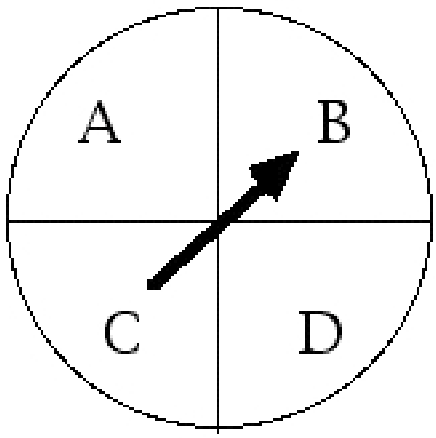
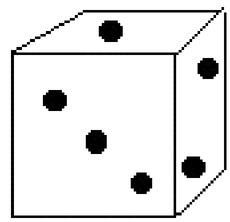
UNIT 3 MODULE 6
INDEPENDENT EVENTS, THE MULTIPLICATION RULES

EXAMPLE 3.6.1
Suppose we have one six-sided die, and a spinner such as is used in a child's game. When we spin the spinner, there are four equally likely outcomes: "A," "B," "C," and "D."





1. An experiment consists of rolling the die and then spinning the spinner. How many different outcomes are possible?

2. What is the probability that the outcome will be "3-C?"

SOLUTIONS

1. There are six equally likely outcomes when we roll the die. There are four equally likely outcomes when we roll the die. According to the Fundamental Counting Principle, if we spin the spinner and roll the die the number of outcomes is
(6)(4) = 24

2. There are 24 equally likely outcomes to the two-part experiment. Exactly one of them is the outcome “3-C.” Thus, the probability that the experiment result will be “3-C” is 1/24.

Suppose we think of the experiment in EXAMPLE 3.6.1 as involving two separate, independent processes, rather than a single two-part process.

Note that when we roll the die, the probability that we will get a “3” is 1/6.

Note also that when we spin the spinner, the probability that we will get a “C” is 1/4.

Finally, note that $\frac{1}{6} \times \frac{1}{4} = \frac{1}{24}$

That is, *the probability that we receive both a “3” on the die and a “C” on the spinner is the same as the probability of getting a “3” on the die* ***multiplied by*** *the probability of getting a “C” on the spinner.*

This illustrates an important property of probability:

THE MULTIPLICATION RULE FOR INDEPENDENT EVENTS
If E and F are *independent* events, then
$\mathbf{P(E\ and\ F) = P(E) \times P(F)}$

EXAMPLE 3.6.2
Recall this (authentic) data from the Natural Resources Defense Council:
40% of bottled water samples are merely tap water.
30% of bottled water samples are contaminated by such pollutants as arsenic and fecal bacteria. If two samples are independently selected, what is the probability that both samples are contaminated by pollutants?

EXAMPLE 3.6.2 SOLUTION

Let E be the event that the first sample is contaminated. Then P(E) = .3.
Let E be the event that the second sample is contaminated. Then P(F) = .3.

We are asked to find P(E and F).

$$\text{P(E and F)} = \text{P(E)} \times \text{P(F)} = .3 \times .3 = .09$$

EXAMPLE 3.6.3
Suppose that survey of hawks reveals that 40% of them agree with the statement "Poodles are tasty." If two hawks are independently selected, what is the probability that neither of them agree that "Poodles are tasty?"
A. .8 B. .6 C. .36 D. .64

INDEPENDENT EVENTS, DEPENDENT EVENTS
Two events A and B are said to be **independent** if they do not influence one another. More formally, this means that the occurrence of one event has no effect upon the probability of the other event.

EXAMPLE 3.6.5
At the entrance to a casino, there are two slot machines. Machine A is programmed so that in the long run it will produce a winner in 10% of the plays. Machine B is programmed so that in the long run it will produce a winner in 15% of the plays.
1. If we play each machine once, what is the probability that we will win on both plays?
2. If we play each machine once, what is the probability that we will lose on both plays?
3. If we play each machine once, what is the probability that we will win on at least one play?

