UNIT 3 MODULE 2
FACTORIALS, PERMUTATIONS AND COMBINATIONS

n! "n factorial"

If *n* is a positive integer, then *n!* is *n* multiplied by all of the smaller positive integers.

Also, 0! = 1

0! = 1
1! = 1
2! = (2)(1) = 2
3! = (3)(2)(1) = 6
4! = (4)(3)(2)(1) = 24
5! = (5)(4)(3)(2)(1) = 120
6! = (6)(5)(4)(3)(2)(1) = 720
7! = (7)(6)(5)(4)(3)(2)(1) = 5,040
8! = (8)(7)(6)(5)(4)(3)(2)(1) = 40,320
9! = (9)(8)(7)(6)(5)(4)(3)(2)(1) = 362,880
10! = (10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 3,628,800
n! is **n** multiplied by all of the positive integers smaller than **n**.

FACT:
n! is the number of different ways to arrange (*permutations of*) **n** objects.

EXAMPLE 3.2.1
There are four candidates for a job. The members of the search committee will rank the four candidates from strongest to weakest. How many different outcomes are possible?

EXAMPLE 3.2.1 SOLUTION
If you were to use the Fundamental Counting Principle, you would need to make four dependent decisions.
1. Choose strongest candidate: *4 options*
2. Choose second-strongest candidate: *3 options*
3. Choose third-strongest candidate: *2 options*
4. Choose weakest candidate: *1 option*

(4)(3)(2)(1) = 24

A shorter way to get this answer is to recognize that the problem is asking us to find the number of ways to arrange (according to relative sutability for the job) four people. By definition, the number of ways to arrange 4 things is 4!
4! = 24

EXAMPLE 3.2.2
In how many ways is it possible for 15 students to arrange themselves among 15 seats in the front row of an auditorium?

EXAMPLE 3.2.3
There are 8 greyhounds in a race. How many different orders of finish (first place through eighth place) are possible?

EXAMPLE 3.2.4

1. The password for Gomer's e-mail account consists of 5 characters chosen from the set {g, o, m, e, r} . How many arrangements are possible, if the password has no repeated characters?

2. How many 5-character passwords are possible if a password may have repeated characters?

EXAMPLE 3.2.5

Gomer has a 20 volume set of World Book Encyclopedia. The 20 volumes are arranged in numerical order. His uncle Aristotle has challenged him to write down every possible arrangement of the 20 books. Aristotle will pay Gomer $10,000 if he can compete the job within 30 days. The only proviso is that if Gomer doesn't complete the job within 30 days, he will have to pay Aristotle 1 penny for every permutation that he has failed to list.

1. How many different arrangements are there?

2. Gomer is a fast worker. Assuming that he can write down 1 million arrangements per second, how long will it take for him to complete the job?

EXAMPLE 3.2.6
Refer to the situation in the previous example.
Use the fundamental counting principle to answer this question:
If Gomer is going to choose 9 of the 20 books, and arrange them on a shelf, how many arrangements are possible?

EXAMPLE 3.2.6 SOLUTION
We can't directly use n! to solve this problem, because in this case he is not arranging the entire set of 20 books. At this point, we must use the Fundamental Counting Principle. Gomer has to make 9 dependent decisions:
1. Choose first book: *20 options*
2. Choose second book: *19 options*
3. Choose third book: *18 options*
4. Choose fourth book: *17 options*
5. Choose fifth book: *16 options*
6. Choose sixth book : *15 options*
7. Choose seventh book: *14 options*
8. Choose eighth book: *13 options*
8. Choose ninth book: *12 options*

According to the Fundamental Counting Principle, the number of different outcomes possible is
(20)(19)(18)(17)(16)(15)(14)(13)(12) = 60,949,324,800 arrangements

There is another way to get the answer to this question, without having to enter 9 numbers into the calculator. It refers to a special formula involving n!:

The PERMUTATION FORMULA
The number of permutations of *n* objects taken *r* at a time:

$$P(n,r) = \frac{n!}{(n-r)!}$$

This formula is used when a counting problem involves ***both****:*
1. Choosing a subset of r elements from a set of n elements; ***and***
2. Arranging the chosen elements.

