

# SHORT HOMOLOGY BASES FOR RIEMANN SURFACES

PETER BUSER AND MIKA SEPPÄLÄ

ABSTRACT. The homological systole of a compact Riemann surface  $X$  is the minimal length of a simple closed non-separating geodesic curve. Since any homology basis of  $X$  must contain curves that intersect any non-separating closed curve, surfaces having small homological systoles cannot have short homology basis. It turns out that this basically the only obstruction to finding short homology basis. We show, in fact, that a compact hyperbolic genus  $g$  Riemann surface  $X$  with homological systole  $\epsilon$  has always a canonical homology basis which consists of curves  $\gamma$  satisfying the length bound

$$\ell(\gamma) \leq (g-1) \cdot 105g + 4 \arcsin\left(\frac{4}{\epsilon}\right) .$$

## 1. INTRODUCTION

A compact Riemann surface of genus  $g$ ,  $g > 1$ , can be decomposed into pairs of pants, i.e., into three hole spheres, by cutting the surface along  $3g - 3$  simple closed non-intersecting geodesic curves. These curves can always be chosen in such a way that their hyperbolic lengths are bounded by  $21g$  ([7]).

First length controlled decompositions of Riemann surfaces into pairs of pants were found by Lipman Bers ([3]). His method did, however, yield a bound that was much larger than the above mentioned  $21g$ .

The same question can be asked about homology bases of Riemann surfaces: is it possible to estimate lengths of closed geodesic curves constituting a basis for the homology of a given genus  $g$ ,  $g > 1$ , Riemann surface? More precisely, one would like to find a canonical homology basis  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$  consisting of curves that are as short as possible.

A canonical homology basis is characterized by the property that the curves  $\alpha_j$  and  $\beta_j$  are simple closed curves, each  $\alpha_j$  intersects  $\beta_j$  exactly at one point, and there are no other intersection points. Such bases are needed when computing period matrices of Riemann surfaces, or when forming a fundamental domain for a uniformizing group. If the curves  $\alpha_j$  and  $\beta_j$  are short, then the sides of the corresponding fundamental domain are also short. This, on the other hand, has potential applications to various computational problems.

After having posed the problem of finding a short homology basis for a given Riemann surface, one observes immediately, that it is not possible to find a universal bound that would depend only on the genus of the Riemann surface in question. For if  $\gamma$  is a short non-separating simple closed geodesic curve, then any homology basis contains a curve that intersects  $\gamma$ . By the Collar Theorem ([6]), any such curve is necessarily long (cf. also Example 3). As the length of  $\gamma$  goes to zero, the

---

1991 *Mathematics Subject Classification.* Primary 30F45, Secondary 57M20.

The research of the first author was supported by the Swiss National Research Foundation.

length of any closed intersecting curve grows towards infinity. Hence one cannot find any length controlled homology basis in which the bound for the lengths of the curves would depend only on the genus.

This leads one to define the *homological systole* of a Riemann surface as the minimal length of simple closed non-separating geodesic curves. The main result of this paper is that one can always find a homology basis consisting of curves whose lengths are bounded by an expression depending only on the homological systole and on the genus of the Riemann surface.

More precisely, we prove, in Section 3:

**Theorem 7.** Let  $X$  be a compact Riemann surface of genus  $g \geq 2$  which has a partition with longest geodesic of length  $L$  and whose homological systole is  $\varepsilon$ . Then there exists a canonical homology basis  $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$  on  $X$  such that any  $\alpha_i$  belongs to the partition and the length  $\ell(\beta_i)$ , of any curve  $\beta_i$ , satisfies

$$(1) \quad \ell(\beta_i) \leq (2g - 2)(2L + 2 \operatorname{arcsinh} \frac{4}{\varepsilon})$$

Observe that in the above the constant  $L$  can be replaced by  $21g$  to get a bound that depends only on the genus  $g$  and on the homological systole  $\varepsilon$  of the Riemann surface in question.

