

# FOLIATIONS AND QUADRATIC DIFFERENTIALS OF RIEMANN SURFACES

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ABSTRACT. Teichmüller's Extremal Mapping Theorem characterizes best possible quasiconformal mappings in a given homotopy class of mappings between compact Riemann surfaces. By TEMT, every homotopy class of mappings between compact Riemann surfaces has a unique quasiconformal mapping with smallest maximal dilatation. The uniqueness of this mapping makes it particularly interesting. This result follows from arguments present already in Grötzsch's result (regarding quasiconformal mappings between quadrilaterals). To that end we need to divide Riemann surfaces into domains that allow the use of arguments similar to those of Grötzsch. This decomposition can be achieved by so called horizontal foliations associated to quadratic differentials.

## 1. FOLIATIONS OF COMPACT SURFACES

1.1. **Measured foliations.** We start by reviewing general theory of foliations of surfaces. For a more detailed discussion of foliations we refer to [5].

Let  $\Sigma$  be a compact and oriented  $C^\infty$ -surface.

**Definition 1.** A non-singular  $C^r$ -foliation  $\mathcal{F}$  of  $\Sigma$  is a maximal  $C^r$ -atlas  $\{(U_i, \varphi_i)\}$  of  $\Sigma$  such that the coordinate transformations are of the form

$$(1) \quad \varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j), \quad \varphi_{ij}(x, y) = (\alpha_{ij}(x, y), \gamma_{ij}(y)).$$

In the above definition  $0 \leq r \leq \infty$ . Unless otherwise stated we will assume in the sequel that  $r = 1$ , i.e., all foliations that we will consider are at least differentiable.

Let  $\mathcal{F}$  be a foliation and  $\{(U_i, \varphi_i)\}$  the maximal atlas corresponding to  $\mathcal{F}$ . For each chart  $(U_i, \varphi_i)$  we can form the function  $f_i = pr \circ \varphi_i : U_i \rightarrow \mathbf{R}$ . Here  $pr : \mathbf{R}^2 \rightarrow \mathbf{R}$ ,  $(x, y) \mapsto y$ , is the projection on the second factor. In view of the transformation rule (1),

$$(2) \quad f_i(p) = \gamma_{ij}(f_j(p))$$

for all  $p \in U_i \cap U_j$ .

The sets  $P_i^y = f_i^{-1}(\{y\})$ ,  $y \in \mathbf{R}$ , are called the plaques of  $\mathcal{F}$  in  $U_i$ . The formula (2) determines how the plaques of  $\mathcal{F}$  in  $U_i$  are glued together with the plaques in an  $U_j$ ,  $U_i \cap U_j \neq \emptyset$ . In this way we obtain, via (2), the leaves  $F_i$  of  $\mathcal{F}$ . We write frequently  $\mathcal{F} = \{F_i\}$ . A foliation is defined by its leaves. If  $\mathcal{F}$  belongs to the class  $C^r$ , then its leaves are 1-dimensional  $C^r$ -submanifolds of  $\Sigma$ .

A more formal definition for the leaves is the following. Let  $\{(U_i, \varphi_i)\}$  be the atlas defining a foliation  $\mathcal{F}$ . The charts  $(U_i, \varphi_i)$  are referred to as the distinguished charts of  $\mathcal{F}$ . Provide  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$  with the topology  $T_0$  which is the product of the natural topology on the first factor and the discrete topology on the second

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These notes follow the presentation of [8].

factor. Then the connected components of the topological space  $(\mathbb{R}^2, T_0)$  are the lines  $y = \text{constant}$ .

Let  $\varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^2$  be a distinguished chart of  $\mathcal{F}$ . On  $U$  there exists a unique topology  $T_U$  such that  $\varphi : (U, T_U) \rightarrow (\varphi(U), T_0)$  is a homeomorphism. The connected components of  $(U, T_U)$  are the plaques of  $\mathcal{F}$ . The family of plaques of  $\mathcal{F}$  is a basis for a topology  $T_L$  on  $\Sigma$ . This topology is called *the leaf topology* because the connected components of the topological space  $(\Sigma, T_L)$  are precisely the leaves of  $\mathcal{F}$  in the sense of the above definition.

