MODULI SPACES OF REAL ALGEBRAIC CURVES

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1. Introduction

Let g, $g > 1$, be an integer, $k = \mathbb{R}$ or $k = \mathbb{C}$, and consider the moduli space

$$M_k^g = \{\text{isomorphism classes of genus } g \text{ algebraic curves over } k\}.$$

The complex moduli space $M_{\mathbb{C}}^g$ has been an object of intensive studies for more than 100 years now. It is known that $M_{\mathbb{C}}^g$ is a normal complex space and a quasiprojective algebraic variety, and the structure of $M_{\mathbb{C}}^g$ is, by now, fairly well understood in comparison with that of $M_{\mathbb{R}}^g$.

The moduli space of real algebraic genus g curves, $M_{\mathbb{R}}^g$, is the object of this review. Some properties of $M_{\mathbb{R}}^g$ can be derived from those of $M_{\mathbb{C}}^g$ but $M_{\mathbb{R}}^g$ is inherently more complex than $M_{\mathbb{C}}^g$, and new methods are needed. For instance, $M_{\mathbb{R}}^g$ does not have an algebraic structure while $M_{\mathbb{C}}^g$ does. Topologically $M_{\mathbb{R}}^g$ is more complicated than $M_{\mathbb{C}}^g$ — the real moduli space is not even connected.

The main result is that $M_{\mathbb{R}}^g$ has $\lfloor \frac{3g+4}{2} \rfloor$ components each of which is a real analytic subset of a complex orbifold. The moduli space $M_{\mathbb{R}}^g$ has a natural compactification obtained by allowing the curves to have double points in such a way that each component of the complement of the double points has a negative Euler characteristic. The moduli space $\overline{M}_{\mathbb{R}}^g$ of these stable genus g real algebraic curves is connected — a property conjectured by Klein.

The aim of this review is to explain the above results. To that end it is necessary to recall results about the topology and the geometry of real algebraic curves. That is done in Sections 2 and 3. These results about the topology of real algebraic curves are needed to count the number of components of $M_{\mathbb{R}}^g$, while the geometric results are needed to understand possible degenerations of real algebraic curves, i.e., to study the compactification $\overline{M}_{\mathbb{R}}^g$ of the real moduli space.

Components of $M_{\mathbb{R}}^g$ will be studied using quasiconformal mappings and classical theory of Teichmüller spaces. An effort has been made, in Sections 4 and 5, to explain these tools. Since these deliberations might not be detailed enough for a novice reader, references to all quoted results are included.

The inclusion $\mathbb{R} \subset \mathbb{C}$ induces a mapping $f : M_{\mathbb{R}}^g \to M_{\mathbb{C}}^g$ which simply forgets the real structure of a real curve. This mapping is not one–to–one, but it is of some interest anyway. The complex space $M_{\mathbb{C}}^g$ has a natural real structure which is induced by the complex conjugation. This real structure is an antiholomoprhic involution $\sigma^* : M_{\mathbb{C}}^g \to M_{\mathbb{C}}^g$ which takes the isomorphism class of a complex algebraic curve onto that of its complex conjugate. It is immediate that the image $f(M_{\mathbb{R}}^g) \subset M_{\mathbb{C}}^g$ is left point–wise fixed by this real structure, i.e., that $f(M^g_{\mathbb{R}})$ is contained in the real part $M^g_{\mathbb{C}}(\mathbb{R})$ of $M^g_{\mathbb{C}}$. It was first observed by Earle that $f(M^g_{\mathbb{R}}) \neq M^g_{\mathbb{C}}(\mathbb{R})$, i.e., that there are complex algebraic curves which are isomorphic to their complex conjugates but are not isomorphic to curves defined by real polynomials. The presence of these complex curves makes the study of $f(M^g_{\mathbb{R}})$ more complicated. It turns out that for $g > 3$, $f(M^g_{\mathbb{R}})$ is that part of $M^g_{\mathbb{C}}(\mathbb{R})$ where the local real dimension of $M^g_{\mathbb{C}}(\mathbb{R})$ is as large as possible, i.e., $3g-3$. Hence $f(M^g_{\mathbb{R}})$ is the quasiregular real part of $M^g_{\mathbb{C}}$ and as such a semialgebraic subset of the quasiprojective complex variety $M^g_{\mathbb{C}}$.

This review does not contain new results. Most of the material presented here dates back to 1970's but some newer material is also included. Several of the results explained here were independently proved by mathematicians in the west and by mathematicians in the former Soviet Union where, in the 1970's, postdoctoral fellows and graduate students usually were not allowed to correspond with their colleagues in the West.

Main references to the theory reviewed here are [26] and [33]. A paper of Earle ([13]), in which he observed that $f(M^g_{\mathbb{R}}) \neq M^g_{\mathbb{C}}(\mathbb{R})$, was an important motivation for these studies. In that paper, Earle also developed Teichmüller space theory for real algebraic curves. That turned out to be an important tool for studying the components of $M^g_{\mathbb{R}}$.

Much of the material included in Sections 4 and 5 is explained in more detail in [22] and in [35].

By the real version of the Torelli theorem, one can study of $M^g_{\mathbb{R}}$ by studying the moduli space of principally polarized real abelian varieties, and vice versa. In that way it is possible to endow the components of $M^g_{\mathbb{R}}$ with semialgebraic structures, a result that appears to be out of reach of the analytic and geometric methods of this review. For details see [34] and Robert Silhol's paper ([37]) in this volume.

2. TOPOLOGY OF REAL ALGEBRAIC CURVES

To fix the notation we recall that a topological model for a real algebraic curve C consists of an oriented compact genus g surface Σ together with an orientation reversing involution $\sigma : \Sigma \to \Sigma$. We assume here that $g > 1$. Real algebraic curves of the topological type (Σ, σ) are complex structures X on Σ such that $\sigma : (\Sigma, X) \to (\Sigma, X)$ is antiholomorphic. Note that whenever we consider a complex structure on an oriented surface Σ we assume that the orientation of this structure agrees with that of Σ.

For a real algebraic curve (Σ, X) we can also consider the associated *Klein surface* (cf. [2]) which is the quotient surface $X/\langle\sigma\rangle$. The fixed–points of the involution σ are boundary points of $X/\langle\sigma\rangle$. The Klein surface associated to a real algebraic curve may be non–orientable. In such a case it does not carry a complex structure but rather a so called dianalytic structure for which the coordinate transition functions are allowed to be either locally holomorphic or antiholomorphic.

Let

$$n = n(\Sigma, \sigma) = \text{ the number of the components of the fixed-point set } \Sigma_\sigma$$

and

$$k = k(\Sigma, \sigma) = 2 - \text{ the number of components of } \Sigma \setminus \Sigma_\sigma.$$

The topological invariants n and k together with the genus g of the surface Σ determine the topological type of (Σ, σ) uniquely. This means that if (Σ, τ) has the same invariants n and k as (Σ, σ), then there is a homeomorphism $f : \Sigma \to \Sigma$ such that $\sigma \circ f = f \circ \tau$. This also means that the quotient surfaces $\Sigma/\langle\sigma\rangle$ and $\Sigma/\langle\tau\rangle$ are homeomorphic. Here $\langle\sigma\rangle$ denotes the group generated by σ.

For a fixed genus g, the invariants n and k satisfy:

- $k = 0$ or $k = 1$
- If $k = 0$, then $0 < n \leq g + 1$ and $n \equiv g + 1 \mod 2$
- If $k = 1$, then $n \leq g$.

These are the only restrictions, and there are $\lfloor \frac{3g+4}{2} \rfloor$ different topological types of real curves of genus g ([18], [19], [38]; for a recent detailed account see Paola Frediani's thesis [14]).

Let $(\Sigma_j, X_j, \sigma_j)$, $j = 1, 2$, be real algebraic curves. They are *real isomorphic* if there is a conformal mapping $h : (\Sigma_1, X_1) \to (\Sigma_2, X_2)$ such that $h \circ \sigma_1 = \sigma_2 \circ h$. This means that real isomorphic real algebraic curves are of the same topological type.

