\left(\int_1^2 \|\tfrac{\partial}{\partial t}A_t^j f\|_2^2\, dt\right)^{1/4}$

$$\lesssim \left(2^{-j/2} + 2^{-j/4}\cdot 2^{j/4}\right)\|f\|_2 \lesssim \|f\|_2,$$

as desired. $\square$

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- Add a dummy variable

$$T_t^j F(x_0, x) := \int_{\mathbb{R}^3} e^{2\pi i 2^j x_0 y_0 \Phi_t(x,y)} a(x_0, x, y_0, y) F(y_0, y) \, dy_0 \, dy$$

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$$T_t^j F(x_0, x) := \int_{\mathbb{R}^3} e^{2\pi i \underbrace{2^j x_0 y_0 \Phi_t(x,y)}_{\tilde{\Phi}_t(x_0, x, y_0, y)}} a(x_0, x, y_0, y) F(y_0, y) \, dy_0 \, dy$$

(Hörmander).

- By T^*T and the Schur test, the problem reduces to bounding

$$K^j(y_0, y, z_0, z) := \int_{\mathbb{R}^3} e^{2\pi i 2^j (\tilde{\Phi}_t(x_0, x; y_0, y) - \tilde{\Phi}_t(x_0, x; z_0, z))} a_t(x_0, x, y_0, y; z_0, z) \, dx_0 \, dx$$

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- The rotational curvature condition allows K^j to be estimated via (non-)stationary phase.

Mixed Hessian of $\tilde{\Phi}_t$

$$\Updownarrow$$

Phong-Stein rotational Curvature Matrix

$$\begin{pmatrix} \Phi_t & \partial_x \Phi_t^T \\ \partial_y \Phi_t & \partial^2_{xy} \Phi_t \end{pmatrix}.$$

Step 3: L^p bounds

Suffices to show:-

- $$\Big\| \sup_{1 \le t \le 2} |A_t^j f| \Big\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-\varepsilon(p) j} \|f\|_{L^p(\mathbb{R}^2)}$$

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- Elementary Sobolev embedding reduces to bounding

$$\int_1^2 \|A_t^j f\|_{L^p(\mathbb{R}^2)}^p \, dt, \quad \int_1^2 \|\tfrac{\partial}{\partial t} A_t^j f\|_{L^p(\mathbb{R}^2)}^p \, dt$$

Bounding $\int_1^2 \|A_t^j f\|_{L^p(\mathbb{R}^2)}^p \, dt$

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- Most difficult part of the argument but we omit the details here...

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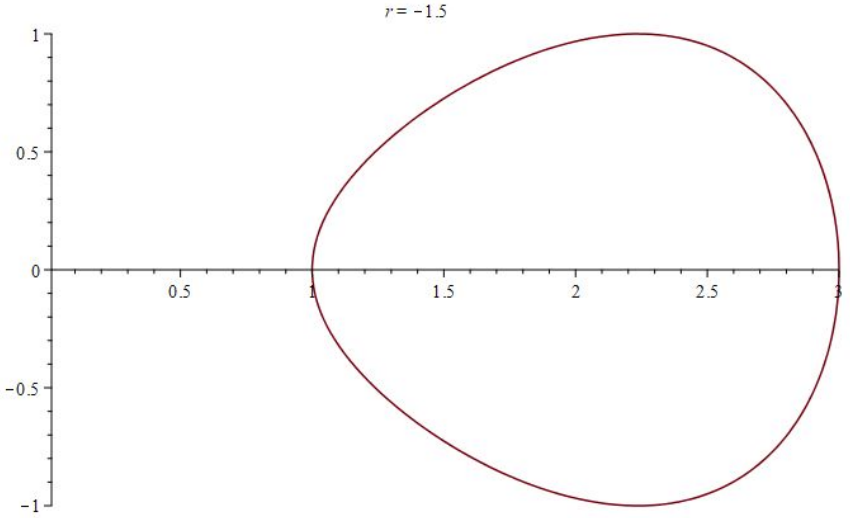

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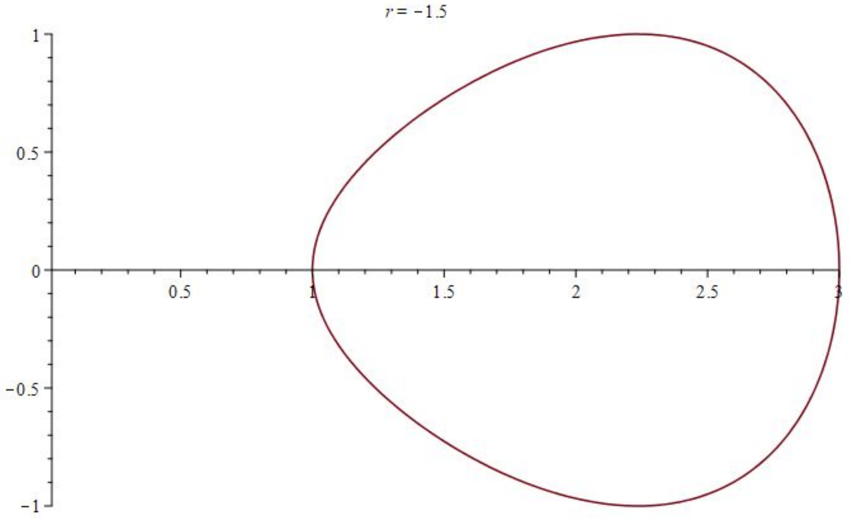

Want to show $\| \sup_{t>0} \mathcal{A}_t |g| \|_{L^p(\mathbb{R}^2)} \lesssim \|g\|_{L^p(\mathbb{R}^2)}$, $p > 2$.