Two foliations  $(\Sigma, \mathcal{F})$  and  $(\Sigma', \mathcal{F}')$  are *homeomorphic* if there exists a homeomorphism  $\Sigma \rightarrow \Sigma'$  which takes the leaves of  $\mathcal{F}$  onto the leaves of  $\mathcal{F}'$ .

It is clear that a leaf  $F$  of a foliation  $\mathcal{F}$  is a Hausdorff space. It is immediate that a leaf has a countable basis for its topology (cf. [5, page 14]). Consequently, each leaf is either  $S^1$  or  $\mathbb{R}$ , since we have assumed that  $\Sigma$  does not have boundary. It follows that each foliation has uncountably many leaves.

**Examples.** Some compact surface (boundary components are now allowed, the surfaces need not be orientable and the genera need not be  $\geq 2$ ) admit non-singular foliations. Here is a complete list of them.

1. An annulus can be foliated in several ways.
2. The Möbius band admits the same type of foliations as the annulus.
3. A torus has several different foliations. The most important ones for our applications are *rational* and *irrational flows* which are defined as follows. Let  $\mathbb{R}^2$  be the universal cover of a torus, and let the covering transformation group be generated by  $(x, y) \mapsto (x + 1, y)$  and  $(x, y) \mapsto (x, y + 1)$ . Foliate  $\mathbb{R}^2$  with lines  $y = \alpha x + r$ ,  $r \in \mathbb{R}$ , where  $\alpha$  is a fixed real number. This foliation of  $\mathbb{R}^2$  is invariant under the cover transformation group. Hence it induces a foliation of the underlying torus. The leaves of this foliation are compact if and only if  $\alpha$  is rational.
4. The Klein bottle is a double of the Möbius band, i.e., take two copies of the Möbius band and glue them together along the boundary to obtain a Klein bottle. Foliations of the Möbius band induce foliations of the Klein bottle.

**Theorem 1.** *Assume that the compact surface  $\Sigma$  without boundary admits a non-singular  $C^0$ -foliation  $\mathcal{F}$ . Then the Euler characteristic  $\chi(\Sigma)$  of  $\Sigma$  vanishes.*

**Proof.** The following argument is due to Kneser (1924). Let  $\mathcal{T}$  be a triangulation of  $\Sigma$  such that following conditions are satisfied.

1. For every vertex  $v$  of  $\mathcal{T}$  there is a distinguished chart  $(U, \varphi)$  of the foliation which contains all the triangles of  $\mathcal{T}$  having  $v$  as vertex.
2. The edges of the triangulation  $\mathcal{T}$  are *transversal* to the foliation  $\mathcal{F}$ , i.e., if  $e$  is an edge in the distinguished chart  $(U, \varphi)$  then the function  $e \rightarrow \mathbb{R}$ ,  $p \mapsto pr \circ \varphi(p)$  is strictly monotone. Recall that  $pr : \mathbb{R}^2 \rightarrow \mathbb{R}$  is the projection on the second variable,  $pr(x, y) = y$ .

A triangulation  $\mathcal{T}$  satisfying the above conditions is *transversal* to the foliation  $\mathcal{F}$ . For the existence of transversal triangulations we refer to the reader's geometric intuition or to [5, Proposition 2.4.4]. Every smooth foliation of a compact surface always admits a transversal triangulation.