Bounds for the lengths of *individual* homologically trivial and non trivial shortest closed curves have been studied for quite a long time. The name “systole” for such curves is due to Berger [1]. We refer to Gromov [9] and [8] for a broad bibliography on the subject and for a large number of new results. For systoles in connection with the Schottky problem we refer to [4], for systoles in connection with arithmetic Fuchsian groups we refer to [10]. As for curve *systems*, short partitions have been investigated in [2], [3], [5], [7], [12], [11], [13].

## 2. HOMOLOGY BASES AND PARTITIONS

In this section we study homology bases and partitions from a purely topological point of view. We begin with the following definition.

A *topologically marked pair of pants* is a compact bordered surface  $Y$  of signature  $(0, 3)$  with boundary curves  $c_1, c_2, c_3$  together with three pairwise disjoint simple arcs  $p_{12}, p_{13}, p_{23}$ , where  $p_{ij}$  has its initial point on  $c_i$ , its end point on  $c_j$  and all other points in the interior of  $Y$ . We call these arcs *connectors*. In the final section, where  $Y$  carries again a hyperbolic metric with geodesic boundary, the connectors will be the usual common perpendiculars decomposing  $Y$  into right angled geodesic hexagons.

A *topologically marked pants decomposition* of a compact orientable surface  $X$  of genus  $g \geq 2$  is a partition  $\mathcal{P}$  of  $X$  with topologically marked pairs of pants. Formally,  $\mathcal{P}$  is understood as the set formed by the  $3g - 3$  partitioning curves and the  $2g - 2$  pairs of pants. For the rest of this section we shall now assume that such a partition  $\mathcal{P}$  is given on  $X$ .

As no metric is specified on  $X$ , there is no measure of shortness of curves. However, one can do the following. We shall say that a simple arc  $v$  on  $X$  is *elementary* if  $v$  is contained in one of the pairs of pants  $Y_k$  of  $\mathcal{P}$  and satisfies the subsequent conditions. A curve will then be considered combinatorially “short” if it can be

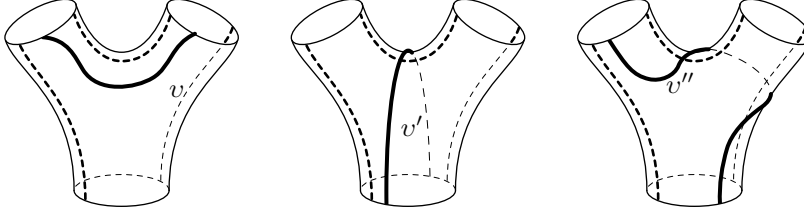


FIGURE 1. Elementary arcs.

decomposed into a sequence of elementary arcs where, moreover, the number of these arcs is “small”.

We denote by  $\check{v}$  the *interior* of  $v$ , that is, the arc without its end points. The conditions for  $v$  on  $Y_k$  are as follows.

- (1) The end points of  $v$  lie on the boundary of  $Y_k$  and  $\check{v}$  is contained in the interior of  $Y_k$ .
- (2)  $\check{v}$  intersects  $p_{12} \cup p_{23} \cup p_{31}$  in at most two points.

As a limit case we also accept the connectors  $p_{ij}$  themselves as elementary arcs. If the end points of  $v$  lie on different boundary components of  $Y_k$ , we shall say that  $v$  is of *type I*. If the end points lie on the same boundary component and  $v$  is not homotopic to an arc on the boundary, then  $v$  is of *type II*. Finally, if  $v$  is homotopic to an arc on the boundary of  $Y_k$  then we shall say that  $v$  is *trivial*. Fig. 1 shows some cases, the dotted lines are the connectors.

For any closed curve  $c$  on  $X$  which is the union of non overlapping elementary arcs the number of these arcs will be denoted by  $\ell_{\mathcal{P}}(c)$ . We shall call  $\ell_{\mathcal{P}}(c)$  the *combinatorial length* of  $c$ .

