Background for Actuarial Models
MAP 4175 and MAP 5177
Bettye Anne Case

Concurrent enrollment. MAP 4170, interest theory and an introduction to contingency theory, is a catalog prerequisite for Actuarial Models. Some of the material of MAP 4170 is a content prerequisite but other can be learned concurrently. Graduate students and some undergraduate students, who have made late changes to their major or who already have a bachelor's degree, may need MAP 4175 or MAP 5177 without MAP 4170 first. Several strong undergraduates, who had not previously attempted MAP 4170, have doubled these courses since Fall 1995, with considerable success. Undergraduates did this "at their own risk" since we cannot assure they have an adequate background.) I took advice of students in working out the first draft of this self-guided tour of concepts with which you will need some familiarity before classes start, and Tapio Boles assisted a rework in 2001.

The following content and exercises will take you from 10 to perhaps 50 hours for completion, depending on your academic background and how thoroughly you work. Note that all numbered equations in this document should be memorized. When you show me a good stab at this work on the **first Fall-term class day** of MAP 4175/5177, I will approve dual enrollment (you will be learning much background material as you go along).

Materials. You will need a Texas Instruments BA-35 Solar calculator and to work for yourself all problems. This is a financial calculator allowed by the Society of Actuaries for its exams. This note refers to particular function keys of this calculator. Be advised that this calculator performs operations as received, and thus it does not recognize the order of operations that most scientific calculators do.

You will also need the book "Theory of Interest and Life Contingencies with Pension Applications" by Michael M. Parmenter. It can be purchased from the FSU Bookstore, Bill's Bookstore, or ordered directly from ACTEX Publications (www.actexmadriver.com). You must have the 1999 third edition version of this book. The paragraphs in this note are preceded by corresponding page numbers and relevant examples from the Parmenter book displayed in brackets "[ ]". Be certain to master all examples referenced.

Basic Interest Theory. [1-2] The **accumulation function** $a(t)$ is the basic interest theory concepts. It is usually introduced alongside the **amount function** $A(t)$. The amount function $A(t)$ can be thought of as the value of money invested over time. $A(0)$, the amount invested at the beginning of the account, is called the **principal**. $A(t)$ is the **accumulated value** of the money account at time t, where t is the number of years from the present. t can also be thought of as the number of years since the inception of the account. The accumulation function $a(t)$ is like an amount function except $a(0)$ is always equal to 1. In other words, the principal in an accumulation function is always \$1. The accumulation function is related to a corresponding amount function in this manner:

$$a(t) = \frac{A(t)}{A(0)} \tag{1.1}$$

[2-4, Example 1.2] The most important variable in interest theory is i. i is used in interest theory as the variable like x in algebra. i represents the annual interest rate. It may be a **simple interest** rate or a **compound interest** rate. Here is the accumulation function for simple interest:

$$a(t) = (1 + i \cdot t) \tag{1.2}$$

[6-7, Example 1.3] As you can infer, simple interest accrues by adding the same amount every year. Most accounts, however, earn compound interest. Compound interest accrues exponentially instead of linearly. Here is the general formula for the accumulation function for compound interest:

$$a(t) = (1 + i)^t \tag{1.3}$$

Thus, if $i = 6$ %, $a(0(= 1$, $a(1) = (1 + .06)^1 = 1.06$, $a(2) = (1.06)^2$.

[8-9] Examine Figure 1.5 on page 9 of the Parmenter book. It plots the accumulation functions for simple and compound interest. As you can see, the simple interest function accrues more rapidly in the first year ($0 < t < 1$) but compound interest grows more rapidly after the first year ($t > 1$).

Calculator Tip: To compute the accumulation of \$1 over 5 years at 10% interest, use one of these calculator sequences:

`10 % + 1 = 2nd` y^x `5 =`

or

`1.1 2nd` y^x `5 =`

or

`1 PV 10 %i 5 N CPT FV`

The first method is a very basic method of starting with an interest rate and compounding it over time. The second method is an intuitive variation of the first. The third method uses the finance keys. Either way you arrive at 1.61051.