Let

FIGURE 1. A transversal triangulation in a distinguished chart.

$$\begin{aligned} n_0 &= \text{the number of vertices of } \mathcal{T} \\ n_1 &= \text{the number of edges of } \mathcal{T} \\ n_2 &= \text{the number of triangles of } \mathcal{T}. \end{aligned}$$

We may assume that each edge belongs to two different triangles. Each triangle has 3 edges. Hence

$$n_1 = \frac{3}{2}n_2.$$

From the fact that  $\mathcal{T}$  is a transversal triangulation it follows, by a geometric consideration (cf. the above figure), that

$$n_0 = \frac{1}{2}n_2.$$

We compute:

$$\chi(\Sigma) = n_0 - n_1 + n_2 = \frac{1}{2}n_2 - \frac{3}{2}n_2 + n_2 = 0.$$

**Remarks.** By means of doubling the above argument can be generalized to compact surfaces with boundary. It follows that the surfaces listed in the above example are the only surfaces that admit non-singular foliations. On the other hand, all non-compact surfaces admit non-singular foliations.

If the foliation  $\mathcal{F}$  belongs to the class  $C^1$ , then it can be viewed as a vector field. In that case the above theorem is a special case of the Poincaré index formula for vector fields.

In order to obtain foliations on surfaces of genus  $g \geq 2$  or  $g = 0$  we have to admit singularities. There are, of course, several different possibilities as illustrated in Fig. 2.

For our applications the only relevant singularities are the ones illustrated in Fig. 3.

We call singularities of type (1), (2), (3), ... *admissible*. A leaf starting from a singular point is a *singular leaf*. The *index*  $P_s$  of a singular point  $s$  is the number of leaves starting from that point. So for the above singularities (1), (2) and (3) the indices are  $P_s = 3, 4,$  and  $5,$  respectively.



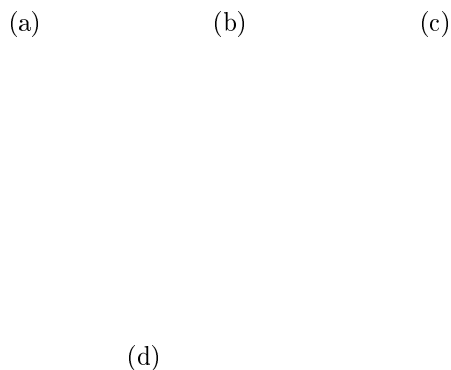


FIGURE 4. Foliations of a pair of pants.

are sufficiently regular, namely those that can be *transversely measured*. Before defining this concept let us consider orientation of foliations.

Deliberate a foliation  $\mathcal{F}$  with admissible singularities. Each leaf of  $F$  of  $\mathcal{F}$  is either  $\mathbb{R}$ ,  $S^1$  or  $\mathbb{R}^+$  (a singular leaf). So there are two possibilities to orientate each leaf. Now, intuitively, the foliation  $\mathcal{F}$  is *orientable* if the orientation of the leaves can be chosen in a coherent manner, that is, in such a way that leaves near to each other are similarly oriented.

Let the foliation  $\mathcal{F}$  be defined by the atlas  $\mathcal{A} = \{(U_i, \varphi_i)\}$ . Recall that the coordinate transformations of  $\mathcal{A}$  are then of the form

$$\varphi_{ij}(x, y) = (\alpha_{ij}(x, y), \gamma_{ij}(y)).$$

For each fixed  $y$ , the mapping  $x \mapsto \alpha_{ij}(x, y)$  is a homeomorphism of an interval in  $\mathbb{R}$  onto another interval. Also  $y \mapsto \gamma_{ij}(y)$  is a homeomorphism of an interval onto an interval.

**Definition 2.** *A foliation  $\mathcal{F}$  is orientable if there exists a subatlas  $\mathcal{A}'$  of  $\mathcal{A}$  such that all the functions  $x \mapsto \alpha_{ij}(x, y)$  associated with the coordinate transformations of  $\mathcal{A}'$  are increasing for all possible values of  $y$ . The foliation  $\mathcal{F}$  is transversely orientable if all the functions  $y \mapsto \gamma_{ij}(y)$  associated to the coordinate transformations of some subatlas  $\mathcal{A}'$  of  $\mathcal{A}$  are increasing.*

Consider a foliation  $\mathcal{F}$  with admissible singularities on a surface  $\Sigma$ . The foliation  $\mathcal{F}$  is not orientable if it has singularities  $s$  with  $P_s \equiv 1 \pmod{2}$ .