Referring to EXAMPLE 3.2.6 above, Gomer is choosing and arranging a subset of 9 elements from a set of 20 elements, so we can get the answer quickly by using the permutation formula, letting n = 20 and r = 9. (That is, the answer to this problem is the number of permutations of 20 things taken 9 at a time.)

$$P(20,9) = \frac{20!}{(20-9)!} = \frac{20!}{11!}$$

= 60,949,324,800

EXAMPLE 3.2.7
There are ten candidates for a job. The search committee will choose four of them, and rank the chosen four from strongest to weakest. How many different outcomes are possible?

EXAMPLE 3.2.8
There are 8 horses in a race. If all we are concerned with are the first, second and third place finishers (the *trifecta*), how many different orders of finish are possible?

EXAMPLE 3.2.9
Suppose we are going to use the symbols {a, b, c, d, e, f, g, h} to form a 5-character "password" having no repeated characters. How many different passwords are possible?

EXAMPLE 3.2.10
There are six greyhounds in a race: Spot, Fido, Bowser, Mack, Tuffy, William.
We are concerned about who finishes first, second and third. How many different 1st-2nd-3rd orders of finish are possible?

A. 120 B. 216 C. 18 D. 15

EXAMPLE 3.2.11

Homer, Gomer, Plato, Euclid, Socrates, Aristotle, Homerina and Gomerina form the board of directors of the Lawyer and Poodle Admirers Club. They will choose from amongst themselves a Chairperson, Secretary, and Treasurer. No person will hold more than one position. How many different outcomes are possible?

A. 336 B. 24 C. 512 D. 21

ASSORTED EXAMPLES:

Many of the examples from Unit 3 Module 1 could be solved with the permutation formula as well as the fundamental counting principle. Identify some of them and verify that you can get the correct solution by using P(n,r).

FACT:

Any problem that could be solved by using P(n,r) could also be solved with the FCP. The advantage to using P(n,r) is that in some cases we can avoid having to multiply lots of numbers.

Conversely, there are problems that can be solved with the FCP but *can't* be solved using P(n,r).

EXAMPLE 3.2.12
Consider the set S = {a, b, c, d, e}.

1. How many different 3-letter code "words" can we form using the letters of set S without using repeated letters? Examples: abc, bca, dec, cde, bda, adb are 6 different code "words."

2. How many different 3-element subsets does S have?

Solution to #1
We can use the FCP, since forming one of these code words requires three decisions:
i. Choose first letter (5 options)
ii. Choose second letter (4 options)
iii. Choose third letter (3 options)
According to the FCP, the number of different outcomes is (5)(4)(3) = 60 code words.

We could also use the permutation formula, since forming a three letter code word requires us to choose and arrange three elements from a set of five elements.

Solution to #2
The answer to this problem is not 60.
Although the code words "abc," "cba," and "bac" are all different from one another, the *subsets* {a, b, c}, {c, b, a} and {b, a, c} are all the same as one another. This means that the number of 3-element subsets must be fewer than 60.

We can list them all:
{a, b, c}
{a, b, d}
{a, b, e}
{a, c, d}
{a, c, e}
{a, d, e}
{b, c, d}
{b, c, e}
{b, d, e}
{c, d, e}

Although there are 60 different 3-element code words with no repeated letters, there are only 10 different 3-element subsets.

There is a formula that allows us to get this result without having to list all of the possible subsets.

We say that "10 is the number of **combinations** of 5 elements taken 3 at a time."

The COMBINATION FORMULA
The number of *combinations* of *n* things taken *r* at a time:

$$C(n,r)=\frac{n!}{(n-r)!r!}$$

*We use this formula when we are choosing a subset of r elements from a set of n elements, and the order in which elements are chosen or listed is **not** significant.*

EXAMPLE How many 5-element subsets are in the set S = {2, 3, 4, 5, 6, 7, 8, 9}?