3. Geometry of real algebraic curves

Let $X = (\Sigma, X)$ be a real algebraic curve of the topological type (Σ, σ). Assume that the genus of the compact topological surface Σ is at least 2. Recall that by the definition, $\sigma : X \to X$ is an antiholomorphic involution. Let $Z = X/\langle\sigma\rangle$ be the associated Klein surface.

Since the genus of the Riemann surface X is > 1, the unit disk D is the universal cover of the Riemann surface X. If G is the cover group, then $X = D/G$. Recall that G is a Fuchsian group consisting of (hyperbolic) Möbius transformations.

The involution $\sigma : X \to X$ lifts to an antiholomorphic self–mapping $s : D \to D$. The lifting s needs not be an involution anymore. It satisfies, however, the condition $s^2 \in G$. The mapping $s : D \to D$ is a glide reflection of D onto itself, i.e., a hyperbolic Möbius transformation followed by a reflection in its axis. R. J. Sibner ([36]) has analyzed in detail the different generating sets for the group $F = \langle G, s\rangle$ generated by the elements of G and by s.

We have

$$X = D/G, \text{ and } Z = D/F. \tag{1}$$

Since the elements of F (and of G) are isometries of the unit disk equipped with the hyperbolic metric, the Riemann surface X and the Klein surface Z both get a hyperbolic metric from that of the unit disk. In this metric, the boundary components of the Klein surface Z are geodesic curves.

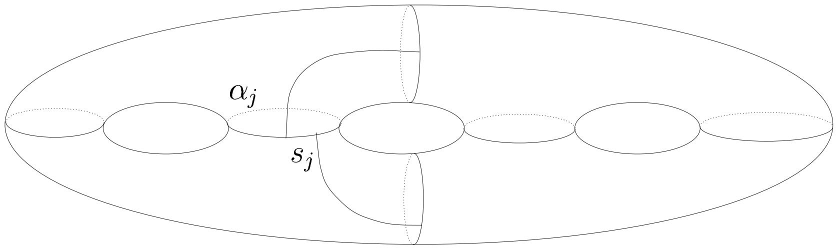

FIGURE 1. The lengths of the decomposing curves α_j together with the respective gluing angles θ_j are the Fenchel–Nielsen coordinates of a Riemann surface. The gluing angles are always between 0 and 2π but the lengths of the decomposing curves can be anything. The gluing angles are defined by dropping perpendicular geodesics to the curve α_j from adjacent decomposing curves. An ordering as well as orientations of the curves are needed to make this precise. Let s_j be the distance, measured in the positive direction along the oriented curve α_j, between the end–points of these perpendicular geodesics, and let ℓ_j denote the length of α_j. Then the gluing angle θ_j is defined by $\theta_j = 2\pi s_j/\ell_j$.

Each compact Riemann or Klein surface of genus > 1 can be built by gluing together *pairs of pants,* i.e., spheres with three geodesic boundary components. Conversely, every such Riemann surface can be decomposed into pairs of pants by disjoint simple closed geodesic curves α_j. The gluings of pairs of pants are specified by so called gluing angles θ_j associated to each decomposing curve α_j. These gluing angles θ_j together with the lengths of the decomposing geodesic curves α_j form the *Fenchel–Nielsen parameters* of the Riemann surface X. These parameters can be extended to Klein surfaces $Z = X/\langle\sigma\rangle$ with minor technical modifications ([27]). For a general discussion with precise definitions of the Fenchel–Nielsen coordinates see [39].

Lipman Bers has shown ([4]) that any hyperbolic compact Riemann surface admits a length controlled decomposition into pairs of pants. In [33] Seppälä extended that result and showed that, in the case of Riemann surfaces with an antiholomorphic involution, i.e., in the case of real algebraic curves, this decomposition can be so chosen that it is invariant under the involution (the complex conjugation).

In [9] Buser and Seppälä derived the following linear bound:

Theorem 1. *Let X be a compact Riemann surface of genus g, $g > 1$. There exist, on X, $3g-3$ disjoint simple closed geodesic curves α_j of length $< 21g$. These curves decompose X into pairs of pants. If X admits an antiholomorphic involution $\sigma : X \to X$, then the curves α_j can be chosen in such a way that the set of these curves is invariant under the mapping σ.*

The above bound, $21g$, for the lengths of the decomposing curves is much better than the one that the methods of Bers would yield. In a recent manuscript, Buser and Seppälä have extended these considerations to study "short" homology bases for Riemann and Klein surfaces ([10]).

This length controlled σ–invariant decomposition of a symmetric Riemann surface into pairs of pants has important applications in the study of compactifications of the moduli space of real algebraic curves.

4. Quasiconformal mappings

Quasiconformal mappings provide convenient tools to study deformations of Riemann surfaces. In this Section we recall the their basic properties including the Teichmüller extremal mapping theorem.

We use the notations

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Furthermore $\partial_\alpha f(z)$ denotes the derivative of f in the direction of a (unit) vector α.

Definition 1. *Let A be an open set in $\mathbb{C}$ and $K \geq 1$. A sense preserving homeomorphism $f : A \to f(A) \subset \hat{\mathbb{C}}$ is K–*quasiconformal *if and only if the following holds:*

- *f in absolutely continuous on lines in A.*
- $\max_\alpha |\partial_\alpha f(z)| \leq K \min_\alpha |\partial_\alpha(z)|$ *almost everywhere in A.*

A mapping f is quasiconformal *if it is K–quasiconformal for some constant K. The smallest number K for which a quasiconformal mapping f is K–quasiconformal, is called the* maximal dilatation *of f.*

Often quasiconformal mappings are defined using their more geometric properties (such as the fact that quasiconformal mappings map small circles onto small ellipses for which the ratio of the axes is bounded). Then the above definition is actually a theorem ([22, Theorem I.3.5]). For our purposes it is, however, convenient to regard this as the definition of quasiconformal mappings in the plane.

The property that f is absolutely continuous on lines guarantees that the mapping f has partial derivatives almost everywhere. Hence the second condition makes sense.

Let $f : A \to f(A)$ be a K–quasiconformal mapping and z a point where f is differentiable and $J_f(z) > 0$.

Since $\max_\alpha |\partial_\alpha f| = |\partial f| + |\overline{\partial} f|$, $\min_\alpha |\partial_\alpha f| = |\partial f| - |\overline{\partial} f|$, the second condition in the definition of quasiconformal mappings is equivalent to

$$|\overline{\partial} f(z)| \leq \frac{K-1}{K+1} |\partial f(z)|. \tag{2}$$

Since the Jacobian

$$J_f(z) = |\partial f(z)|^2 - |\overline{\partial} f(z)|^2$$

is positive, $\partial f(z) \neq 0$, and we can form the quotient

$$\mu_f(z) = \frac{\overline{\partial} f(z)}{\partial f(z)}. \tag{3}$$

Definition 2. *The function* μ_f *which is defined almost everywhere by the formula (3) is* the complex dilatation *of the quasiconformal mapping* f*.*

Observe:

- μ_f is a Borel measurable function.
- By (2), $|\mu_f(z)| \leq k = \frac{K-1}{K+1} < 1$ almost everywhere.

Definition 3. *A Borel measurable function* $\mu : A \to \mathbb{C}$ *which satisfies*

$$\text{ess sup}_z \, \|\mu(z)\| < 1$$

is a Beltrami differential *in the domain* A*. Let* μ *be a Beltrami differential in* A*. The differential equation*

$$\overline{\partial} f = \mu \partial f \tag{4}$$

is called a Beltrami equation.

For a holomorphic mapping f, μ_f vanishes identically, and (4) reduces to the Cauchy–Riemann equation $\overline{\partial} f = 0$.

Theorem 2. *A homeomorphism* $f : A \to f(A)$ *is* K*–quasiconformal if and only if the partial derivatives of* f *are locally in* L^2 *and satisfy the condition (2) for almost all* z*.*

Proof. [22, Theorem I.4.1].

Let f and g be quasiconformal mappings of a domain A with complex dilatations μ_f and μ_g, respectively. Direct computation yields the transformation rule

$$\mu_{f \circ g^{-1}}(\zeta) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \left(\frac{\partial g(z)}{|\partial g(z)|} \right)^2, \ \ \zeta = g(z), \tag{5}$$

which is valid for almost all $z \in A$, and hence for almost all $\zeta \in g(A)$.