We now show that there exist combinatorially short homology bases.

**Theorem 1.** There exists a canonical homology basis  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$  on  $X$  with the following properties.

- (1) Each  $\alpha_i$  belongs to  $\mathcal{P}$ .
- (2) Each  $\beta_i$  satisfies  $\ell_{\mathcal{P}}(\beta_i) \leq 2g - 2$ .
- (3) No  $\beta_i$  intersects a separating curve of  $\mathcal{P}$ .

*Proof.* If  $\mathcal{P}$  contains a separating curve  $\eta$ , we cut  $X$  open along  $\eta$  to obtain two bordered surfaces  $\Sigma', \Sigma''$  of signatures  $(g', 1)$  and  $(g'', 1)$  respectively, with  $g' + g'' = g$ . If one of the components  $\Sigma'$  or  $\Sigma''$  is again separated by some curve in  $\mathcal{P}$ , then we cut the component open along this curve, and so on. After finitely many steps we obtain surfaces  $\Sigma_1, \dots, \Sigma_n$  of signatures  $(g_1, m_1), \dots, (g_n, m_n)$ , where  $g_1 + \dots + g_n = g$  and where for  $k = 1, \dots, n$ , no curve of  $\mathcal{P}$  in the interior of  $\Sigma_k$  separates  $\Sigma_k$ . (If the above  $\eta$  does not exist, then  $n = 1$  and  $\Sigma_1 = X$ .) This preliminary procedure will guarantee point (3) of the theorem.

Now let  $\Sigma_k$  be one of the components with  $g_k \neq 0$  and denote, by  $\mathcal{P}_k$ , the pants decomposition of  $\Sigma_k$  induced by  $\mathcal{P}$ . We cut  $\Sigma_k$  open along a non separating curve of  $\mathcal{P}_k$  into a connected surface  $\Sigma_k^1$  of signature  $(g_k - 1, m_k + 2)$ . If  $g_k - 1 \neq 0$  then  $\mathcal{P}_k$  contains a curve not separating  $\Sigma_k^1$  and we cut  $\Sigma_k^1$  open along it to obtain a surface  $\Sigma_k^2$  of signature  $(g_k - 2, m_k + 4)$ , and so on. This procedure yields a surface  $S_k$  of signature  $(0, m_k + 2g_k)$  tessellated with  $2g_k - 2 + m_k$  pairs of pants of  $\mathcal{P}$ . We now denote by  $\mathcal{P}_k$  the corresponding pants decomposition of  $S_k$ . The combinatorial

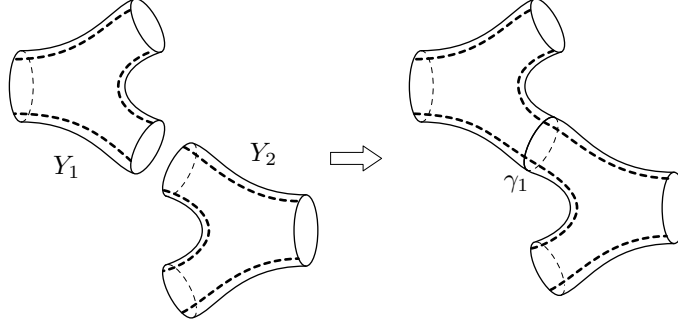


FIGURE 2. Constructing the boundary lane.

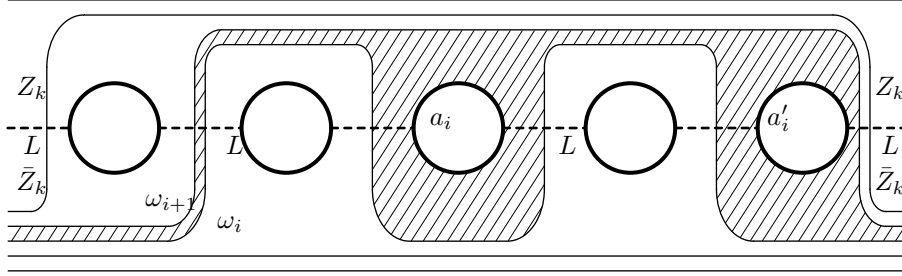
scheme of  $\mathcal{P}_k$  is a three regular graph  $\mathcal{G}_k$  without closed edge paths.  $\mathcal{G}_k$  has  $m_k + 2g_k$  half edges and  $2g_k + m_k - 3$  edges.