SOLUTION We choose 5 elements from a set of 8 elements. The order in which we select or list the elements is not important, so this is a combination problem.

$$C(8,5)=\frac{8!}{(8-5)!5!}=\frac{8!}{3!5!}=\frac{40320}{(6\times 120)}$$

$$=\frac{40320}{720}$$

= 56 different 5-element subsets

EXAMPLE 3.2.13
Recall this problem from earlier: There are six greyhounds in a race: Spot, Fido, Bowser, Mack, Tuffy, William. How many different 1st-2nd-3rd place orders of finish are possible? We saw that the answer was P(6,3) = 120.

New question: After the race, three dogs will be randomly chosen for veterinary examination. How many different three-dog groups are possible?

EXAMPLE 3.2.14

A pizzeria is offering a special: for $6 you get a four-topping pizza. The choices for toppings are pepperoni, sausage, olives, mushrooms, anchovies, peppers, and onions. How many different 4-topping combinations are possible (assuming that no topping can be repeated on a pizza)?

EXAMPLE 3.2.15 Classic example of combinations

1. The Florida Lotto Saturday night drawing used to work like this:

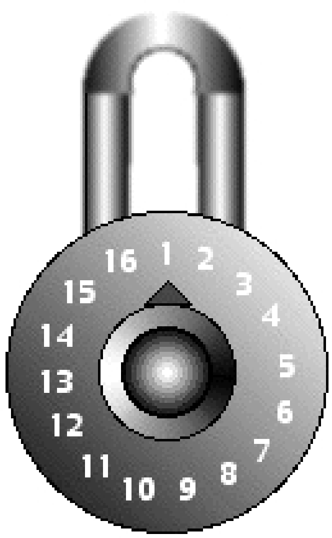
There are 49 ping-pong balls in a machine, each bearing a number from 1 to 49. The machine randomly spits out 6 ping-pong balls. If the numbers on the ping-pong balls match the six numbers that you chose, YOU WIN!

How many different outcomes are possible?

2. Now, the Lotto works like this: there are 53 balls instead of 49. How many outcomes are possible under this new scheme?

EXAMPLE 3.2.16

Poor choice of words: the device that we commonly call a *combination lock* would more accurately be called a *permutation lock* or a *fundamental counting principle lock.*



Why? Because, for one of these locks, the correct "combination" is determined not only by the numbers that are selected, but also by the *order* in which they are selected.

1. Suppose "combination" lock has a dial whose numbers are 1 through 16. Assuming that repeated numbers are allowed within a "combination," how many different 3-number "combinations" are possible? (For example, 10-13-10, 8-12-2, 2-12-8 are three different possibilities.)

2. How many possibilities would there be if repeated numbers were not allowed?

GENERAL WARNING: Our everyday use of the word "combination" does not always agree with the mathematical meaning of that word.

EXAMPLE 3.2.17

There are nine people on the Board of Directors of the Gomermatic Investment Corporation (GIC). At the onset of their monthly board meeting, each person shakes hands with each of the other people. How many handshakes occur?

EXAMPLE 3.2.18

1. Harpo, Groucho, Chico, Zeppo, Gummo, Karl, and Skid have won three tickets to the opera. They will randomly choose three people from their group to attend the opera. How many outcomes are possible?

2. Suppose that instead of choosing 3 people to attend the opera, they decide instead to choose 4 people to *not* attend. How many outcomes are possible?

EXAMPLE 3.2.18 solution

1. There are 7 people from whom to choose, and we are choosing three of them. Because all three people are receiving the same reward (they get to go to the opera), the order in which they are chosen or listed is not important. (For instance, if Harpo, Groucho and Chico go to the opera it’s the same as if Chico, Harpo and Groucho go to the opera.) Thus, this is a combination problem.
C(7,3) = 35

2. Following the approach in #1 above, we should see that answer will be C(7,4). Another way to get the answer is to understand that the answer to this problem should be exactly the same as the answer to problem #1. Why? Because every time we select a three person group to go to the opera, we are automatically choosing a four-person group to not go to the opera. That is, for each of the 35 different 3-person groups in the answer to #1, there is a corresponding 4-person group in the answer to #2. Thus there must be exactly 35 4-person groups. Indeed, C(7,4) = 35.