If all singularities of  $\mathcal{F}$  are of even order, then  $\mathcal{F}$  is orientable and it defines a global vector field with singularities on  $\Sigma$  (take the tangent vectors of the leaves of  $\mathcal{F}$ ). The foliation  $\mathcal{F}$  is obtained as flow lines of this vector field.

Let  $\mathcal{F}$  be now any foliation with admissible singularities on  $\Sigma$ . Near a point  $p \in \Sigma \setminus \text{sing}\mathcal{F}$  the foliation  $\mathcal{F}$  can be viewed as solutions to a differential equation. Namely, take a distinguished chart  $(U, \varphi)$  containing the point  $p$ . In this chart consider the differential  $dy$ . Leaves of  $\mathcal{F}$  in  $U$  are the integral curves of the differential equation  $dy = 0$ . In this way the foliation  $\mathcal{F}$  defines and is defined by a differential form  $\omega_p$  near the point  $p$ .

Assume that the foliation  $\mathcal{F}$  has a singular point  $s$  with  $P_s \equiv 1 \pmod{2}$ . Then  $\mathcal{F}$  is not orientable. Take a point  $p$  which is not a singular point of  $\mathcal{F}$ . Let  $\omega_p$  be the differential form corresponding to  $\mathcal{F}$  near  $p$ . We can continue  $\omega_p$  along paths that do not pass through singular points of  $\mathcal{F}$ . If we continue  $\omega_p$  along a closed path going around one singular point of  $\mathcal{F}$  with an odd order, then we end up with the differential form  $-\omega_p$ . This reflects the fact that the ‘local orientation’ of  $\mathcal{F}$  is reversed when we walk around this path. So the locally defined differential forms  $\omega_p$  do not define a global differential form. But their squares  $\omega_p^2$  do! They define a quadratic differential and the singularities of  $\mathcal{F}$  correspond to the zeros of the quadratic differential. For our applications this connection between quadratic differentials and foliations is important.

If a foliation  $\mathcal{F}$  is sufficiently regular, then we can measure how many leaves a transversal curve  $\alpha$  cuts.

**Definition 3.** A measured foliation  $\mathcal{F}$  on  $\Sigma$  is a foliation with admissible singularities such that the atlas  $\mathcal{A} = \{(U_i, \varphi_i)\}$  defining  $\mathcal{F}$  apart from the singular points satisfies the following condition: the coordinate transformation functions of  $\mathcal{A}$  are of the form

$$(3) \quad \varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j),$$

$$(4) \quad \varphi_{ij}(x, y) = (\alpha_{ij}(x, y), \pm y + c_{ij})$$

where  $c_{ij}$  is a constant.

**Remark.** Two different measured foliations may coincide as foliations. In fact, if the atlas  $\{(U_i, \varphi_i)\}$  defines a measured foliation, then  $\{(U_i, r\varphi_i)\}$ ,  $r \in \mathbb{R}$ ,  $r \neq 0, 1$ , defines another measured foliation. As foliations these measured foliations are the same.

Let  $\mathcal{F}$  be a measured foliation and  $\alpha : [0, 1] \rightarrow \Sigma$  a curve. In view of (4) the number

$$\ell_{\mathcal{F}}(\alpha) = \text{the total variation of the } y \text{ - coordinate along } \alpha$$

is well defined; it is called *the transversal length of  $\alpha$* .

The notation  $\int_{\alpha} \mathcal{F}$  is also used for the transversal length of  $\alpha$ .

Assume that a measured foliation  $\mathcal{F}$  is defined by the atlas  $\mathcal{A} = \{(U_i, \varphi_i)\}$ . If  $f : \Sigma \rightarrow \Sigma$  is a homeomorphism then  $f^*(\mathcal{A}) = \{(f(U_i), \varphi_i \circ f^{-1})\}$  is an atlas of  $\Sigma$  which defines another measured foliation  $f^*(\mathcal{F})$  of  $\Sigma$ .