Since  $\mathcal{G}_k$  has no closed edge paths (i.e.  $\mathcal{G}_k$  is a tree), the surface  $S_k$  may be reconstructed out of the pants in  $\mathcal{P}_k$  by starting with a first pair of pants, say  $Y_1$ , then paste the neighboring pair of pants  $Y_2$  along the corresponding boundary component of  $Y_1$ , say along  $\gamma_1$ , to obtain a surface of signature  $(0, 4)$  as shown in Fig.2, then paste  $Y_3$  along  $\gamma_2$  to get a surface of signature  $(0, 5)$ , and so on.  $S_k$  is finished after  $2g_k + m_k - 3$  steps.

The procedure thus described allows us to construct a sequence of curves on  $S_k$  which we shall call a *boundary lane* and which is obtained inductively as follows (see Fig.2). On  $Y_1$  the boundary lane is formed by the three arcs  $p_{12}, p_{13}, p_{23}$ . On  $Y_1 \cup Y_2$  we connect the  $p_{ij}$  of  $Y_1$  and the  $p'_{kl}$  of  $Y_2$  which meet  $\gamma_1$  by inserting two disjoint arcs on  $\gamma_1$ . The four curves obtained up to now yield the boundary lane on  $Y_1 \cup Y_2$ . This is shown by the four dotted arcs on the right hand side in Fig.2. Proceeding in this way for each new pair of pants which is added, we obtain a boundary lane on  $S_k$  which is a union  $L = \lambda_1 + \dots + \lambda_q$ ,  $q = m_k + 2g_k$ , of pairwise disjoint simple arcs, where  $\lambda_1$  goes from a first boundary component of  $S_k$  to a second one,  $\lambda_2$  goes from the second boundary component to a third one, etc. Finally,  $\lambda_q$  goes back to the first boundary component. Since  $S_k$  has genus 0,  $L$  separates  $S_k$  into two topological disks  $Z_k$  and  $\bar{Z}_k$ . Observe that with an arbitrarily small homotopy  $L$  may be deformed such that the deformed lane consists of  $3(2g_k + m_k - 2)$  arcs of type I. Fig.3 shows part of  $L$  schematically together with additional curves which we define next.

For  $i = 1, \dots, g_k$ , we denote by  $a_i$  and  $a'_i$  the pair of boundary components of  $S_k$  which were obtained at step  $i$  during the cutting process (i.e. pasting together  $a_i$  and  $a'_i$  for  $i = 1, \dots, g_k$  yields  $\Sigma_k$ ). Then we draw for each  $i = 1, \dots, g_k + 1$  a closed curve  $\hat{\omega}_i$  which goes along  $L$  and surrounds each boundary curve  $c$  of  $S_k$  in the following way. If  $c$  is none of the  $a_j, a'_j$ , then  $\hat{\omega}_i$  goes around  $c$  in  $Z_k$ . If  $c$  is  $a_j$  or  $a'_j$  and  $i \leq j$ , then  $\hat{\omega}_i$  goes around  $c$  in  $Z_k$  as well. If  $c$  is  $a_j$  or  $a'_j$  and  $i > j$ , then  $\hat{\omega}_i$  goes around  $c$  in  $\bar{Z}_k$ . The  $\hat{\omega}_i$  consist of arcs of  $L$  and of arcs on the boundary of  $S_k$ .