Now is a good time to familiarize yourself with the finance keys. A typical loan, money account, or the like involving periodic payments has five key attributes: duration, interest rate, present value, payment amount, and future value. The purpose of the finance keys is to take four known attributes and compute one unknown. For example, consider a loan of \$10000. If you make 5 annual payments of \$2000 at an interest rate of 8%, what will be the balance still owed on the loan at the end of the 5 payments? In other words, if PV = 10000, N = 5, PMT = 2000, and $i = 8\%$, what is the future value (FV)? In order to find it, use the finance keys:

`5 N 8 %i 10000 PV 2000 PMT CPT FV`

The calculator should display 2960.078848, signifying that the balance due after your fifth annual payment would be \$2960.08. For more information on the finance keys, study the BA-35 Quick Reference Card and practice examples.

[2,13-14] Equation 1.3 above deals with the **effective** = annual rate of interest. An effective rate is based on one yearly compounding. A **nominal** rate may be compounded several times per year. For example, consider a car loan with monthly payments may be quoted at a nominal rate $i^{(12)}$. In order for the balance to be accurate, the interest on the account must be compounded monthly. In standard notation the effective interest rate is represented as i, and the nominal interest rate is accurately represented as $i^{(m)}$, where m equals the number of compounding periods in a year. Here is the accumulation function for interest compounded at a nominal rate of $i^{(m)}$:

$$a(t) = \left(1 + \frac{i^{(m)}}{m}\right)^{m \cdot t} \tag{1.4}$$

[14, Example 1.9] The nominal rate is often called the annual percentage rate, or APR. There are two secondary keys (in yellow) on the TI BA-35 marked "▷EFF" and "▷APR". The former will convert the nominal interest rate to a yearly or effective interest rate, which is equivalent to the effective rate given, and the latter will do the converse. Detailed instructions on how to use these keys are on the BA-35 Quick Reference Card. We can also convert from one rate to the other algebraically. Here is the relationship between the nominal rate $i^{(m)}$ and the effective rate i:

$$(1 + i) = \left(1 + \frac{i^{(m)}}{m}\right)^{m} \tag{1.5}$$

[15-16] From this we can solve for i and $i^{(m)}$ to derive formulas for the effective rate and nominal rate, respectively:

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^{m} - 1 \tag{1.6}$$

$$i^{(m)} = m\left((1 + i)^{\frac{1}{m}} - 1\right) \tag{1.7}$$

Example: A credit card compounds interest on the outstanding balance monthly at an APR of 12%. Calculate the effective annual rate of the credit card. (This amount is legally required to be stated.)

Solution: In this example, the nominal rate $i^{(m)} = 12\%$ and $m = 12$, because there are 12 months in a year. To compute the effective rate:

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^{m} - 1$$

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1$$

$$i = \left(1 + \frac{12\%}{12}\right)^{12} - 1$$
$$i = (1.01)^{12} - 1$$
$$i = 0.12682503$$

Calculator Solution:

`12 2nd ▷EFF 12 = %`

The calculator should output 0.12682503. Thus we see that a nominal rate of 12% compounded monthly has the same yield as an effective rate of interest of approximately 12.682503% (compounded only once yearly).

[16-17] Examine this table (taken from Table 1.1 on page 16 of the Parmenter book):

m	1	2	5	10	50	$m \to \infty$
$i^{(m)}$	.12	.1166	.1146	.1140	.1135	.113328685...

This table assumes the effective annual rate of interest is $i = .12$, and it displays the nominal rate of interest $i^{(m)}$ for various values of m. As you can see, higher values of m correspond to lower values of $i^{(m)}$ for a given effective rate i. However, as m becomes very large, the nominal interest rate $i^{(m)}$ does not decrease unbounded. The limit that $i^{(m)}$ approaches is designated as δ:

$$\lim_{m\to\infty} i^{(m)} = \ln(1+i) = \delta \tag{1.6}$$

[17, Example 1.11] The **force of interest**, δ, can be thought of as the rate at which continual compounding ($m = \infty$) would have the same yield as an effective rate i. The effective rate i is related to the force of interest δ in this equation:

$$i = e^{\delta} - 1 \tag{1.7}$$

[18] More generally, given an accumulation function $a(t)$:

$$\delta = \frac{a'(t)}{a(t)} = \frac{d}{dt}(\ln a(t)) \tag{1.8}$$

We see that if $a(t) = (1+i)^t$ then by this definition and taking the derivative we have

$$\delta = \frac{a'(t)}{a(t)} = \frac{(1+i)^t \cdot \ln(1+i)}{(1+i)^t} = \ln(1+i)$$

hence $1 + i = e^0$ as (1.7) above.