EXAMPLE 3.2.19

Gomer has eight pet wolverines. He has won a gift certificate to Wally's Wolverine World, that entitles him to one free wolverine massage, one free wolverine shampoo, and one free wolverine manicure. Gomer will randomly select which wolverines receive the treats described above.

1. How many different outcomes are possible, if we assume that no wolverine will receive more than one treat?

A. 512 B. 256 C. 56 D. 336

2. How many different outcomes are possible, if we assume that it **may** be possible for a wolverine to receive more than one treat?

A. 512 B. 256 C. 56 D. 336

EXAMPLE 3.2.20

Euclid's Auto Body Shoppe is trying to unload some unwanted paint by offering the following special deal: for $99 you get a two-tone paint job, using one color for the top and a different color for the body. The available colors are: hot pink, puke green, Gator orange, flaming chartreuse. How many different paint schemes are possible?

A. 8 B. 16 C. 12 D. 7

EXAMPLE 3.2.21

A jar contains a penny, a nickel, a dime, a quarter, a half-dollar, and a silver dollar. Three coins are selected (without replacement) and their monetary sum is determined. How many different monetary sums are possible? (Examples: dime, quarter, penny: 36¢; nickel, half-dollar, dollar: $1.55.)

A. 36 B. 120 C. 60 D. 20

EXAMPLE 3.2.22

Is it A. P(8,3); B. C(8,3); C. 8^3; or D. 3^8 ?

There are 8 kittens in a pet shop:

1. Three kittens will be randomly selected and donated to a nursing home. How many different three-kitten collections are possible?
2. One kitten will be chosen for a rabies vaccination, one kitten will be chosen for a distemper shot, and one kitten will be declawed. In how many different ways can the choice of kittens be made, if it is possible that more than one of these treatments may be given to the same kitten?
3. One kitten will be chosen for a rabies vaccination, another kitten will be chosen for a distemper shot, and a third kitten will be declawed. In how many different ways can the choice of kittens be made?
4. Each kitten will be given either a red, blue, or green ribbon. How many outcomes are possible?

EXAMPLE 3.2.23

Is it P(9,5), C(9,5) or 9^5?

Each item below can be answered with either **A.** P(9,5); **B.** C(9,5); or **C.** 9^5

Choose the correct answer in each case.

1. The number of 5-element subsets in a 9-element set = ____
2. The number of 5-symbol codes that can be formed using the symbols {@, /, $,¿, &, ß, £, á, ? } if repeated symbols are allowed = _____.
3. The number of different ways to choose 5 people from a set of 9 and position them in 5 seats = _____.
4. The number of ways to choose a chairperson, associate chairperson, parliamentarian, treasurer, and secretary from a list of 9 candidates = _____ (assuming that no person will hold more than one office).
5. The number of different 5-person groups that can be chosen from a collection of 9 people = _____.
6. The number of 4-element subsets in a 9-element set = _____.

EXAMPLE 3.2.24

1. The heavy-metal band Death Maggot normally performs 10 songs during a concert. One night, however, they found that they would only have enough time to perform 7 songs, due to the fact that their opening act got called back for three encores. If the band randomly chooses 7 songs to play, how many different outcomes are possible? (All we care about is which 7 songs are chosen.)

2. On the other hand, if they randomly choose 3 songs not to play, how many different outcomes are possible?

3. Assuming that they are only going to play 7 songs from their 10-song repertoire, how many different **arrangements** are possible?

WORLD WIDE WEB NOTE

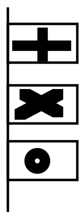
For practice problems involving permutations, combinations or the fundamental counting principle visit the companion website and try THE FUNDAMENTALIZER.

PRACTICE EXERCISES

1. How many different outcomes are possible if a coin is tossed 5 times?
(examples: HTTHT, THTHT, and TTHTT are three different outcomes.)