If two singular points of the foliation  $\mathcal{F}$  are joined to each other by a singular leaf  $F$  then we may consider a mapping  $g : \Sigma \rightarrow \Sigma$  which maps  $\Sigma \setminus F$  homeomorphically onto  $\Sigma \setminus \{\text{a point}\}$  and which maps the leaf  $F$  onto the point where we punctured  $\Sigma$ . Just like above we define the foliation  $g^*(\mathcal{F})$ . It is called *a Whitehead deformation of  $\mathcal{F}$* .

**Definition 4.** Two measured foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are equivalent in the sense of Whitehead if  $\mathcal{F}_2$  is a Whitehead deformation  $g^*(\mathcal{F}_1)$  of  $\mathcal{F}_1$  and if the deforming

FIGURE 5. A Whitehead deformation.

mapping  $g : \Sigma \rightarrow \Sigma$  is homotopic to the identity mapping of  $\Sigma$ . The set of Whitehead equivalence classes of measured foliations on  $\Sigma$  is denoted by  $\mathcal{MF}(\Sigma)$ .

Consider the set

$$\mathcal{S} = \{\text{homotopy classes of closed curves on } \Sigma\},$$

and define

$$(5) \quad L : \mathcal{MF}(\Sigma) \rightarrow [0, \infty)^{\mathcal{S}},$$

$$(6) \quad [\mathcal{F}] \mapsto (\mathcal{S} \rightarrow [0, \infty), [\alpha] \mapsto \inf \{\ell_{\mathcal{F}}(\beta) \mid \beta \approx \alpha\}).$$

The definition of the Whitehead equivalence relation is motivated by the following important result ([1, 6.V.2 Théorème, page 110]).

**Theorem 3.** *The mapping  $L : \mathcal{MF}(\Sigma) \rightarrow [0, \infty)^{\mathcal{S}}$  is injective and the image of  $\mathcal{MF}(\Sigma)$  in  $[0, \infty)^{\mathcal{S}}$  does not contain the origin.*

By Theorem 3 we may identify  $\mathcal{MF}(\Sigma)$  with its image in  $[0, \infty)^{\mathcal{S}}$ . Let us do that now.

On  $[0, \infty)^{\mathcal{S}}$  we have the product topology. As its subspace also  $\mathcal{MF}(\Sigma)$  is a topological space.

**1.2. Geometry of quadratic differentials.** Assume now that  $\Sigma$  is a compact and oriented surface of genus  $\geq 2$ . Let  $X$  be a complex structure on  $\Sigma$ . Let  $\mathcal{A} = \{(U_i, \varphi_i)\}$  be an atlas of  $X$ .

Recall that a holomorphic quadratic differential on  $X$  is given by a collection  $\omega = \{w_i\}$  of holomorphic functions  $w_i : U_i \rightarrow \mathbf{C}$  subject to the following condition: on  $\varphi_j(U_i \cap U_j)$  we have

$$(7) \quad w_i(\varphi_i^{-1} \circ \varphi_{ij}) \left( \frac{\partial \varphi_{ij}}{\partial z} \right)^2 = w_j(\varphi_j^{-1}).$$

Since the genus of the Riemann surface  $X$  is at least 2, we can write  $X = U/G$ , where  $U$  is the upper half-plane and  $G$  is a Fuchsian group acting in  $U$ .

A quadratic differential  $\omega$  on  $X$  lifts to a function  $w : U \rightarrow \mathbf{C}$  satisfying

$$(8) \quad w(g(z)) g'(z)^2 = w(z), \quad g \in G.$$

A holomorphic function  $w$  in  $U$  satisfying (8) is called a *holomorphic quadratic differential of the group  $G$* .

We use the notations

$$\begin{aligned} D^2(G) &= \text{holomorphic quadratic differentials of } G\}, \\ D^2(X) &= \text{holomorphic quadratic differentials on } X\}. \end{aligned}$$

Consider a quadratic differential  $\omega \in D^2(X) \setminus \{0\}$ . On the Riemann surface

$$X' = X \setminus \{ \text{the zeros of } \omega \} .$$

we define *the natural local variables*  $\xi_\omega$  associated to  $\omega$  as follows.