STOP! At this point you must be sure you understand the concept of force of interest for compound interest intuitively as compounding "infinitely often", in terms of nominal interest. Be certain you understand and can work out Examples 1.7, 1.8, 1.9, 1.11; and problems [pp. 26–27] — note answers in back of book — #21, 22, 23(a–c), 24, 29, 31.

[11-13, Example 1.5] A third variable (after i and δ) used to describe interest is the rate of **discount**, d. Discount can be thought of as the reverse effect of interest. For example, consider an account with \$100 in 2000 and \$110 in 2001. If 2000 is considered the base year and we are looking forward, then \$10 is the amount of interest that accrued in the year 2000 and the interest rate i is $\frac{10}{100} = 10\%$. However, if 2001 is considered the base year and we are looking backward, then \$10 is the amount of *discount* that takes us from 2001 to 2000 and the discount rate d is $\frac{10}{110} = 9.09\%$. d can be understood in the following relations:

$$d = \frac{i}{1+i} \tag{1.10}$$

$$i = \frac{d}{1-d} \tag{1.11}$$

Example: Let $i = 25\%$. Suppose you invest in stocks and earn a return of i in the first year. (1) How much will you have, relative to your initial investment, if you lose i in the second year? (2) At what rate of loss in the second year would you break even?

Solution: (1) If you gain i and lose i you will have $a(2) = (1+i)(1-i) = (1.25)(0.75) = 93.75\%$ of your initial investment. (2) If you gain i you can afford to lose d and still break even:

$$a(2) = (1+i)(1-d)$$

$$a(2) = (1+i)\left(1 - \frac{i}{1+i}\right)$$

$$a(2) = (1+i)\left(\frac{1+i-i}{1+i}\right)$$

$$a(2) = (1+i-i)$$

$$a(2) = 1$$

$$d = \frac{i}{1+i} = \frac{0.25}{1+0.25} = 0.20 = 20\%$$

Solving for d, we get 20%. Thus, if you gain 25% in the first year and lose 20% in the second year, you will have the same amount as you started with.

[10-11, Example 1.4] The fourth and final notation that may be used to describe compound interest is v. In order to have \$1 in your account one year from now at an effective rate i you must invest \$$v$ today. The variable v is defined relative to the other three key interest variables where i is the effective yearly compound interest rate:

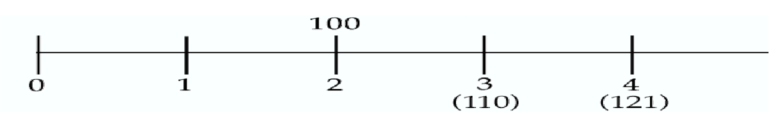
$$v = \frac{1}{1+i} = 1 - d = e^{-\delta} \tag{1.12}$$

[6,10,12,20] Thus the accumulation function for compound interest can be defined in four key ways:

$$a(t) = (1+i)^t = e^{\delta \cdot t} = (1-d)^{-t} = v^{-t} \tag{1.13}$$

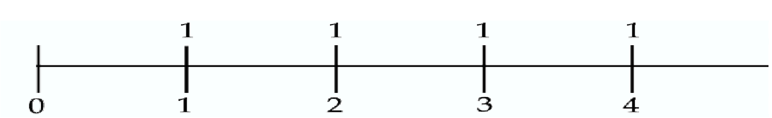
[22-28] Now we have reached the end of the necessary material from Chapter 1 of the Parmenter book, so take this opportunity to work some of the problems from the end of the chapter. Turn to page 23-27 and do the following exercises and any you may have failed to do earlier: 2, 9, 10, 13, 17, 20, 29 (a, b, and c). Note that in exercise 2 you must deal with a non-integer value of t, and that by April 7 there will have been 96 days for interest to accrue (31 + 29 + 30 + 6). Part (c) of exercise 2 uses simple interest (linear interpolation) during the last year of compounding; if you were to use compound interest for the entire life of the account, you would have \$24247.31 (verify this). Look over the information in this note and Chapter 1 of Parmenter if you have difficulty completing these problems.