The above transformation rule can be easily computed but it has deep consequences. Writing $(\partial g(z)/|\partial g(z)|)^2 = a$, $\mu_g(z) = b$, $\mu_f(z) = \xi$ and $\mu_{f \circ g^{-1}} = \omega$, the formula (5) becomes

$$\omega = \frac{a\xi - ab}{1 - \xi \overline{b}}.$$

The mapping $\xi \mapsto \omega$ is a Möbius–transformation mapping the unit disk onto itself. We conclude that *the complex dilatation of* $\mu_{f \circ g^{-1}}$ *depends holomorphically on the complex dilatation* μ_f. This fact plays a crucial role when defining the complex structure of a Teichmüller space.

The relationship between quasiconformal mappings and their Beltrami differentials or complex dilatations is very close. For our purposes they are simply two views of one object. That follows from the following important result.

Theorem 3 (Existence Theorem)**.** *Let μ be a measurable function in a domain A. Assume that $\|\mu\|_\infty < 1$. There exists a quasiconformal mapping of A whose complex dilatation agrees with μ almost everywhere in A.*

Proof. [20, p. 136] or [21, p. 191].

The concept of quasiconformality extends naturally for Riemann and Klein surfaces.

Definition 4. *A homeomorphism $f : X \to X'$ between Klein surfaces is K–*quasiconformal *if it is locally K–quasiconformal in the following sense: for each point $P \in X$ there is a connected open neighborhood U of P, a dianalytic local variable $z : U \to z(U) \subset \mathbf{C}$ and a dianalytic local variable $w : V \to w(V)$ of a neighborhood V of $f(P)$ such that the mapping $w \circ f \circ z^{-1}$ is a K–quasiconformal homeomorphism of a neighborhood of $z(P)$.*

Computing locally we may form the Beltrami differential of a quasiconformal mapping $f : X \to X'$ between Klein surfaces: If $\{(U_i, z_i)|\, i \in I\}$ is a dianalytic atlas of X (whose charts U_i are connected), then for each index $i \in I$ there is a dianalytic chart (V_{j_i}, w_{j_i}) of X' such that $f(U_i) \subset V_{j_i}$ and $w_{j_i} \circ f \circ z_i^{-1}$ is quasiconformal in $z_i(U_i)$.

Let τ_i be the complex dilatation of this quasiconformal mapping. Consider the family of functions $\mu_i = \tau_i \circ z_i^{-1}$ associated to the different dianalytic charts of X. The function μ_i is a complex valued L^∞–function defined on U_i and $\|\mu_i\|_\infty \leq \frac{K-1}{K+1} < 1$.

To see how the different functions μ_i are related to each other, consider two intersecting dianalytic charts (U_i, z_i) and (U_j, z_j) of X. Define the function $T_{ij} : U_i \cap U_j \to \mathbb{C}$ setting

$$T_{ij} = (\partial(z_i \circ z_j^{-1}) + \overline{\partial}(z_i \circ z_j^{-1})) \circ z_j.$$

The following transformation formula is a straightforward computation: If $z_i \circ z_j^{-1}$ is holomorphic at $z_j(P)$, $P \in U_i \cap U_j$, then

$$\mu_j(P) = \mu_i(P)\frac{\overline{T_{ij}(P)}}{T_{ij}(P)}. \tag{6}$$

If $z_i \circ z_j^{-1}$ is antiholomorphic at $z_j(P)$, then

$$\overline{\mu_j(P)} = \mu_i(P)\frac{\overline{T_{ij}(P)}}{T_{ij}(P)}. \tag{7}$$

Definition 5. *We say that a collection $\mu = \{\mu_i|\, i \in I\}$ of measurable functions $\mu_i : U_i \to \mathbb{C}$ associated to dianalytic charts (U_i, z_i) of X is a (–1,1)–*differential of the Klein surface X *if the functions μ_i satisfy the transformation rules (6) and (7). If, in addition,*

$$\sup_{i \in I} \|\mu_i\|_\infty < 1$$

then μ *is* a Beltrami differential of the Klein surface X. *Let us use the notation* $D^{(-1,1)}(X)$ *for the space of (–1,1)–differentials of* X *and the notation* $Bel(X)$ *for the space of the Beltrami differentials of* X.

For a Riemann surface X, (i.e., for an orientable Klein surface X) $D^{(-1,1)}(X)$ is a complex Banach–space and $Bel(X)$ is its open unit ball. If X is not orientable, then, since the transformation rule (7) has to hold, elements of $Bel(X)$ cannot be multiplied by complex numbers. Hence, for a non–orientable Klein surface X, $Bel(X)$ is the open unit ball of a real Banach space.

By the above remarks, the complex dilatation of a quasiconformal mapping of a Klein surface X is a Beltrami differential of X. The following theorem is an immediate application of the existence and the uniqueness theorems of plane quasiconformal mappings:

Theorem 4. *Let* μ *be a Beltrami differential of a Klein surface* $X = (\Sigma, X)$ *then there exists a dianalytic structure* X_μ *of the topological surface* Σ *such that the identity mapping* $(\Sigma, X) \to (\Sigma, X_\mu)$ *is quasiconformal with the complex dilatation* μ.

If $f_i : X \to f_i(X)$, $i = 1, 2$, *are two* μ*–quasiconformal mappings of* X, *then there exists an isomorphism* $g : f_1(X) \to f_2(X)$ *such that* $f_2 = g \circ f_1$.

Proof. Let $\mathcal{U} = \{(U_i, z_i) \mid i \in I\}$ be a dianalytic atlas of (Σ, X). Let $\mu_i : U_i \to \mathbb{C}$ be the measurable function associated to the chart (U_i, z_i) by the Beltrami differential μ.

By Theorem 3 there exists $\mu \circ z_i^{-1}$–quasiconformal mappings $f^i_\mu : z_i(U_i) \to \mathbb{C}$. Choose one for each index $i \in I$.

Then by the transformation rule (5) $\mathcal{U}_\mu = \{(U_i, f^i_\mu \circ z_i \mid i \in I\}$ is dianalytic atlas of Σ. Let X_μ be the dianalytic structure defined by this atlas. By the definition it is now clear that the identity mapping $(\Sigma, X) \to (\Sigma, X_\mu)$ is μ–quasiconformal proving the first statement.

The second statement follows directly from the transformation rule (5). ☐

Let $X = U/G$ where G is a reflection group. A (–1,1)–differential μ of X lifts to a function $\mu : U \to \mathbb{C}$. The transformation rules (6) and (7) are equivalent with the following formulae:

$$\mu = (\mu \circ g)\frac{\overline{\partial g}}{\partial g} \tag{8}$$

for orientation preserving Möbius–transformations $g \in G$ and

$$\overline{\mu} = (\mu \circ \sigma)\frac{\overline{\overline{\partial}\sigma}}{\overline{\partial}\sigma} \tag{9}$$

for glide–reflections $\sigma \in G$.

Definition 6. *A measurable function* μ *satisfying the transformation rules (8) and (9) with respect to the elements of a reflection group* G, *is* a *(–1,1)–*differential of the group G. *If, in addition,* $\|\mu\|_\infty < 1$, *then* μ *is* a

Beltrami differential of the group G. *We use the notation* $D^{(-1,1)}(G)$ *for (–1,1)–differentials of a reflection group G, and the notation* $Bel(G)$ *for Beltrami differentials of* G.

Provided that the group G does not contain orientation reversing elements $D^{(-1,1)}(G)$ is a complex Banach space and $Bel(G)$ is its open unit ball. If G contains also glide–reflections, $D^{(-1,1)}(G)$ is a real Banach space.

There is a real analytic homeomorphism between the unit disk and the complex plane. Nevertheless, there are no quasiconformal mappings between them. This follows rather easily from the quasi–invariance of conformal invariants under quasiconformal mappings. The case of compact Klein surfaces is, however, different: homeomorphic compact Klein surfaces are also quasiconformally equivalent by the following result.