Now we use small homotopies to deform each  $\hat{\omega}_i$  into a closed curve  $\omega_i$  contained in the interior of  $S_k$  such that  $\omega_1, \dots, \omega_{g_k+1}$  are pairwise disjoint and such that each  $\omega_i$  is a union of elementary arcs. It is not difficult to check that this is possible

FIGURE 3. Domains of signature  $(0, 4)$  along the boundary lane.

except for the case where  $S_k$  consists of a single pair of pants. In this particular case the  $\omega_i$  are homotopic to one of the boundary components and we shall say that they “consist of 0 elementary arcs”.

Since  $S_k$  consists of  $2g_k + m_k - 2$  pairs of pants and has  $2g_k + m_k$  boundary components none of which is intersected by the  $\omega_i$ , we can perform the above homotopies such that any  $\omega_i$  consists of

$$3(2g_k + m_k - 2) - (2g_k + m_k) = 4g_k + 2m_k - 6$$

elementary arcs (where some of them may be trivial arcs).

The curves are shown in Fig. 3 where  $\omega_1$  is the lowest curve,  $\omega_2$  is above  $\omega_1$ ,  $\omega_3$  is above  $\omega_2$ , etc. For  $i = 1, \dots, g_k$ , the four curves  $\omega_i, \omega_{i+1}, a_i$  and  $a'_i$  bound a domain  $\Omega_i$  of signature  $(0, 4)$  as shown by the shaded area. The domains  $\Omega_1, \dots, \Omega_{g_k}$  do not overlap each other.

For each  $i$  we draw a curve  $b_i$  in  $\Omega_i$  going from a point on  $a_i$  to an equivalent point (with respect to the pasting) on  $a'_i$ . Taking the shorter of the two paths along  $L$  from  $a_i$  to  $a'_i$  we achieve that  $b_i$  consists of at most  $2g_k + m_k - 2$  elementary arcs.

Since the  $\Omega_i$  do not overlap, the  $b_i$  are pairwise disjoint. It follows that on  $\Sigma_k$  the  $a_i$  and  $b_i$  are closed curves with the intersection properties as required for a canonical homology basis. Since  $2g_k + m_k - 2 \leq 2g - 2$ , this proves the theorem.  $\square$

Observing that the constructions and length estimates in the preceding proof depend only on the components  $\Sigma_k$  we actually have the following more detailed version of Theorem 1, where we also admit surfaces with boundary.

For the statement of the theorem we note that for a bordered surface  $S$  of signature  $(g, m)$  a *canonical homology basis* is a curve system  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \rho_1, \dots, \rho_{m-1}$ , where  $\rho_1, \dots, \rho_{m-1}$  are boundary components and  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$  have the configuration of a canonical homology basis on a compact unbordered surface of genus  $g$ .

**Theorem 2.** Let  $\mathcal{P}$  be a pants decomposition of the compact orientable surface  $\Sigma$  of signature  $(g, m)$ ,  $g \geq 1$ , denote by  $\Sigma_1, \dots, \Sigma_n$  the components obtained by cutting  $\Sigma$  open along all separating curves occurring in  $\mathcal{P}$ , and let  $\#\Sigma_k$  be the number of pairs of pants of  $\mathcal{P}$  in  $\Sigma_k$ ,  $k = 1, \dots, n$ . Then there exists a canonical homology basis  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \rho_1, \dots, \rho_{m-1}$  on  $\Sigma$  such that

- (1)  $\alpha_1, \dots, \alpha_g$  and  $\rho_1, \dots, \rho_{m-1}$  belong to  $\mathcal{P}$ .
- (2) each  $\beta_i$  is contained in some  $\Sigma_k$  and satisfies  $\ell_{\mathcal{P}}(\beta_i) \leq \#\Sigma_k$ .  $\square$

We point out that the upper bound  $2g - 2$  in Theorem 1 for the combinatorial length of all  $\beta_k$  is the best possible. For this we consider the following.