[29-35] Before we move on to annuities, it is important briefly to understand the major concept in chapter 2 of the Parmenter book, **equation of value**. For interest > 0 %, \$1 today is not worth the same as \$1 last year or \$1 next year. As long as the interest rate is positive, a given amount of money is always worth less in the future and worth more in the past. For example, \$100 in 2002 has value less than \$100 in 2001. Having said that, we can *equate* different amounts of money at set time intervals by using interest theory. For example, at an effective annual interest rate of 10%, \$100 at $t = 2$ has value of \$110 at $t = 3$, and has value of \$121 at $t = 4$:



$$\$100(1 + 10\%) = \$110 = 100(1 + 10\%)^2 = \$121$$

Annuities. [44-45] An **annuity** is a series of payments and we consider only those at equal intervals. The value of an annuity changes over time due to interest accumulation, so usually one expresses an annuity in terms of its **present value**, $a_{\overline{n}|i}$ (pronounced a n angle i), or its **accumulated value**, $s_{\overline{n}|i}$. Consider an **annuity immediate** of \$1 per year for n years. The first payment occurs at the end of the first period at time $t = 1$. The present value is v at $(t = 0)$ of \$1 paid at $t = 1$. The present value of \$1 paid at $t = 2$ is v^2. The present value of \$1 paid at $t = n$ is v^n. The present value of a series of four payments of \$1 each is equivalent to an annuity immediate of duration n:



$$a_{\overline{4}|i} = v + v^2 + v^3 + v^4 \tag{2.3}$$

[45] Because this is a geometric series, the annuity can be rewritten more generally:

$$a_{\overline{n}|i} = \frac{1 - v^n}{i} \tag{2.4}$$

[45, Example 3.4] The **accumulated value** is the value of the annuity at $t = n$. To convert an amount from its value at $t = 0$ to its value at $t = n$, we multiply by $(1+i)^n$. Therefore, the accumulated value of an annuity immediate is related to its present value in this way:

$$s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i} \tag{2.5}$$

which yields

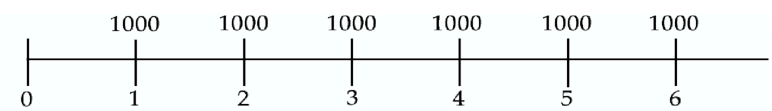
$$s_{\overline{n}|i} = (1+i)^n \left(\frac{1 - v^n}{i}\right) == \frac{(1+i)^n - 1}{i} \tag{2.6}$$

Note, the "i" may be omitted from the annuity expressions. If so, then i is assumed to be the interest rate per period.

$$a_{\overline{n}|} = a_{\overline{n}|i} \qquad s_{\overline{n}|} = s_{\overline{n}|i} \tag{2.7}$$

Example: How much would you have to deposit today at an effective interest rate of 8% to withdraw $1000 per year for 6 years, with the first payment at the end of the current year?

Solution: First of all, we must recognize that $1000 per year is 1000 times greater than an annuity of $1 per year. The number of years, n, is 6. The interest rate, i, is 8%. With this information, we can calculate the solution as 1000 times the present value of an annuity immediate:



$$1000a_{\overline{n}|} = 1000\left(\frac{1 - v^n}{i}\right)$$

$$1000a_{\overline{6}|} = 1000\left(\frac{1 - (\frac{1}{1.08})^6}{0.08}\right)$$

$$1000a_{\overline{6}|} = 4622.879664 = \text{the value at } t = 0$$

Calculator Solution (in Finance mode):

```
6   N   8   %i   1000   PMT   CPT   PV
```

The calculator should output 4622.879664. Thus, you would need to deposit $4622.88 in order to withdraw $1000 at the end of each year for 6 years.