Theorem 5. *Let X and Y be compact Klein surfaces. Each homeomorphism $f : X \to Y$ is homotopic to a quasiconformal mapping.*

Proof. It is well known that each homeomorphism $f : X \to Y$ is homotopic to a diffeomorphism. The dilatation quotient of such a diffeomorphism can be computed at each point of X. It does not depend on the choices of the local variables, and it is a continuous function on X. Since X is compact, this function has a finite maximum K on X. Hence a diffeomorphism homotopic to f is a quasiconformal mapping of X. There are many ways to find a diffeomorphism homotopic to a given homeomorphism. For more details and a direct construction we refer to [22, Theorem V.1.5.]. ☐

Let X and Y be homeomorphic compact Klein surfaces. By Theorem 5 there are quasiconformal mappings $X \to Y$. Teichmüller considered the problem of finding, in a given homotopy class of mappings $X \to Y$, one with the smallest maximal dilatation.

Theorem 6 (Teichmüller extremal mapping theorem)**.** *Let X and Y be compact Riemann surfaces of genus $g > 1$ and $f : X \to Y$ a homeomorphism. There exists always a unique quasiconformal mapping $F : X \to Y$ such that the following holds:*

- *F is homotopic to the mapping f.*
- *If $g : X \to Y$ is a K_g–quasiconformal mapping and homotopic to f, then $K_g \geq K_F$, where K_g is the maximal dilatation of g and K_F is that of F. If $K_g = K_F$, then also $g = F$.*

Teichmüller gave also an explicit description of the geometry of such an extremal mapping F. The most delicate part of this result is the uniqueness of the extremal mapping. Detailed proofs for the above results of Teichmüller can be found, for instance, in [22, Chapter V].

Observe that non–classical Klein surfaces are not usually considered in this context but the above result holds also for them. In other words we have: *A homotopy class of a homeomorphism between compact non–classical compact Klein surfaces of genus $g > 1$ always contains a unique quasiconformal mapping having the smallest maximal dilatation.*

It is out of the scope of this presentation to prove these results. A clear exposition can be found, for instance, in the monograph of Olli Lehto [22]. Observe that, in the above theorems, the uniqueness part is not anymore true if we drop the assumption that the genus is at least two.

5. Teichmüller spaces

5.1. **Definitions.** There are many alternative definitions for the Teichmüller space. We will next discuss some of these equivalent definitions.

Use the notation

$$\mathcal{M}(\Sigma) = \{\text{complex structures of } \Sigma \text{ which agree with the orientation of } \Sigma\}.$$

For a non–orientable surface $S = \Sigma/\langle\sigma\rangle$, we set

$$\mathcal{M}(S) = \{\text{dianalytic structures of } S\}.$$

Dianalytic structures differ from analytic structures in that the coordinate transition functions may also be antiholomorphic. Observe that the fixed–points of σ correspond to boundary points of S. The dianalytic structures of S are required to be such that they extend to the boundary. This implies that a dianalytic structure of S lifts to a usual complex structure of Σ.

The group $\text{Homeo}(\Sigma)$ of homeomorphic self–mappings of Σ acts on $\mathcal{M}(\Sigma)$ in the following way. Let $\alpha \in \text{Homeo}(\Sigma)$ and $X \in \mathcal{M}(\Sigma)$. Then $\alpha^*(X) \in \mathcal{M}(\Sigma)$ is defined requiring the mapping

$$\alpha : (\Sigma, \alpha^*(X)) \to (\Sigma, X) \tag{10}$$

be either holomorphic or antiholomorphic depending on whether α is orientation preserving or not. In the same way we define the action of $\text{Homeo}(S)$ on $\mathcal{M}(S)$.

It is clear, by the definitions, that the fixed-point set $\mathcal{M}(\Sigma)_{\sigma^*}$ consists of real algebraic curves of the topological type (Σ, σ).

Consider

$$\text{Homeo}_0(\Sigma) = \{f \in \text{Homeo}(\Sigma) \mid f \text{ homotopic to the identity}\},$$

and

$$\text{Homeo}_0(S) = \{f \in \text{Homeo}(S) \mid f \text{ homotopic to the identity}\},$$

Definition 7 (of Teichmüller spaces)**.** *The set theoretic quotient*

$$T(\Sigma) = \mathcal{M}(\Sigma)/\text{Homeo}_0(\Sigma)$$

is the Teichmüller space *of the surface Σ. We use also the notation T^g for the Teichmüller space of a genus g surface Σ. In the same way we define the Teichmüller space $T(S)$ of the surface $S = \Sigma/\langle\sigma\rangle$ as the quotient*

$$T(S) = \mathcal{M}(S)/\text{Homeo}_0(S).$$

Observe that homotopic self–mappings of an oriented surface are simultaneously orientation preserving. Consequently, if Σ is oriented, then $\text{Homeo}_0(\Sigma)$ is a subgroup of the group $\text{Homeo}_+(\Sigma)$ of orientation preserving homeomorphic self–mappings of Σ.

Definition 8 (of the modular group). *For an oriented surface Σ, the group* $\Gamma(\Sigma) = \mathrm{Homeo}_+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$ *is the* modular group *or the* mapping class group *of the surface Σ. For non–orientable surfaces S* the modular group *is defined setting* $\Gamma(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$. *We use also the notation Γ^g for the modular group of a genus g surface Σ.*

Definition 9 (of the moduli space). *For an oriented surface Σ, the quotient* $M(\Sigma) = \mathcal{M}(\Sigma)/\mathrm{Homeo}_+(\Sigma)$ *is the* moduli space . The moduli space *of a non–orientable surface S is* $M(S) = \mathcal{M}(S)/\mathrm{Homeo}(S)$. *We use also the notation M^g for the moduli space of a smooth genus g surface Σ.*

It follows from the above definitions that the modular group acts on the Teichmüller space, and that $M(\Sigma) = T(\Sigma)/\Gamma(\Sigma)$. An orientation reversing self–mapping f of Σ induces, likewise, a mapping $f^* : T(\Sigma) \to T(\Sigma), [X] \mapsto [f^*(X)]$, where the analytic structure $f^*(X)$ of Σ is defined requiring (10) be antianalytic.

Let $X, Y \in \mathcal{M}(\Sigma)$ be two analytic structures of a fixed compact surface Σ. By Theorem 5 there are quasiconformal mappings $(\Sigma, X) \to (\Sigma, Y)$ homotopic to the identity mapping of Σ. For such a quasiconformal mapping f, let K_f denote the maximal dilatation of f.

Definition 10 (of the Teichmüller metric). *The distance between two points $[X]$ and $[Y]$ of the Teichmüller space $T(\Sigma)$ in* the Teichmüller metric τ *of the $T(\Sigma)$ is defined by*

$$\tau([X],[Y]) = \inf\{\frac{1}{2}\log K_f \,|\, f \in \mathrm{Homeo}_0(\Sigma)\}.$$

The Teichmüller space $T(\Sigma)$ together with the Teichmüller metric is homeomorphic to an Euclidean space ([22, Theorem 9.2, page 241]). This is one of the deep consequences of the Teichmüller extremal mapping theorem that cannot be presented here. We refer to Chapter V of the monograph of Olli Lehto ([22]) for a complete and detailed treatment.

We observe, nevertheless, that the elements of the modular group $\Gamma(\Sigma)$ are isometries of the Teichmüller metric. This is an immediate consequence of the definitions. Hence they are, in particular, homeomorphic self–mappings of the Teichmüller space.

5.2. Teichmüller spaces of Beltrami differentials.

By Theorems 4 and 5 we can associate, to each Beltrami differential μ of a Klein surface X, a Klein surface X_μ, and vice versa. Consequently, Teichmüller spaces can be defined also in terms of Beltrami differentials.

To be more precise, choose a point $[X] \in T(\Sigma)$, which we will refer to as *the origin* of the Teichmüller space. Consider the space $Bel(X)$ of Beltrami differentials of X. Each $\mu \in Bel(X)$ defines a unique $X_\mu \in \mathcal{M}(\Sigma)$ such that the identity mapping of Σ is a μ– quasiconformal mapping $X \to X_\mu$. We say that two Beltrami differentials μ_1 and μ_2 of X are *equivalent*, $\mu_1 \approx \mu_2$, if the homotopy class of the identity mapping of Σ contains an isomorphism $(\Sigma, X_{\mu_1}) \to (\Sigma, X_{\mu_2})$.