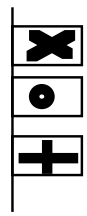
A. 10 B. 32 C. 20 D. 25

2. Ships of the navy of Outer Tyrania communicate at sea via code signals transmitted by flags, as follows: each ship has six flags (the same set of six flags is on every ship); a code is formed by choosing three flags and arranging them, from top to bottom, on the mainmast. How many different codes are possible? Example:



"don't pull that big plug in the bottom of the boat"

differs from



"if you sail too close to the edge, you'll fall off"

A. 120 B. 18 C. 40 D. 10

3. A couple is expecting the arrival of a baby girl. They'll name the child by choosing a first and middle name from this list of their favorite girls' names: Ann, Beth, Carrie, Donna, Edna. The first name will be different from the middle name. (Example: Ann Beth, Beth Ann, and Edna Beth are three different names. Beth Beth is not a valid name.) How many different names are possible?

A. 25 B. 32 C. 20 D. 10

4. Ships of the navy of Inner Tyrania communicate at sea via a method similar to the one described in #2 above, except: each ship has seven flags, and the code is determined exclusively by the 3 flags chosen, and not by the order in which they are arranged on the mast. How many different codes are possible?

A. 35 B. 210 C. 343 D. 128

5. There are 8 teams in a football conference. Each team must play all of the other teams one time. How many games will be played?

A. 16 B. 64 C. 56 D. 28

6. Al, Beth, Chuck, Dora, and Ed have won 3 tickets to the opera. They will randomly choose which 3 of them get to attend the performance. How many different 3-person groups are possible? (Note: you should realize that in this case, the outcome BCA, for instance, is the same as the outcome CAB.)

A. 10 B. 20 C. 243 D. 60

6*. Groucho, Harpo, Chico, Zeppo and Gummo have won 2 tickets to the opera. They will randomly choose which two of them have to attend. How many different 2-person groups are possible? A. 10 B. 20 C. 243 D. 60

7. The uniform for a marching band consists of: 2 different hats, 2 different types of shoe, 3 different jackets, and 3 kinds of trousers. How many different uniform configurations are possible, assuming that a uniform configuration is determined by the hat, shoe, jacket and trouser?
A. 10 B. 24 C. 36 D. 210

8. In how many different ways is it possible to answer an 8-question true/false quiz?
A. 28 B. 256 C. 56 D. 16

9. 6 diplomats, representing 6 different nations, meet for a peace conference. At the outset, each diplomat shakes hands once with each other diplomat. How many handshakes occur?

A. 12 B. 36 C. 64 D. 15

10. The ships of the navy of Middle Tyrania communicate via a method identical to that described in #2 above, except that all six of the flags are arranged on the mast in order to form a code. How many different codes are possible?
A. 120 B. 36 C. 720 D. 12

11. A 'combination' lock has a dial bearing the numbers 1 through 20. How many different 3-number 'combinations' are possible if there are no restrictions on the 3 numbers (example: 19-5-1, 5-19-1, 3-3-12 are three different, valid 'combinations').
A. 6840 B. 8000 C. 60 D. 1140

12. Referring to #11 above, how many different possibilities are there if the only restriction is that a 'combination' cannot have any repeated number (example: 5-17-5 is not valid)?
A. 6840 B. 8000 C. 20!3! D. 20! - 3!

13. P(9,5) = the number of:
A. 5-element subsets in a 9-element set
B. ways for 5 people from a group of 9 to be arranged on a bench
C. 4-element subsets in a 9-element set
D. all of these

14. A couple is expecting the birth of a baby boy. They will name the boy by choosing a first name and a middle name from this list of their favorite boys' names:
Billy, Bobby, Buster, Bubba. They have not ruled out the possibility that the child's first and middle names will be the same (example: Billy Buster, Buster Billy, and Bubba Bubba are three different, valid possibilites). How many different names are possible?
A. 12 B. 144 C.16 D. 64

15. Same as #14 above, except the first and middle names will be different.
A. 12 B. 144 C. 16 D. 6

16. There are seven greyhounds running in a race. How many different
1st-2nd-3rd place orders of finish are possible?
A. 210 B. 243 C. 35 D. 18

17. There are seven greyhounds running in a race. After the race, three of the dogs will be randomly selected for veterinary examination. How many different groups of 3 dogs are possible?
A. 210 B. 243 C. 35 D. 18