[48-49] Another type of annuity is the **annuity due**. An annuity due is just like an annuity immediate except the first payment occurs at time $t = 0$ instead of time $t = 1$. The present value of an annuity due is represented $\ddot{a}_{\overline{n}|}$ and the accumulated value ($t = n$) is represented $\ddot{s}_{\overline{n}|}$. Because the payments in an annuity due occur one year earlier in time, the present value is discounted one year less than an annuity immediate. Thus, to equate the two types of annuities, use the following relationship:

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|} \tag{2.8}$$

[49] We can use this relationship to obtain a general formula for the present value of an annuity due. Below are the algebraic steps:

$$\ddot{a}_{\overline{n}|} = (1+i)\left(\frac{1 - v^n}{i}\right)$$

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{\frac{i}{1+i}}$$

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d} \tag{2.9}$$

[49] The accumulated value of an annuity due is similarly related to the accumulated value of an annuity immediate:

$$\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|} \tag{2.10}$$

[49] Attempt to derive the general formula for the accumulated value of an annuity due. Here is what you should get:

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} \tag{2.11}$$

[52-53] Annuities don't always have finite length. An annuity whose payments continue forever is called a **perpetuity**. Consider a wealthy philanthropist who wants to set up an annual stipend for a university. He could do so by purchasing a perpetuity; it would continue to make stipend payments even after his death. If we let $n = \infty$ then we get the following annuity formula:

$$a_{\overline{\infty}|} = \frac{1 - v^\infty}{i}$$

[53] However, $v^\infty = 0$ for all positive rates of interest. As a result, the following is the general formula for a perpetuity immediate, the payments of which begin at the end of the first year:

$$a_{\overline{\infty}|} = \frac{1}{i} \tag{2.12}$$

[53] If the payments begin immediately at time $t = 0$, then we have a **perpetuity due**, represented $\ddot{a}_{\overline{\infty}|}$. Using what you know about annuities due, attempt to derive a general formula for a perpetuity due. This is what you should get:

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d} \tag{2.13}$$

[57-58] Consider an annuity in which payments of $1 aren't made at the beginning of the year or at the end of the year but *continuously* throughout the year. This is called a **continuous annuity**. The present value of a continuous annuity is represented as $\overline{a}_{\overline{n}|}$ (pronounced a bar n angle). Here are the general formulas for the present value and accumulated value of a continuous annuity, respectively:

$$\overline{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1 - v^n}{\delta} \tag{2.14}$$

$$\overline{s}_{\overline{n}|} = \int_0^n (1+i)^t dt = \frac{(1+i)^n - 1}{\delta} \tag{2.15}$$

Example: Imagine you could withdraw money continuously from a savings account earning 8% effective (like an intravenous solution dripping fractions of pennies), and your IV drains a total of $1000 each year for the next 6 years. How much must be in the account now for the account to be empty after 6 years?

Solution: Because $1000 is withdrawn from the account each year, the value of the account is 1000 times a continuous annuity with $n = 6$ and $i = 8\%$.

$$1000\overline{a}_{\overline{n}|} = 1000\left(\frac{1 - v^n}{\delta}\right)$$

$$1000\overline{a}_{\overline{n}|} = 1000\left(\frac{1 - \left(\frac{1}{1.08}\right)^6}{ln(1.08)}\right)$$

$$1000\overline{a}_{\overline{n}|} = 4805.423207$$

Thus, the present value of the account must be $4805.42.

Finally, consider the three types of annuities together: immediate, due, and continuous. For any given length and interest rate, an annuity due is worth the most, an annuity immediate is worth the least, and a continuous annuity is in the middle. This is true because money is worth more in the past and less in the future. Recall that an annuity due has payments in the beginning of the year, so the payments have the longest time possible to accrue interest over the life of the annuity. An annuity immediate has payments at the end of the year, so the payments have the least amount of time to accrue interest. A continuous annuity has payments spread evenly over the course of a year, so the payments have less time than an annuity due to accrue interest but more time than an annuity immediate. This concept is represented in the following inequality for $i > 0$:

$$\ddot{a}_{\overline{n}|} > \overline{a}_{\overline{n}|} > a_{\overline{n}|} \tag{2.16}$$

Now we have completed the important annuity concepts from Chapter 3 of the Parmenter book, so take the time to work some of the problems from the end of Chapter while the material is fresh in your mind. Turn to page 66 and do exercises 3, 6, 19, and 56. Look over the material from Chapter 3 if you have difficulty with these problems.