Definition 11. *The set* $T(X) = Bel(X)/\approx$ *is* the Teichmüller space of Beltrami differentials of X.

It is an immediate consequence of these definitions that

$$T(X) \to T(\Sigma), [\mu] \mapsto [X_\mu], \tag{11}$$

is a bijection between these two Teichmüller spaces.

The Teichmüller metric of $T(X)$ is the pull back of the Teichmüller metric of $T(\Sigma)$ under the mapping (11).

Theorem 7. *Teichmüller space* $T(X)$ *is connected.*

Proof. Let μ represent an arbitrary point of $T(X)$. Then $t \mapsto [t\mu]$ is a path connecting the point $[\mu]$ to the origin of $T(X)$. Note that this path depends on the choice of μ ([40]). ☐

Lars Ahlfors showed as early as in 1959([1]) that, for an orientable surface Σ without boundary, the Teichmüller space $T(\Sigma)$ has a natural complex structure. The construction of Ahlfors was based on considering periods of Abelian differentials. It is not possible to present it here. We will, however, describe a way to decide which functions are holomorphic on $T(\Sigma)$. *Let us assume now that* Σ *is a compact and oriented surface without boundary.*

It is convenient to consider the Teichmüller space $T(X)$ of Beltrami differentials of X instead of $T(\Sigma)$. Let $\pi : Bel(X) \to T(X)$ be the projection. This is a continuous mapping with respect to the L^∞–metric on $Bel(X)$ and the Teichmüller metric on $T(X)$ (cf. e.g. [22, III.2.2]). Hence, for any open $U \subset T(X)$, $\pi^{-1}(U)$ is open in $Bel(X)$.

Holomorphic functions on $T(X)$. Let $U \subset T(X)$ be open. We declare a function $f : U \to \mathbb{C}$ *holomorphic* if the composition $f \circ \pi$ is a holomorphic function on the open set $\pi^{-1}(U)$ of the complex Banach space of (–1,1)–differentials of X.

In this way $T(X)$ becomes first *a ringed space.* It is not, *a priori,* clear that the above definition actually gives a good complex structure on $T(X)$. That is, however, the case if Σ is an oriented surface which does not have boundary (cf. e.g. [22, Chapter V]). This complex structure of $T(X)$ is then transported to a complex structure of $T(\Sigma)$ by requiring that the mapping (11) is holomorphic.

We still have to check that this complex structure of $T(\Sigma)$ does not depend on the choice of the origin X of the Teichmüller space $T(\Sigma)$. But that is an immediate consequence of the transformation formula (5) and the remark made after it.

The elements of the modular group are biholomorphic automorphisms of the Teichmüller space. *Royden has shown (*[29, Theorem 1 on page 281] *and* [30, Theorem 2 on page 379]*), in fact, that for surfaces of genus* $g > 2$, $\Gamma(\Sigma)$ *is the full group of holomorphic automorphisms of* $T(\Sigma)$. In the same way we verify that the mapping $\sigma^* : T(\Sigma) \to T(\Sigma)$, induced by an orientation reversing mapping $\sigma : \Sigma \to \Sigma$ (cf. formula (10)), is an antiholomorphic self–mapping of the Teichmüller space (for details see [31, 5.10]).

Theorem 8. *The Teichmüller space of compact genus g Riemann surfaces, $g > 1$, is a complex manifold of complex dimension $3g - 3$. The elements of the modular group Γ^g are holomorphic automorphisms of T^g. For $g > 2$, Γ^g is the full group of holomorphic automorphisms of T^g.*

This result can be extended to general finite dimensional Teichmüller spaces of oriented surfaces without boundaries. For a proof of this result we refer to the monograph of F. W. Gardiner [15, 9.2].

5.3. Teichmüller spaces of real algebraic curves.

In order to study Teichmüller spaces of real algebraic curves of a given topological type (Σ, σ) we have two choices. We may consider the quotient surface $S = \Sigma/\langle\sigma\rangle$ and reproduce the classical theory there. Alternatively we may work on the oriented compact surface Σ and make the constructions σ-equivariantly. Both approaches lead to the same result.

Let us first consider $S = \Sigma/\langle\sigma\rangle$. In this case $T(S)$ is *not* going to be a complex manifold. The Teichmüller space is, instead, a real analytic manifold.

The surface Σ is a compact oriented surface without boundary together with a projection $\pi : \Sigma \to S$ that is a ramified double covering mapping. It is ramified precisely at the points lying over the boundary points of S. The covering group of π is generated by an orientation reversing involution σ of Σ for which $\pi \circ \sigma = \pi$. The fixed point set of the involution σ corresponds to the boundary points of S.

Above we have observed that the mapping $\sigma : \Sigma \to \Sigma$ induces an antiholomorphic self–mapping σ^* of $T(\Sigma)$. This mapping is an involution since σ is an involution.

For any $X \in \mathcal{M}(S)$ let $\pi^*(X)$ be the complex structure of Σ which agrees with the orientation of Σ and for which the projection

$$\pi : (\Sigma, \pi^*(X)) \to (S, X)$$

is dianalytic. It is immediate that

$$\pi^* : T(S) \to T(\Sigma), [X] \mapsto [\pi^*(X)],$$

is a well–defined mapping of Teichmüller spaces. It is not difficult to show that the mapping $\pi^* : T(S) \to T(\Sigma)$ is an isometry with respect to the corresponding Teichmüller metrics. Hence it is, in particular, a homeomorphism of $T(S)$ onto $\pi^*(T(S))$.

Theorem 9. *Assume that $S = \Sigma/\langle\sigma\rangle$ is a non–classical surface of genus g, $g > 1$. We have*

$$\pi^*(T(S)) = T(\Sigma)_{\sigma^*}$$

Proof. Let X be a complex structure of S. The complex structure $\pi^*(X)$ of Σ has the (defining) property that $\sigma : (\Sigma, \pi^*(X)) \to (\Sigma, \pi^*(X))$ is an antiholomorphic involution. This implies that

$$[\pi^*(X)] \in T(\Sigma)_{\sigma^*}, \text{ i.e. that } \pi^*(T(S) \subset T(\Sigma)_{\sigma^*}.$$

To prove the converse inclusion, take a point $[Y] \in T(\Sigma)_{\sigma^*}$. By the definition this means that there is a holomorphic mapping $f : (\Sigma, Y) \to (\Sigma, \sigma^*(Y))$ which is homotopic to the identity mapping of Σ. (Recall that $\sigma^*(Y)$ is defined as that complex structure of Σ for which the mapping $\sigma : (\Sigma, \sigma^*(Y)) \to (\Sigma, Y)$ is antiholomorphic.)

The construction implies that $\tau = \sigma \circ f : (\Sigma, Y) \to (\Sigma, Y)$ is an antiholomorphic mapping. Then $\tau^2 : (\Sigma, Y) \to (\Sigma, Y)$ is a holomorphic mapping. Since σ is an involution and f is homotopic to the identity, we conclude that τ^2 is also homotopic to the identity. Since the genus of the Riemann surface (Σ, Y) is at least 2, the only holomorphic automorphism of (Σ, Y) that is homotopic to the identity is the identity itself. This implies that $\tau : (\Sigma, Y) \to (\Sigma, Y)$ is an antiholomorphic *involution* proving the theorem. □

For a surface $S = \Sigma/\langle\sigma\rangle$ that is either non–orientable or has a non–empty boundary (or both) we now identify $T(S)$ with its image $\pi^*(T(S)) = T(\Sigma)_{\sigma^*}$ in $T(\Sigma)$. Since σ^* is an antiholomorphic involution of $T(\Sigma)$, its fixed–point set, $\pi^*(T(\Sigma))$, is a real analytic manifold.

The Teichmüller spaces $T(S)$ were studied in the 1970's by several authors (C. Earle [13], S. Natanzon [26], and M. Seppälä [31]).