**Example 3.** Start with a pair of pants all of whose boundary geodesics have the same length  $\varepsilon$  and paste two copies of it together in order to obtain a surface  $\Omega$  of signature  $(1, 2)$  with boundary components  $\gamma_L$  and  $\gamma_R$ . Then take  $g - 1$  copies  $\Omega_1, \dots, \Omega_{g-1}$  of  $\Omega$  with respective boundary geodesics  $\gamma_{L,i}, \gamma_{R,i}$ ,  $i = 1, \dots, g - 1$ .

We construct a “necklace”  $N$  out of these copies by pasting  $\Omega_i$  to  $\Omega_{i+1}$  along  $\gamma_{R,i}$  and  $\gamma_{L,i+1}$ , for  $i = 1, \dots, g - 2$  and by pasting  $\Omega_{g-1}$  to  $\Omega_1$  along  $\gamma_{R,g-1}$  and  $\gamma_{L,1}$ . The resulting surface  $N$  has genus  $g$ . We let  $\gamma_1, \dots, \gamma_{g-1}$  be the geodesics in  $N$  obtained from  $\gamma_{R,1}, \dots, \gamma_{R,g-1}$  respectively. They are all non separating.

If  $\mathcal{B}$  is a canonical homology basis on  $N$  then by the lemma below, at least one curve  $b \in \mathcal{B}$  has intersection number  $(b, \gamma_1) \neq 0$ . On the surface  $N'$  obtained by cutting  $N$  open along  $\gamma_1$  the curve  $b$  is cut into a number of arcs, one of which connects the two boundary components with each other. This follows from the fact that  $(b, \gamma_1) \neq 0$ . We conclude that  $b$  crosses at least  $2g - 2$  times a pair of pants.

**Lemma 4.** Let  $\mathcal{B}$  be a homology basis on a compact orientable surface  $X$ . For any non separating simple closed curve  $c$  on  $X$  there exists  $b \in \mathcal{B}$  with  $(b, c) \neq 0$ .

*Proof.* Otherwise  $(c, \beta) = 0$  for any cycle  $\beta$ , but  $c$  is non separating and there exists a cycle  $c'$  with  $(c, c') = 1$ .  $\square$

### 3. LENGTH ESTIMATES

Let us now derive metric length estimates. All partitioning curves are assumed to be closed geodesics, and on any pair of pants the connectors are the common perpendiculars between the boundary geodesics.

Replacing the curves constructed in the preceding section by geodesics in their homotopy classes we obtain short homology bases in the sense of the hyperbolic metric. In order to get length estimates we proceed as follows. First we replace any elementary arc of the given curve by a homotopic arc with the same end points. By the minimal intersection property of geodesics, this new arc is also elementary and we shall give a length estimate for it in Proposition 5. Thus we have a piecewise geodesic curve with controlled length and the smooth closed geodesic in its homotopy class is then even shorter.

**Proposition 5.** Let  $Y$  be a pair of pants such that all boundary geodesics have lengths between  $\varepsilon$  and  $L$ , where  $0 < \varepsilon \leq L$ . Then any elementary geodesic arc  $\eta$  on  $Y$  has length  $\ell(\eta) \leq 2L + 2 \operatorname{arcsinh} \frac{4}{\varepsilon}$ .

*Proof.* The three common perpendiculars  $p_{ij}$  between the boundary geodesics  $\gamma_1, \gamma_2, \gamma_3$  of  $Y$  decompose  $Y$  into two isometric right angled hyperbolic geodesic hexagons  $G$  and  $G'$  (cf. [6]), where  $G$  has the succession of sides  $a_1, p_{12}, a_2, p_{23}, a_3, p_{13}$ . With these symbols we shall also denote the lengths of the sides.

Consider an elementary geodesic arc  $\eta$  on  $Y$ . If  $\eta$  is trivial then  $\eta$  is a simple arc on the boundary and has length less than  $L$ .