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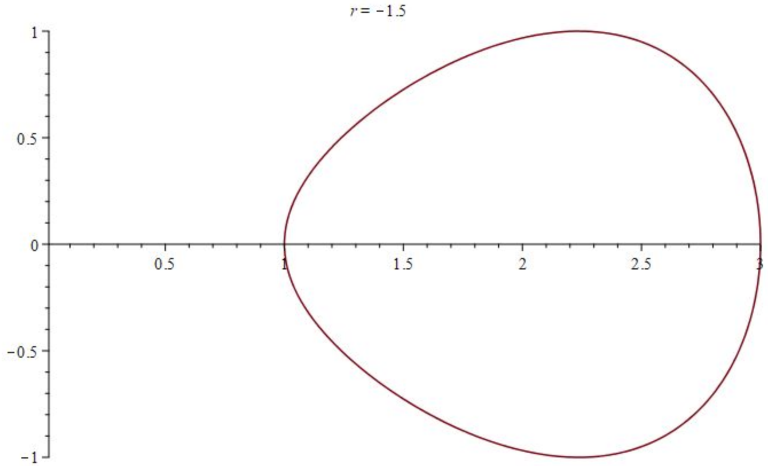

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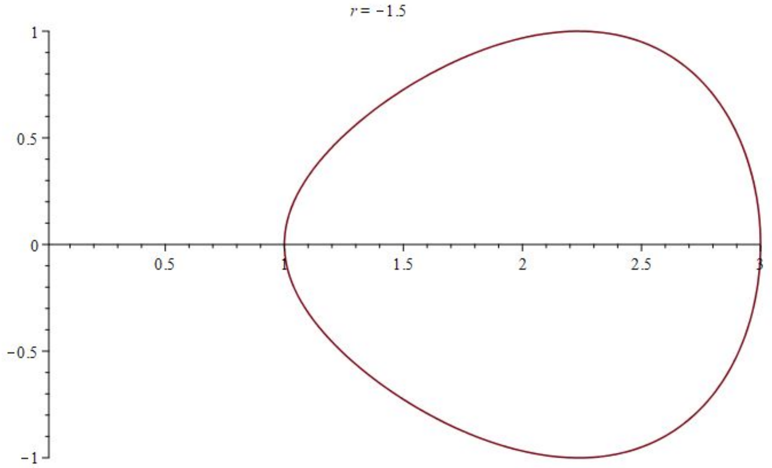

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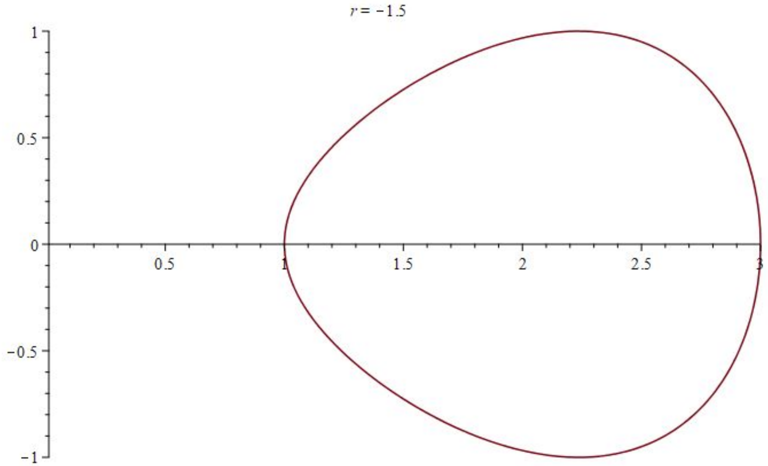

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Failure of the cinematic curvature condition

- L^p theory is 'non-quantitative' : just need

$$\Big\| \sup_{1 \leq t \leq 2} |A_t^j f| \Big\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-\varepsilon(p) j} \| f \|_{L^p(\mathbb{R}^2)}$$

for some $p > 2$ and some $\varepsilon(p) > 0$.

(c.f. Kung).

The most serious issue is...

The vanishing of the Rotational Curvature...

Recall: • Rotational Curvature: $\Phi_t(x,y)=0 \Rightarrow \det\begin{pmatrix}\Phi_t & \partial_x\Phi_t^T \\ \partial_y\Phi_t & \partial^2_{xy}\Phi_t\end{pmatrix}(x,y) \neq 0$

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Prototypical example:

$$\int e^{i\lambda(x_1 y_1 + \dots + x_{n-1} y_{n-1} + x_n \frac{y_n^2}{2})} f(y)\, dy \qquad \begin{pmatrix} I_{n-1} & \vec{0}^T \\ \vec{0} & y_n \end{pmatrix}$$

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1/6 loss in the exponent!

Cannot be improved!

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- Smallness compensates precisely with the $2^{j/6}$ loss!
- Similar phenomena observed in higher dimensions by Müller–Seeger.

HAPPY

BIRTHDAY

ANDREAS!