Let  $p \in X'$ , and let  $\varphi : U \rightarrow \mathbb{C}$  be a local variable of  $X'$  at  $p$ , for which  $\varphi(p) = 0$  and  $U$  is simply connected. Assume that the holomorphic function  $w : U \rightarrow \mathbb{C}$  corresponds to  $\omega$  at  $(U, \varphi)$ . Then  $w$  is a holomorphic function which does not take the value 0. Hence we can consider  $\sqrt{w}$  in  $U$ . It is well defined up to sign. Choose one of the two alternatives.

Let  $q \in U$ . Define

$$(9) \quad \xi_\omega(q) = \int_0^{\varphi(q)} \sqrt{w \circ \varphi^{-1}(z)} dz.$$

The function  $\xi_\omega$  is holomorphic in  $U$ . It is also non-constant. Therefore we can assume, restricting  $U$  if necessary, that  $\xi_\omega : U \rightarrow \mathbb{C}$  is injective. Hence  $\xi_\omega : U \rightarrow \xi_\omega(U) \subset \mathbb{C}$  is a local variable of  $X'$  at the point  $p$ . All local variables  $\xi_\omega$  obtained from  $\omega$  in this manner form *the natural  $\omega$ -atlas of  $X'$* . In terms of this atlas  $\omega = (d\xi_\omega)^2$ .

Let  $\mathcal{A}_\omega = \{(U_i, \xi_i)\}$  be the natural  $\omega$ -atlas of  $X'$ . Assume that  $U_i \cap U_j \neq \emptyset$ .

**Lemma 1.** *The coordinate transformations  $\xi_{ij}$  of the natural  $\omega$ -atlas are of the form*

$$\xi_{ij}(z) = \xi_i \circ \xi_j^{-1}(z) = \pm z + c_{ij}$$

where  $c_{ij}$  is a constant depending on the indices  $i$  and  $j$ .

**Proof.** We can assume that the point  $p$  that is used to define  $\xi_i$  by (4.6) belongs to  $U_i \cap U_j$ . Hence we may further assume that the same point  $p \in U_i \cap U_j$  is used when defining  $\xi_j$  by the formula (9).

Let  $\varphi_i : U_i \rightarrow \varphi_i(U_i)$  and  $\varphi_j : U_j \rightarrow \varphi_j(U_j)$  be local variables with  $\varphi_i(p) = \varphi_j(p) = 0$ . Assume that  $\omega$  associates the holomorphic functions  $w_i : U_i \rightarrow \mathbb{C}$  and  $w_j : U_j \rightarrow \mathbb{C}$  to the charts  $(U_i, \varphi_i)$  and  $(U_j, \varphi_j)$ , respectively.

Consider a point  $q \in U_i \cap U_j$ . By the definition,

$$\xi_j(q) = \int_0^{\varphi_j(q)} \sqrt{w_j \circ \varphi_j^{-1}(z)} dz.$$

Using (7) we get

$$\xi_j(q) = \int_0^{\varphi_j(q)} \sqrt{w_i \circ (\varphi_i^{-1} \circ \varphi_{ij})(z)} \left( \frac{\partial \varphi_{ij}}{\partial z} \right) dz.$$

By the change of variable the latter integral equals

$$\int_0^{\varphi_i(q)} \sqrt{w_i \circ \varphi_i^{-1}(z)} dz = \pm \xi_i(q).$$

The  $\pm$  sign here reflects the different choices of the sign of the square root in (9). The constant  $c_{ij}$  corresponds to the choice of the point  $p$  in (9). So the above lemma reduces simply to a change of variable in an integral.

In view of Lemma 1, the natural  $\omega$ -atlas of  $X'$  is a measured foliation of  $X'$ . Denote it by  $\mathcal{F}_\omega$ . Let  $(U, \varphi)$  be any chart of  $X'$  and let  $w : U \rightarrow \mathbb{C}$  be the function associated to  $(U, \varphi)$ . Then the leaves of  $\mathcal{F}_\omega$  are just integral curves of the equation  $\text{Im} \sqrt{w \circ \varphi^{-1}(z)} d\varphi = 0$ .