18. A jar contains a penny, a nickel, a dime, a quarter, and a half-dollar and silver dollar. Two coins are selected (without replacement) and their monetary sum is determined. How many different monetary sums are possible? (Examples: dime, quarter: 35¢; nickel, half-dollar: 55¢ .)
A. 36 B. 64 C. 30 D. 15

19. In a jail cell, there are 5 Democrats and 6 Republicans. Four of these people will be chosen for early release. How many 4-person groups are possible?
A. 330 B. 7920 C. 150 D. 1,663,200

20. Same as #19, except that 2 Democrats and 2 Republicans will be chosen.
A. 160 B. 150 C. 50 D. 600

21. Same as #19, except 4 people will be chosen for a police line-up. How many different line-ups are possible?
A. 330 B. 7920 C. 150 D. 1,663,200

22. Same as #21, except 2 Democrats and 2 Republicans will be chosen for a police line-up, with the Democrats on the left and the Republicans on the right.
A. 160 B. 150 C. 50 D. 600

23. C(15,3) = the number of
A. ways to choose a President, Vice-President, and Secretary from a 15-member club.
B. 3-element subsets in a 15-element set. C. 12-element subsets in a 15-element set.
D. B & C are both correct. E. none of these.

24. In a pet shop, there are 6 kittens. One kitten will be declawed, another kitten will be given a rabies vaccination, and yet another will be given a distemper shot. In how many ways can the selection be made? (This means, for instance, that if Fluffy gets the distemper shot, Buffy gets declawed and Whiskers gets the rabies vaccine, then this selection differs from one where Buffy gets the distemper shot, Whiskers gets declawed and Fluffy gets the rabies vaccine.)
A. 18 B. 20 C. 60 D. 120

25. In a pet shop, there are 6 kittens. Three kittens will be donated to a nursing home. How many different 3-kitten groups are possible?
A. 18 B. 20 C. 60 D. 120

ANSWERS TO LINKED EXAMPLES

EXAMPLE 3.2.2	$15! \approx 1.31 \times 10^{12}$			
EXAMPLE 3.2.3	8! = 40,320			
EXAMPLE 3.2.4	1. 5! = 120	2. (5)(5)(5)(5)(5) = 3125		
EXAMPLE 3.2.5	1. $20! \approx 2.43 \times 10^{18}$	2. About 77,000 years		
EXAMPLE 3.2.7	P(10,4) = 5,040			
EXAMPLE 3.2.8	P(8,3) = 336			
EXAMPLE 3.2.9	P(8,5) = 6720			
EXAMPLE 3.2.10	A			
EXAMPLE 3.2.11	A			
EXAMPLE 3.2.13	C(6,3) = 20			
EXAMPLE 3.2.14	C(7,4) = 35			
EXAMPLE 3.2.15	1. C(49,6) = 13,983,816	2. C(53,6) = 22,957,480		
EXAMPLE 3.2.16	1. 4096	2. 3360		
EXAMPLE 3.2.17	C(9,2) = 36			
EXAMPLE 3.2.19	1. P(8,3) = 336	2. (8)(8)(8) = 512		
EXAMPLE 3.2.20	P(4,2) = 12			
EXAMPLE 3.2.21	C(6,3) = 20			
EXAMPLE 3.2.22	1. B	2. C	3. A	4. D
EXAMPLE 3.2.23	1. B	2. C	3. A	4. A
	5. B	6. B		
EXAMPLE 3.2.24	1. C(10,7) = 120	2. C(10,3) = 120 = C(10,7)		
	3. P(10,7) = 604800			

ANSWERS TO PRACTICE EXERCISES

1. B	**2.** A	**3.** C	**4.** A	**5.** D	**6.** A
6*. A	**7.** C	**8.** B	**9.** D	**10.** C	**11.** B
12. A	**13.** B	**14.** C	**15.** A	**16.** A	**17.** C
18. D	**19.** A	**20.** B	**21.** B	**22.** D	**23.** D
24. D	**25.** B				