Let (Σ, σ) and (Σ, τ) be two different topological models for real algebraic curves of genus g. Then the respective Teichmüller spaces, $T(\Sigma)_{\sigma^*}$ and $T(\Sigma)_{\tau^*}$ are both real analytic manifolds. Since the topological types are different, the mappings σ^* and τ^*, of the Teichmüller space $T(\Sigma)$ onto itself, are not conjugate in the modular group. Buser and Seppälä (1998, [11]) have, however, constructed, for any such σ and τ, a real analytic diffeomorphism $d : T(\Sigma) \to T(\Sigma)$ such that $\sigma^* \circ d = d \circ \tau^*$. Thus any two real structures σ^* and τ^* of the Teichmüller space are conjugate in the group of real analytic diffeomorphisms of $T(\Sigma)$. This means, in particular, that the Teichmüller spaces of the respective real algebraic curves are diffeomorphic as real analytic manifolds (this can actually be shown also without this conjugating map d).

6. Moduli spaces

6.1. Complex orbifolds.

To describe the structure of the various moduli spaces we next introduce the concept of *complex orbifolds* (cf. Ratcliffe [28, §13.2]).

Let G be the group of biholomorphic self–mappings of $\mathbb{C}^n$, and let M be a Hausdorff space. An $(\mathbb{C}^n, G)$–*orbifold chart* for M is a pair (U, ϕ) where

- $U \subset M$ is an open set, and
- ϕ is a homeomorphism of U onto $\phi(U) \subset \mathbb{C}^n/\Gamma_U$ where Γ_U a finite subgroup of G.

A collection of $(\mathbb{C}^n, G)$–orbifold charts

$$\mathcal{A} = \{(U_i, \phi_i : U \to \mathbb{C}^n/\Gamma_i) \mid i \in I\}$$

is an $(\mathbb{C}^n, G)$–*orbifold atlas* if

- $M = \cup_{i \in I} U_i$

- If U_i and U_j intersect, then the *coordinate change function*
$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$
has the property that if x and y are points of $\mathbb{C}^n$ such that
$$\phi_j(\phi_i^{-1}(\Gamma_i x)) = \Gamma_j y,$$
then there is an element g of G such that $gx = y$ and g lifts $\phi_j \circ \phi_j^{-1}$ in a neighborhood of x, i.e.,
$$\phi_j \circ \phi_i^{-1}(\Gamma_i w) = \Gamma_j g w$$
for all w in a neighborhood of x.

Locally, a complex orbifold M is the quotient of an open set in $\mathbb{C}^n$ mod the action of finite group of holomorphic automorphisms of $\mathbb{C}^n$. A function $f : M \to \mathbb{C}$ is a *holomorphic function of the complex orbifold* if, near every point, it lifts to a holomorphic mapping of an open set of $\mathbb{C}^n$ into $\mathbb{C}$. In the same way we define *real analytic* functions of a complex orbifold. The condition regarding the change of variable implies that the above definitions do not depend on the choice of the local variable.

6.2. **Moduli spaces of real curves of a given topological type.** The *moduli space* of the surface Σ is

$$M(\Sigma) = \mathcal{M}(\Sigma)/\mathrm{Homeo}_+(\Sigma), \tag{12}$$

where $\mathrm{Homeo}_+(\Sigma)$ denotes the subgroup of the group $\mathrm{Homeo}(\Sigma)$ consisting of those homeomorphisms which preserve the orientation. The *modular group* $\Gamma(\Sigma) = \mathrm{Homeo}_+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$ acts on the Teichmüller space $T(\Sigma)$, and excluding certain special cases, $\Gamma(\Sigma)$ acts properly discontinuously as the full group of holomorphic automorphisms of $T(\Sigma)$ ([30]).

Consider the Teichmüller space $T(\Sigma)$. Let X be a complex stucture on Σ. For any choice of X, we can form the Bers' embedding ([3]) of $T(\Sigma)$ into the space of quadratic differentials on (Σ, X). The construction of this embedding is rather technical. What matters for us is that the space of quadratic differentials of a compact genus g Riemann surface is $\mathbb{C}^{3g-3}$. Bers' construction implies that the subgroup
$$\Gamma_X = \{\gamma \in \Gamma(\Sigma) \mid \gamma([X]) = [X]\}$$
of the modular group fixing the point $[X] \in T(\Sigma)$ corresponds to a group of linear mappings of the space of quadratic differentials of (Σ, X) onto itself. Also, since the group $\Gamma(\Sigma)$ acts properly discontinuously on $T(\Sigma)$, every group Γ_X is finite.

It then follows, from the definitions, that the moduli space $M(\Sigma) = T(\Sigma)/\Gamma(\Sigma)$ is a complex orbifold.

In the definition of the moduli space of real algebraic curves of a given topological type (Σ, σ) it is, as in the case of Teichmüller spaces, convenient to consider the quotient surface $S = \Sigma/\langle\sigma\rangle$ rather than working on Σ. The moduli space of the possibly non–orientable surface S is simply

$$M(S) = \mathcal{M}(S)/\mathrm{Homeo}(S). \tag{13}$$

The space $M(S)$ is also referred to as *the moduli space of Klein surfaces of the topological type* S.

Complications start at this point. As in the case of Teichmüller spaces, consider the mapping

$$\rho : M(S) \to M(\Sigma), [X] \mapsto [X^d] \tag{14}$$

defined by the liftings of dianalytic structures X of $S = \Sigma/\langle\sigma\rangle$ to analytic structures X^d of Σ. This mapping just forgets the real structure of the real algebraic curve (i.e., point in $M(S)$) in question.

Some symmetric Riemann surfaces X^d admit several antiholomorphic involutions ([5], [6], [7], [8], [17], [23], [24], [25]), hence the mapping $\rho : M(S) \to M(\Sigma)$ is not one-to-one.

More precisely, the mapping ρ is not one–to–one for the following reason. Define first the modular group $\Gamma(S)$ of the possibly non–orientable surface S in the usual way by setting

$$\Gamma(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S).$$

Then $\Gamma(S)$ acts on the Teichmüller space $T(S)$, and clearly $M(S) = T(S)/\Gamma(S)$. Any homeomorphic self–mapping f of the surface $S = \Sigma/\langle\sigma\rangle$ lifts to a unique orientation preserving homeomorphism F of Σ. This lifting satisfies $F \circ \sigma = \sigma \circ F$. In this way, elements of $\Gamma(S)$ lift to elements of $\Gamma(\Sigma)$, and we can map $\Gamma(S)$ into $\Gamma(\Sigma)$. It is immediate, that $\Gamma(S)$ becomes, in this way, the centralizer of σ^* in $\Gamma(\Sigma)$. In particular, the lifting of $\Gamma(S)$ is *not* the full modular group $\Gamma(\Sigma)$. Consequently, the mapping $\rho : M(S) \to M(\Sigma)$ is not going to be one–to–one.

Let $\Gamma(\Sigma, \sigma) = \{\gamma \in \Gamma(\Sigma) \mid \sigma^*\gamma = \gamma\sigma^*\}$ be the centralizer of σ^* in $\Gamma(\sigma)$. Then we may consider the intermediate moduli space $M(\Sigma, \sigma) = T(\Sigma)/\Gamma(\Sigma, \sigma)$. The diagram

$$\begin{tikzcd}
T(S) \arrow[r, "\rho"] \arrow[d, "\pi"] & T(\Sigma) \arrow[d, "\pi"] \\
T(S)/\Gamma(S) = M(S) \arrow[r, "\rho"] & M(\Sigma, \sigma) = T(\Sigma)/\Gamma(\Sigma, \sigma)
\end{tikzcd} \tag{15}$$

commutes, and now the induced mapping $\rho : M(S) \to M(\Sigma, \sigma)$ is one–to–one.