Let us now consider a zero of  $\omega$ . Assume that the point  $p$  is a zero of order  $m$ . Then, for some local variable  $(U, \varphi)$  at  $p$ ,  $\omega = (\varphi)^m (d\varphi)^2$ . Here we have assumed that  $\varphi(p) = 0$ . Hence, near  $p$ , the foliation  $\mathcal{F}_\omega$  corresponds to the integral curves of  $\text{Im} z^{m/2} dz = 0$ . It is a matter of computation to check that the point  $p$  is then an admissible singularity of  $\mathcal{F}_\omega$  with index  $P_p = m + 2$ . We conclude that  $\mathcal{F}_\omega$  is a measured foliation on the Riemann surface  $X$ .

Note that if  $\omega$  is a quadratic differential, then  $-\omega$  is such as well.

**Definition 5.** *The measured foliation  $\mathcal{F}_\omega$  is the  $\omega$ -horizontal foliation of  $X$ . The measured foliation  $\mathcal{F}_{-\omega}$  is referred to as  $\omega$ -vertical foliation of  $X$ .*

By Theorem 2,

$$2\chi(X) = \sum_{s \in \text{sing } \mathcal{F}_\omega} (2 - P_s).$$

By the above consideration the singular points  $s$  of  $\mathcal{F}_\omega$  are zeros of order  $m_s = P_s - 2$  of the quadratic differential  $\omega$ . Hence

$$\sum m_s = -2\chi(X) = -2(2 - 2g) = 4g - 4$$

where  $g$  is the genus of  $X$ . This is the familiar formula for the number of the zeros of a holomorphic quadratic differential on a Riemann surface. Theorem 2 gives therefore a purely topological proof for the analytic formula.

**Definition 6.** *A quadratic differential  $\omega$  is a Strebel differential if the leaves of  $\mathcal{F}_\omega$  are compact.*

**Remark.** One can show that if critical leaves of  $\mathcal{F}_\omega$  are compact, then all leaves of  $\mathcal{F}_\omega$  are compact. Therefore a quadratic differential is a Strebel differential if the critical leaves of its horizontal foliation are compact.

Even though Strebel-differentials are rather special, they are dense in  $D^2(X)$  [2].

Consider a measured foliation  $\mathcal{F}$  on a Riemann surface  $X$ . Assume that the leaves of  $\mathcal{F}$  are compact. Consider the singular leaves  $F_1, F_2, \dots, F_n$ . Take the complement  $X^* = X \setminus \cup_{k=1}^n F_k$ .  $X^*$  is a disjoint union of interiors of Riemann surfaces  $X_1, X_2, \dots, X_m$  with boundary.

The measured foliation  $\mathcal{F}$  on  $X$  induces a measured foliation  $\mathcal{F}_l$  on  $X_l$ ,  $l = 1, 2, \dots, m$ . The boundary curves of  $X_l$  are leaves of this foliation. Consequently, the double  $X_l^d$  of  $X_l$  gets a measured foliation from that of  $X_l$ . Recall that the double  $X_l^d$  of  $X_l$  is formed taking two copies of  $X_l$  and identifying their corresponding boundary points.

The Riemann surface  $X_l^d$  is a compact and oriented Riemann surface without boundary. The above defined measured foliation of  $X_l^d$  does not have any singularities. Hence, by Theorem 1,  $\chi(X_l^d) = 0$ . The Riemann surface  $X_l$  is orientable, compact and it has a non-empty boundary. Furthermore,  $\chi(X_l) = \frac{1}{2}\chi(X_l^d) = 0$ . We conclude that  $X_l$  is an annulus.

An annulus can be pictured as a cylinder. Gluing the cylinders  $X_1, X_2, \dots, X_m$  together along the boundary gives us the original surface  $X$ . Some possible configurations are presented in the picture below.

$$g = 2, m = 1 \qquad g = 2, m = 2$$

$$g = 2, m = 3$$

FIGURE 6.