EXAMPLE 3.6.5 SOLUTION
1. Let A be the event that we win when we play Machine A. Then P(A) = .10.
Let B be the event that we win when we play Machine B. Then P(B) = .15.
We are trying to find P(A and B).
$P(\text{A and B}) = P(A) \times P(B) = .1 \times .15 = .015$

2. In this case we are trying to find $P(A' \text{ and } B')$.
Since P(A) = .1, $P(A') = .9$ (The complements rule).
Likewise, since P(B) = .15, $P(B') = .85$
Thus, $P(A' \text{ and } B') = P(A') \times P(B') = .9 \times .85 = .765$

3. In this case we are trying to find P(A or B). We cannot solve this directly by using the multiplication rule for independent events, but there are two different ways to get the correct answer indirectly.

First, recall from logic that the condition “A or B” is the opposite (in logic we called it “negation,” in probability we call it “complement”) of the condition “not A and not B.” That means that we can use the answer to problem #2 above to get the answer to this problem. According to the complements rule,

$$P(\text{A or B}) = 1 - P(A' \text{ and } B') = 1 - .765 = .235$$

Alternatively, we could and use this formula from UNIT 3 MODULE 5:

P(E or F) = P(E) + P(F) – P(E and F). This will allow us use the answer from Problem #1 above.

P(A or B) = P(A) + P(B) – P(A and B) = .10 + .15 – .015 = .235

Notice again that we have solved Problem #3 twice, using two different approaches, each of which shows that the answer is .235.

EXAMPLE 3.6.6

According to a study in 1992 by the U.S. Department of Agriculture, 80% of commercially grown celery samples and 45% of commercially grown lettuce samples contain traces of agricultural poisons (insecticides, herbicides, fungicides).

If Homer eats one serving (one sample) of celery and one serving of lettuce:

1. What is the probability that both the celery and the lettuce contain traces of agricultural poisons?
2. What is the probability that neither serving contains traces of agricultural poisons?
3. What is the probability that at least one of the servings contains traces of agricultural poisons?

EXAMPLE 3.6.7
Real data (as of 1999):
Each day, 7% of the US population eat a meal at McDonald's.
If two people are randomly and independently selected, what is the probability that...
1. ...both people will eat a meal at McDonald's today?
2. ...neither person will eat a meal at McDonald's today?
3. ...at least one of them will eat a meal at McDonald's today?

GENERAL NOTE

In any situation in which two or more individuals are chosen from a large population of unspecified size, we will assume that the selections are independent events.

EXAMPLE 3.6.8

Suppose the table below shows the results of a survey of TV viewing habits:

Hours of viewing per week	Percent of respondents
5 or fewer	4%
5.1 – 10	8%
10.1 – 15	10%
15.1 – 20	18%
20.1 – 25	22%
25.1 – 35	30%
More than 35	8%

Assume that Homer and Gomer are a couple of randomly selected, independent guys.
According to the data in the table above, what is the probability that:
1. Homer views TV for 5 or fewer hours per week, and Gomer views TV for 10.1 - 20 hours per week?
2. Homer views TV for 35 or fewer hours per week, and so does Gomer?

Still assuming that Homer and Gomer are a couple of randomly selected, independent guys:
3. What is the probability that neither of them falls in the 15.1 - 20 hours per week category?
4. What is the probability that at least one of them views TV for 15.1 - 20 hours per week?

EXAMPLE 3.6.9
A university awards scholarships on the basis of student performance on a certain placement test. The table below indicates the distribution of scores on that test.

Score	Scholarship	Percentage
0-200	None	13%
201-300	None	23%
301-400	None	26%
401-500	Partial	12%
501-600	Partial	11%
601-700	Partial	9%
701-800	Full	6%

If Homer and Gomer are a couple of randomly selected, independent guys, what is the probability that neither of them received a scholarship?