Now let  $\eta$  be of type II with end points on  $\gamma_1$  as shown in Fig. 4.  $\eta$  is homotopic with fixed end points to a piecewise geodesic curve  $c'Bc''$ , where  $c'$  and  $c''$  are arcs on  $\gamma_1$  and  $B$  is the arc of length  $2b$  formed by the common orthogonals on  $G$  and  $G'$  from  $\gamma_1$  to  $p_{23}$ . Since the interior of  $\eta$  intersects the union  $p_{12} \cup p_{23} \cup p_{13}$  at most twice, the total length of  $c$  and  $c'$  is at most  $\ell(\gamma_1) \leq L$ . From the trigonometry of

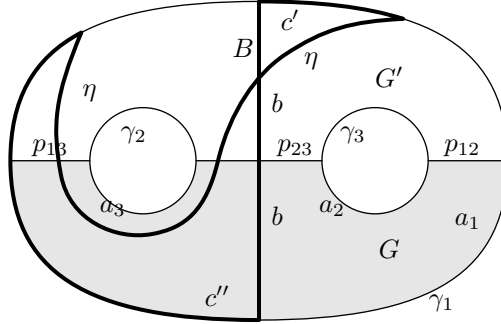


FIGURE 4. Arc of type II on a pair of pants

hexagons (cf. [6]) it follows that  $b$  becomes maximal when  $\gamma_1$  assumes the minimal length  $\varepsilon$  and  $\gamma_2$  and  $\gamma_3$  assume the maximal length  $L$ . In this extremal case  $b$  is given by the formula  $\cosh \frac{L}{2} = \sinh b \sinh \frac{\varepsilon}{4}$ . After some calculus involving monotonicity of derivatives we then get

$$(2) \quad b \leq \frac{L}{2} + \operatorname{arcsinh} \frac{4}{\varepsilon}.$$

This settles the length estimate for arcs of type II, so let  $\eta$  be of type I, say with end points on  $\gamma_2$  and  $\gamma_3$ .

Here  $\eta$  is homotopic with fixed end points to the piecewise geodesic curve  $c_2 p_{23} c_3$ , where  $c_2, c_3$  are arcs on  $\gamma_2, \gamma_3$  respectively. The length of  $p_{23}$  becomes maximal when  $\gamma_2$  and  $\gamma_3$  have length  $\varepsilon$  and  $\gamma_1$  has length  $L$ . In this extremal case we have the formula  $\cosh \frac{L}{4} = \sinh \frac{\varepsilon}{2} \sinh \frac{1}{2} p_{23}$  from which we deduce the inequality

$$(3) \quad p_{23} \leq \frac{L}{2} + 2 \operatorname{arcsinh} \frac{2}{\varepsilon}.$$

If  $\eta$  intersects  $p_{23}$  then the interior of  $\eta$  intersects  $p_{12} \cup p_{13}$  at most once. Hence the total length of  $c_2$  and  $c_3$  is at most  $\frac{3}{2}L$  and we are done in this case. If  $\eta$  does not intersect  $p_{23}$  then the total length of  $c_2$  and  $c_3$  may reach  $2L$  so that we must refine the estimates a bit. Using the trigonometric formulas for hexagons we get  $\cosh \eta = g(c_2, c_3, a_1)$ , where

$$g(x, y, z) = \frac{\operatorname{ch} 2x \operatorname{ch} 2y}{\operatorname{sh} x \operatorname{sh} y} (\operatorname{ch} z + \operatorname{ch} x \operatorname{ch} y) - \operatorname{sh} 2x \operatorname{sh} 2y.$$

An elementary argument now shows that for any  $x, y$  with  $\frac{\varepsilon}{2} \leq x, y \leq \frac{L}{2}$  we have  $g(x, y, \frac{L}{2}) \leq 2L + \operatorname{arcsinh} \frac{4}{\varepsilon}$ .  $\square$

**Example 6.** If in Example 3 we take pairs of pants with all boundary geodesics of length  $\varepsilon$ , then the distance  $\lambda$  between two boundary geodesics of a pair of pants satisfies  $\operatorname{sh} \frac{\lambda}{2} \operatorname{sh} \frac{\varepsilon}{2} = \operatorname{ch} \frac{\varepsilon}{4}$ . It follows that any homology basis on the surface  $N$  constructed in the example has at least one curve of length  $\geq 4(g-1) \operatorname{arcsinh} \frac{2}{\varepsilon}$ . Hence, in certain cases the upper bound in Theorem 7 is close to optimal.