Consider the measured foliation  $\mathcal{F}_l$  on  $X_l$ . Let  $\theta_l$  be an arc connecting the two boundary components of  $X_l$ . Consider its homotopy class  $[\theta_l]$  which consists of arcs whose end-points lay on the different boundary components of  $X_l$  and which are freely homotopic to  $\theta_l$ .

Use the notation

$$\ell_{\mathcal{F}_l}([\theta_l]) = \inf\{\ell_{\mathcal{F}_l}(\theta) \mid \theta \in [\theta_l]\}.$$

The number  $\ell_{\mathcal{F}_l}([\theta_l])$  is the *height of the cylinder*  $X_l$ . The Whitehead-equivalence class of the foliation  $\mathcal{F}$  is defined by the curves  $\sigma_l$  and the heights  $\ell_{\mathcal{F}_l}([\theta_l])$ .

Assume now that  $\omega$  is a Strebel-differential on  $X$ . Let  $\mathcal{F}_\omega$  be the  $\omega$ -horizontal foliation of  $X$ , and let  $X_1, X_2, \dots, X_m$  be the corresponding cylinders. In view of Lemma 1, the cylinders  $X_l$  acquire Euclidean metrics; the natural local variables  $\xi_\omega$  of  $\omega$  actually transform the flat metric of the plane to that of  $X \setminus \{\text{the zeros of } \omega\}$ .

It follows that the Riemann surfaces  $X_1, X_2, \dots, X_m$  are straight Euclidean cylinders. The Euclidean height of the cylinder  $X_l$  equals the above defined number  $\ell_{\mathcal{F}_\omega}([\theta_l])$ . The top and the bottom of the cylinder have the same length.

We have shown that each holomorphic quadratic differential defines a canonical measured foliation, the associated horizontal foliation. The converse is not true. There are measured foliations which are not horizontal foliations of any holomorphic quadratic differential.

To get an example of such a measured foliation take two cylinders foliated by horizontal circles. Give some measure on these foliations. Glue the cylinders together as indicated in Fig. 7.

In this way we obtain a measured foliation of a Riemann surface  $X$  of genus 2. If this foliation were an  $\omega$ -horizontal foliation of a quadratic differential  $\omega$ , the top and the bottom of the cylinders would have the same Euclidean length. This would imply that

FIGURE 7. Example of Hubbard and Masur.

$$\begin{aligned} l_1 + l_6 &= l_1 + l_2 + l_3 + l_4 \\ l_4 + l_5 &= l_2 + l_3 + l_5 + l_6. \end{aligned}$$

This system has no positive solutions.

The above example is due to Hubbard and Masur ([6]). We have, however, the following result which has been proved in several steps by Hubbard and Masur [6], Marden and Strebel [7] and Gardiner [3] (see also [4]).

**Theorem 4.** *The mapping*

$$D^2(X) \setminus \{0\} \rightarrow \mathcal{MF}(\Sigma), \omega \mapsto [\mathcal{F}_\omega],$$

*is a homeomorphism.*

Instead of studying  $\mathcal{MF}(\Sigma \subset [0, \infty)^S$  it is, for our purposes, more fruitful to study the image of  $\mathcal{MF}(\Sigma)$  in the projective space  $\mathbf{P}[0, \infty)^S$  under the projection  $\pi : [0, \infty)^S \setminus \{0\} \rightarrow \mathbf{P}[0, \infty)^S$ . Use the notation  $\mathbf{PMF}(\Sigma)$  for  $\pi(\mathcal{MF}(\Sigma))$ . Theorem 4 has now the following immediate corollary.

**Corollary 1.**  *$\mathcal{MF}(\Sigma)$  is homeomorphic to  $\mathbb{R}^{6g-6} \setminus \{0\}$ . The space  $\mathbf{PMF}(\Sigma)$  of projective equivalence classes of measured foliations of  $\Sigma$  is homeomorphic to  $S^{6g-7}$ .*

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