Recall the following scenario from Unit 3 Module 5:

EXAMPLE 3.5.1*

According to a recent article from the *New England Journal of Medical Stuff* ,
63% of cowboys suffer from saddle sores,
52% of cowboys suffer from bowed legs,
40% suffer from both saddle sores and bowed legs.

What is the probability that a randomly selected cowboy...

4. ...has saddle sores and bowed legs?

Let's answer this question again, using the Multiplication Rule for Independent events.

Let E be the event "the randomly selected cowboy has saddle sores." Then P(E) = .63.
Let F be the event "the randomly selected cowboy has bowed legs." Then P(F) = .52.
According to the multiplication rule,

$$P(\text{E and F}) = P(E) \times P(F) = .63 \times .52 = .3276$$

This seems very nice, until we notice that the data provided in the problem states directly that P(E and F) = .40.

The question now becomes: What's wrong here?
Why does the Multiplication Rule not give the correct answer?
Does this mean that the Multiplication Rule is not reliable? Is this evidence of a rip in the very fabric of space/time, perhaps signaling the impending destruction of the universe as we know it?

HACKING MATHEMATICS

CONDITIONAL PROBABILITY
On the Impossibility of Tuesday
A dialogue

Gomer: Do you know what day it is?
Homer: It's Tuesday.
Gomer: Are you sure?
Homer: Sure I'm sure.
Gomer: Really? But it can't be Tuesday, can it?
Homer: Of course it's Tuesday. Yesterday was Monday, today is Tuesday.
Gomer: But that's exactly the problem.
Homer: What problem?
Gomer: The problem of Tuesday. It can't be Tuesday.
Homer: Whatever.
Gomer: Look, there are seven days in a week, right?
Homer: Last time I checked.
Gomer: So if I just woke up from a coma--
Homer: --that would be a nice change--
Gomer: --if I just woke up from a coma, and didn't know what day it was, the probablity that today is Tuesday would be one seventh, right?
Homer: Right; one out of seven.
Gomer: But in order for today to be Tuesday, yesterday must have been Monday.
Homer: It follows.
Gomer: Actually, it precedes. But the probability that yesterday was Monday is also one seventh, so the probability that yesterday was Monday AND today is Tuesday is only one seventh of one seventh...
Homer: ...The multiplication rule.
Gomer: Right, so that's only one out of 49. And it gets worse. In order for yesterday to have been Monday, the day before yesterday would have had to have been Sunday...
Homer: ...so the probability that the day before yesterday was Sunday, AND yesterday was Monday, AND today is Tuesday...
Gomer: ...is one seventh of one seventh of one seventh...
Homer: ...which is only, let's see, (*mumbles, makes calculations in the air with finger, carries the one, et c*) one out of 343, I guess. Dang. Maybe today isn't Tuesday, after all.
Gomer: Now, I'm especially worried, because it occured to me that in order for the day before yesterday to have been Sunday, the day before that would have had to have been Saturday, so (let's work backward here) the probability that today is Tuesday AND yesterday was Monday AND the day before yesterday was Sunday AND the day before that was Saturday...
Homer: ...is one seventh of one seventh of one seventh of one seventh, which is ...
Gomer: ...one out of 2401.
Homer: Hey, you're pretty quick with that.
Gomer: Well, I've been researching the matter. In fact, I found out that if you take this back as far as a week and a half, it's obvious that the probability that today is Tuesday is only about one in 282 million.
Homer: A virtual impossibility!
Gomer: So, I wonder what day it is.
Homer: Me too, now that you've explained the situation to me.
Gomer: One thing's for sure.
Homer: You've got that right. One thing's for sure: today isn't Tuesday.
Gomer: Exactly.
Homer: It's a virtual impossibility.
Gomer: There is a bright side, though.
Homer: And that is?
Gomer: Well, I was supposed to have a math test on Tuesday, but I haven't been studying.
Homer: Clearly, you've had more important fish to fry.
Gomer: Well put; I've been wrestling with this "impossibility of Tuesday" issue for quite a while. At least one thing is virtually certain: my math test can't be today.
Homer: It's virtually impossible.
Gomer: That's a relief.
Homer: Every cloud has its silver lining. I have get going, though. I have term paper due tomorrow.