Theorem 1 and Proposition 5 yield now:

**Theorem 7.** Let  $X$  be a compact Riemann surface of genus  $g \geq 2$  which has a partition with longest geodesic of length  $L$  and whose homological systole is  $\varepsilon$ . Then there exists a canonical homology basis  $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$  on  $X$  such that any  $\alpha_i$  belongs to the partition and the length  $\ell(\beta_i)$ , of any curve  $\beta_i$ , satisfies

$$(4) \quad \ell(\beta_i) \leq (2g - 2)(2L + 2 \operatorname{arcsinh} \frac{4}{\varepsilon})$$

## REFERENCES

- [1] Marcel Berger. Systoles et applications selon Gromov. *Astérisque*, (216):Exp. No. 771, 5, 279–310, 1993. Séminaire Bourbaki, Vol. 1992/93.
- [2] Lipman Bers. Finite dimensional Teichmüller spaces and generalizations. *Bull. Amer. Math. Soc.*, 5(2):131 – 172, September 1981.
- [3] Lipman Bers. An Inequality for Riemann Surfaces. In Isaac Chavel and Hersel M. Farkas, editors, *Differential Geometry and Complex Analysis*, pages 87 – 93. Springer-Verlag, Berlin-Heidelberg-New York, 1985.
- [4] P. Buser and P. Sarnak. On the period matrix of a Riemann surface of large genus. *Invent. Math.*, 117(1):27–56, 1994. With an appendix by J. H. Conway and N. J. A. Sloane.
- [5] Peter Buser. Riemannsche Flächen und Längenspektrum vom trigonometrischen Standpunkt aus, 1980. Habilitationsschrift, Universität Bonn.
- [6] Peter Buser. *Geometry and Spectra of Compact Riemann Surfaces*. Birkhäuser Verlag, Basel-Boston-New York, 1992.
- [7] Peter Buser and Mika Seppälä. Symmetric pants decompositions of Riemann surfaces. *Duke Math. J.*, 67(1):39–55, 1992.
- [8] M. Gromov. Metric invariants of Kähler manifolds. In *Differential geometry and topology (Alghero, 1992)*, pages 90–116. World Sci. Publishing, River Edge, NJ, 1993.
- [9] Mikhael Gromov. Systoles and intersystolic inequalities. In *Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992)*, volume 1 of *Sémin. Congr.*, pages 291–362. Soc. Math. France, Paris, 1996.
- [10] Paul Schmutz. Systoles of arithmetic surfaces and the Markoff spectrum. *Math. Ann.*, 305(1):191–203, 1996.
- [11] Paul Schmutz Schaller. Extremal Riemann surfaces with a large number of systoles. In *Extremal Riemann surfaces (San Francisco, CA, 1995)*, volume 201 of *Contemp. Math.*, pages 9–19. Amer. Math. Soc., Providence, RI, 1997.
- [12] Paul Schmutz Schaller. Geometry of Riemann surfaces based on closed geodesics. *Bull. Amer. Math. Soc. (N.S.)*, 35(3):193–214, 1998.
- [13] Mika Seppälä. Moduli spaces of stable real algebraic curves. *Ann. scient. Éc. Norm. Sup.*, 4<sup>e</sup> série(24):519 – 544, 1991.

DÉPARTEMENT DE MATHÉMATIQUES, ECOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, CH-1007 LAUSANNE

*E-mail address:* `buser@masg1.epfl.ch`

DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FL 32306

*E-mail address:* `Mika.Seppala@fsu.edu`