On the other hand, $M(\Sigma, \sigma) = T(\Sigma)/\Gamma(\Sigma, \sigma)$ is a quotient of a complex manifold via the action of a properly discontinuous subgroup $\Gamma(\Sigma, \sigma)$ of the modular group. In view of the considerations regarding the orbifold structure of the ordinary moduli space, it follows that $M(\Sigma, \sigma)$ is also a complex orbifold. Now the image of $M(S)$ in $M(\Sigma, \sigma)$ is a projection of a real analytic submanifold $T(\Sigma)_{\sigma^*}$ of $T(\Sigma)$. Hence it is a real analytic subset of the orbifold $M(\Sigma, \sigma)$ ([32]) in the sense that it is locally the zero set of a real analytic function on $M(\Sigma, \sigma)$.

The situation here is more complicated than what one realizes at first. These complications are illustrated by the following example (cf. [16]).

Consider the finite complex plane $\mathbb{C}$, and the real analytic subset $\mathbb{R}$ consisting of the real points. The group generated by the involution $s(z) = -z$ acts on $\mathbb{C}$. The quotient space $X = \mathbb{C}/\langle s\rangle$ is then an orbifold with a singularity at the origin. The projection is given by

$$\mathbb{C} \to X,\ z \mapsto w = z^2. \tag{16}$$

A holomorphic or a real analytic function f in $\mathbb{C}$ defines a holomorphic or a real analytic function on $X = \mathbb{C}/\langle s\rangle$ if and only if $f \circ s = f$.

The real analytic subset $\mathbb{R}$ of $\mathbb{C}$ is the zero–set of the s invariant real analytic function $(\operatorname{Im} z)^2$. Here $\operatorname{Im} z$ denotes the imaginary part of the complex number z. The projection of $\mathbb{R} \subset \mathbb{C}$ in $X = \mathbb{C}/\langle s\rangle$ is the non–negative real line. It is the zero–set of the real analytic function $(\operatorname{Im} \sqrt{w})^2$. Observe that this is a real analytic function in X, it is not a real analytic function in $\mathbb{C}$. The quotient space X has a cone singularity at the origin. The projection of the real analytic subset $\mathbb{R} \subset \mathbb{C}$ has also a singularity at the origin. This time the singularity is a boundary point. The projection of $\mathbb{R}$ in X is, nevertheless, a real analytic subspace of the complex orbifold X.

Observe that, in this example, the mapping $z \mapsto z^2$ of the Riemann surface $\mathbb{C}$ onto $\mathbb{C}/\langle s\rangle$ is holomorphic but not biholomorphic. The mapping is not conformal at the origin. Hence, as Riemann surfaces, $\mathbb{C} \neq \mathbb{C}/\langle s\rangle$.

The same phenomena is present in the moduli spaces of real curves. They are real analytic subsets of certain complex orbifolds, but they do have, in general, boundary points or, rather, cone points. These singular points correspond to singular points of the complex orbifold (see [16]).

6.3. **Moduli spaces of real curves of a given genus.**

Let $g > 1$ be an integer. Consider *the moduli space of real algebraic curves of genus g*

$$M^g_{\mathbb{R}} = \{\text{isomorphism classes of real algebraic curves of genus } g\}.$$

This space is our main object of interest.

By considerations of Section 2, there are $m(g) = \lfloor \frac{3g+4}{2} \rfloor$ different topological types of real curves of genus g.

By considerations of the preceding Section, the moduli space $M(S)$ of real curves of the topological type $S = \Sigma/\langle\sigma\rangle$ is a connected real analytic subset of a complex orbifold. It is clear that real curves of the same genus but of different topological types cannot be real isomorphic. It follows that $M^g_{\mathbb{R}}$ is the disjoint union of the $m(g)$ moduli spaces $M(S)$.

It is of some interest to consider the mapping

$$f : M^g_{\mathbb{R}} \to M^g$$

which simply forgets the real structure of a real curve, and maps the real isomorphism class of a real algebraic curve C onto the complex isomorphism class of the same curve (viewed as a complex curves).

We have

Theorem 10. *Smooth real algebraic genus g curves in the moduli space of complex genus g curves form a connected subset, i.e., $f(M^g_{\mathbb{R}})$ is a connected subset of M^g.*

This follows from the observations that real hyperelliptic curves form a connected subset of M^g. For a proof see [12].

Any orientation reversing involution $\sigma : \Sigma \to \Sigma$ of the genus g compact topological surface Σ induces an antiholomorphic involution $\sigma^* : M^g \to M^g$. Observe that the induced involution σ^* does not depend on the choice of the involution $\sigma : \Sigma \to \Sigma$. For if $\tau : \Sigma \to \Sigma$ is another involution, then

$$\tau = \tau \circ \sigma \circ \sigma = g \circ \sigma$$

where g is an orientation preserving automorphism of Σ. Hence $\tau^* = \sigma^* \circ g^* = \sigma^*$ since $g^* \in \Gamma^g = \Gamma(\Sigma)$.

It follows that

$$f(M^g_{\mathbb{R}}) \subset M^g_{\sigma^*} = \text{ the fixed–point set of } \sigma^*.$$

The complex moduli space M^g is a complex space together with the real structure σ^*. The fixed–point set $M^g_{\sigma^*}$ is the real part of M^g. The conclusion is that real curves are contained in the real part of the complex moduli space. A surprise is that there are complex curves which have real moduli but which curves are not isomorphic to real curves. Such curves have non–trivial automorphisms. This argument yields the following:

Theorem 11. *Real algebraic curves of genus g, $g > 3$, in the moduli space of complex algebraic genus g curves form the quasiregular real part of the real structure σ^* of the complex space moduli M^g.*

For a proof see [32]. The quasiregular real part consists of those real points where the local dimension of the real part is as large as possible (i.e. $3g - 3$).

These results characterize the space of smooth real algebraic curves in the moduli space of complex algebraic curves.

6.4. **Moduli spaces of stable real curves.** A topology can be defined on the moduli space of smooth genus g, $g > 1$, Riemann surfaces by using the Fenchel–Nielsen coordinates (see Section 3 or [39]). To that end, fix a decomposition of a topological genus g surface into pairs of pants. Let α_j, $j = 1, 2, \dots, 3g - 3$, be the decomposing curves. We now take a geometric point of view to moduli, and consider, instead of complex structures, the associated hyperbolic metrics. Without loss of generality we may assume that all hyperbolic metrics under consideration are such that the decomposing curves α_j are simple closed geodesic curves.

Let now $\ell_j(X)$ denote the length of the geodesic curve α_j on the Riemann surface X. Let $\theta_j(X)$ be the corresponding gluing angle (cf. Figure 1). It turns out that the parameters ℓ_j and θ_j are real analytic functions on the Teichmüller space $T(\Sigma)$ (cf. [39]). Hence the topology of the Teichmüller space (and that of the moduli space) could also have been defined by requiring the Fenchel–Nielsen parameters be continuous.

These parameters are useful when picturing degenerations of Riemann surfaces. Let $([X_k])$ be a degenerating sequence in the moduli space $M^g = M(\Sigma)$. By considerations of Section 3, we may choose, for each X_k, a decomposition of Σ into pairs of pants in such a way that the lengths of the decomposing curves are always $< 21g$. Important here is that the lengths are bounded; the bound $21g$ is of less interest.

Now there are only finitely many combinatorially different decompositions of Σ into pairs of pants. Hence, by passing to a subsequence $([X_{k_i}])$, we may assume that the lengths of the decomposing curves of a *fixed* pants decomposition of Σ are bounded.

This means that we may assume, again by passing to a subsequence, that the sequences of real numbers, formed by the Fenchel–Nielsen coordinates,

$$\ell_j(X_{k_i}) \text{ and } \theta_j(X_{k_j})$$

converge to finite limits ℓ_j^∞ and θ_j^∞. These limits specify the limiting Riemann surface. If $\ell_j^\infty = 0$ for some j, then the curve α_j gets pinched to a node. The limiting Riemann surface is a stable Riemann surface with nodes (see the exposition of Lipman Bers [3]).

The same description of degeneration applies to real algebraic curves. That is based on the fact that if X is a genus g, $g > 1$, Riemann surface admitting an antiholomorphic involution σ, then a length controlled pants decomposition can always be chosen so that it is σ invariant.

