Numerical uniformization

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3. **improve the above so that $\Omega = D$, the unit disk (Fuchsian uniformization)**

History of numerical uniformization

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Applies to general hyperelliptic plane curves C of genus g defined by

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We use it for curves with all the points λ_j real (or ∞).

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- elements of G map the upper half–plane onto itself
- G acts properly discontinuosly in a domain Ω so that $\Omega/G = C$.

The method also yields an approximation of the projection $\Omega \to C$.

Extension of Myrberg's method

- **Conjugate the construction by explicit quasiconformal mappings between a hyperelliptic curve C (with real λ_j's) and a general curve of genus g.**

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- General problem: find Teichmüller mappings between given algebraic curves.

Fuchsian uniformization

Given a hyperelliptic curve C (with real λ_j's). Use Myrberg's method to to find a Fuchsian group G of the second kind acting in a domain Ω such that $\Omega/G = C$.

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Find a Fuchsian group F of the first kind acting in the upper half–plane U so that U/F is a algebraic curve $near$ the curve C.

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Find a Fuchsian group F of the first kind acting in the upper half–plane U so that U/F is a algebraic curve *near* the curve C.

Conjugate F with an explicit quasiconformal mapping to get a Fuchsian group of the first kind F^μ acting in U so that $U/F^\mu = C$.

Other current work

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P. Buser and R. Silhol: explicit geometric methods to find Fuchsian groups uniformizing given curves.
M. Wagner: implementations and approximations of accessory parameters.

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