Studying the possible degenerations, one can further prove ([33]) a conjecture of Klein (1892, [19, Page 8]):

Theorem 12. *The moduli space of stable real algebraic curves of genus g, $g > 1$, is a connected and compact Hausdorff space.*

It is an interesting open problem to study, in more detail, the structure of the moduli spaces $\mathcal{M}_{\mathbb{R}}^g$ and of $\overline{M}_{\mathbb{R}}^g$, in particular to compute homology and cohomology of these spaces.

References

[1] Lars V. Ahlfors. The complex analytic structure of the space of closed Riemann surfaces. In Rolf Nevanlinna et. al., editor, *Analytic Functions*, pages 45 – 66. Princeton University Press, 1960.

[2] Norman L. Alling and Newcomb Greenleaf. *Foundations of the theory of Klein surfaces.* Number 219 in Lecture Notes in Mathematics. Springer–Verlag, Berlin–Heidelberg–New York, 1971.

[3] Lipman Bers. Finite dimensional Teichmüller spaces and generalizations. *Bull. Amer. Math. Soc.*, 5(2):131 – 172, September 1981.

[4] Lipman Bers. An Inequality for Riemann Surfaces. In Isaac Chavel and Hersel M. Farkas, editors, *Differential Geometry and Complex Analysis*, pages 87 – 93. Springer–Verlag, Berlin–Heidelberg–New York, 1985.

[5] S. A. Broughton, E. Bujalance, A. F. Costa, J. M. Gamboa, and G. Gromadzki. Symmetries of Accola-Maclachlan and Kulkarni surfaces. *Proc. Amer. Math. Soc.*, 127(3):637–646, 1999.

[6] E. Bujalance and A. F. Costa. On symmetries of p-hyperelliptic Riemann surfaces. *Math. Ann.*, 308(1):31–45, 1997.

[7] E. Bujalance, A. F. Costa, and D. Singerman. Application of Hoare's theorem to symmetries of Riemann surfaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 18(2):307–322, 1993.

[8] Emilio Bujalance, Grzegorz Gromadski, and David Singerman. On the number of real curves associated to a complex algebraic curve. *Proc. Amer. Math. Soc.*, 120(2):507–513, 1994.

[9] Peter Buser and Mika Seppälä. Symmetric pants decompositions of Riemann surfaces. *Duke Math. J.*, 67(1):39 – 55, 1991.

[10] Peter Buser and Mika Seppälä. Short homology bases of Riemann surfaces. *Topology*, 2001.

[11] Peter Buser and Mika Seppälä. Real structures of Teichmüller spaces, Dehn twists, and moduli spaces of real curves. *Math Z*, 1999.

[12] Peter Buser, Mika Seppälä, and Robert Silhol. Triangulations and moduli spaces of Riemann surfaces with group actions. *Manuscripta math*, 88:209–224, 1995.

[13] C. J. Earle. On moduli of closed Riemann surfaces with symmetries. In *Advances in the Theory of Riemann Surfaces. Annals of Mathematics Studies 66*, pages 119 – 130, Princeton, New Yersey, 1971. Princeton University Press and University of Tokyo Press.

[14] Paola Frediani. Real algebraic functions, real algebraic curves and their moduli spaces, 1998. Thesis, University of Pisa.

[15] Frederick P. Gardiner. *Teichmüller Theory and Quadratic Differentials.* A Wiley–Interscience of Series of Texts, Monographs, and Tracts. John Wiley & Sons, New York Chichester Brisbane Toronto Singapore, 1987.

[16] J. Huisman. Real quotient singularities and nonsingular real algebraic curves in the boundary of the moduli space. *Compositio Mathematica*, 118(1):43–60, 1999.

[17] Milagros Izquierdo and David Singerman. Pairs of symmetries of Riemann surfaces. *Ann. Acad. Sci. Fenn. Math.*, 23(1):3–24, 1998.

[18] Felix Klein. Über eine neue Art von Riemannschen Flächen. *Math. Annalen*, 10, 1876.

[19] Felix Klein. Über Realitätsverhältnisse bei der einem beliebigen Geschlechte zugehörigen Normalkurve der φ. *Math. Annalen*, 42, 1892.

[20] O. Lehto. Quasiconformal homeomorphisms and Beltrami equations. In W. J. Harvey, editor, *Discrete Groups and Automorphic Functions*, pages 121 – 142. Academic Press, 1977.

[21] O. Lehto and K. I. Virtanen. *Quasiconformal mappings in the plane*, volume 126 of *Die Grundlehren der mathematischen Wissenschaften.* Springer–Verlag, Berlin–Heidelberg–New York, 1973. Translated from the German by K. W. Lucas. 2nd ed.

[22] Olli Lehto. *Univalent Functions and Teichmüller Spaces*, volume 109 of *Graduate Texts in Mathematics.* Springer-Verlag, Berlin–Heidelberg–New York, 1986.

[23] S. M. Natanzon. Automorphisms of the Riemann surface of an M-curve. *Funktsional. Anal. i Prilozhen.*, 12(3):82–83, 1978.

[24] S. M. Natanzon. The order of a finite group of homeomorphisms of a surface onto itself, and the number of real forms of a complex algebraic curve. *Dokl. Akad. Nauk SSSR*, 242(4):765–768, 1978.

[25] S. M. Natanzon. Real frames of complex algebraic curves, and Coxeter groups. *Uspekhi Mat. Nauk*, 51(6(312)):215–216, 1996.

[26] S.M. Natanzon. Moduli of real algebraic curves. *Uspehi Mat. Nauk*, 30:1(181):251 – 252, 1975. (Russian).

[27] Juha Pöyhönen. On Fenchel–Nielsen type coordinates for Teichmüller spaces of Klein surfaces. *Ann. Acad. Sci. Fenn. Ser. A I. Mathematica Diss.*, 72:1 – 34, 1988.

[28] John G. Ratcliffe. *Foundations of Hyperbolic Manifolds.* Springer-Verlag, New York, 1994.

[29] H. L. Royden. Automorphisms and isometries of Teichmüller space. In Cabiria Andreian Cazacu, editor, *Proceedings of the Romanian-Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings, Brasov 1969*, pages 273 – 286. Publishing House of the Academy of the Socialist Republic of Romania, 1971.

[30] H.L. Royden. Automorphisms and isometries of Teichmüller space. In Lars V. Ahlfors et al., editor, *Advances in the Theory of Riemann Surfaces*, volume 66 of *Ann. of Math. Studies*, pages 369–383. 1971.

[31] Mika Seppälä. Teichmüller Spaces of Klein Surfaces. *Ann. Acad. Sci. Fenn. Ser. A I Mathematica Dissertationes*, 15:1 – 37, 1978.

[32] Mika Seppälä. Quotients of complex manifolds and moduli spaces of Klein surfaces. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, 6:113 – 124, 1981.

[33] Mika Seppälä. Moduli spaces of stable real algebraic curves. *Ann. scient. Éc. Norm. Sup.*, 4^{e} série(24):519 – 544, 1991.

[34] Mika Seppälä and Robert Silhol. Moduli spaces for real algebraic curves and real abelian varieties. *Math. Z.*, 201:151 – 165, 1989.

[35] Mika Seppälä and Tuomas Sorvali. *Geometry of Riemann Surfaces and Teichmüller Spaces*. Number 169 in Mathematics Studies. North–Holland, 1992.

[36] R. J. Sibner. Symmetric fuchsian groups. *Am. J. Math.*, 90:1237–1259, 1968.

[37] Robert Silhol. Period Matrices and the Schottky Problem. In Emilio Bujalance et. al., editor, *Topics on Riemann Surfaces and Fuchsian Groups*. Cambridge University Press, 2001.

[38] Guido Weichhold. Über symmetrische Riemannsche Flächen und die Periodizitätsmodulen der zugehörigen Abelschen Normalintegrale erstes Gattung. *Leipziger Dissertation*, 1883.

[39] Scott Wolpert. The Fenchel–Nielsen deformation. *Ann. Math., II. Ser.*, 115:501–528, 1982.

[40] Li Zhong. Nonuniqueness of Geodesics in Infinite Dimensional Teichmüller spaces. *Research Report*, (29):1 – 25, 1990.

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