Gomer: Wednesday?
Homer: Yeah. Wednesday morning, eight o'clock sharp.
Gomer: But it's virtually impossible that tomorrow will be Wednesday...

We will discuss the "impossibility of Tuesday" after we've introduced some preliminary facts.

EXAMPLE 3.6.10
An IRS auditor has a list of 12 taxpayers whose tax returns are questionable. The inspector will choose 2 of these people to be audited. Eight of the taxpayers are Floridians and 4 are Georgians. What is the probability that both people selected will be Floridians?

EXAMPLE 3.6.10 SOLUTION
Let E be the event that the first person selected is a Floridian, and let F be the event that the second person is also a Floridian. Note that the probability of F is affected by whether or not E occurs.

When we select the first person, the probability that he/she is Floridian is 8/12, because eight of the twelve people are Floridians.
P(E) = 8/12

Assuming that the first person selected was a Floridian, there will be eleven people left in the pool, seven of whom are Floridians. Thus, the probability that the second person is a Floridian, given that the first person was a Floridian, is 7/11.

P(F|E) = 7/11

We are trying to find the probability of both events occurring, so we should multiply:

$$\text{P(E and F)} = \text{P(E)} \times \text{P(F, given E)} = \frac{8}{12} \times \frac{7}{11} = \frac{56}{132} \approx .424$$

The previous example suggests the following fact:

THE MULTIPLICATION RULE FOR *DEPENDENT* EVENTS.
If E and F are **dependent events**, then
$P(\text{E and F}) = P(\text{E}) \times P(\text{F, given E})$

This rule is especially useful when we have a two-step experiment where the outcome of the first step affects the possible outcomes for the second step, such as the previous example.

INDEPENDENT vs. DEPENDENT EVENTS
In a case where two or more items are selected from a *large population of unspecified size,* we will assume that the selections are INDEPENDENT. In a case where two or more items are selected from a *small population of specified size,* we will assume that the selections are DEPENDENT.

EXAMPLE 3.6.11
In his pocket, Gomer has 3 red, 5 orange and 2 blue M&Ms.
If he randomly chooses two M&Ms, what is the probability that both will be red?
A. .6 B. .09 C. .067 D. .52

EXAMPLE 3.6.12
The Skuzuzi Kamikaze sport/utility vehicle is manufactured at two plants, one in Japan and one in the US. Sixty percent of the vehicles are made in the US, while the others are made in Japan. Of those made in the US, 5% will be recalled due to a manufacturing defect. Of those made in Japan, 3% will be recalled.
Find the probability that a vehicle will be...
1. ...made in the US and not recalled.
2. ...made in Japan and recalled.

EXAMPLE 3.6.13

The state insurance commission revealed the following information about the Preferential Insurance Company's homeowners' insurance: 10% of the policy-holders have filed more than 5 claims over the past two years; 60% of these people have had their insurance canceled; 90% of the policy-holders have filed 5 or fewer claims over the past two years; 15% of these people have had their insurance canceled.

What is the probability that a policy holder filed more than 5 claims over the past 2 years and had his/her insurance canceled?

Once again, let's turn our attention to this scenario from Unit 3 Module 5:

EXAMPLE 3.5.1**

According to a recent article from the *New England Journal of Medical Stuff* ,
63% of cowboys suffer from saddle sores,
52% of cowboys suffer from bowed legs,
40% suffer from both saddle sores and bowed legs.

What is the probability that a randomly selected cowboy has saddle sores and bowed legs? We know that the answer is .40, because that information is stated directly in the problem. Earlier we saw, however, that if we try to derive this answer by using the Multiplication Rule for Independent Events, we fail because (.63)(.52) is NOT equal to .40.

Let's try again, using the correct version of the Multiplication Rule.

$$\text{P(S.S. and B.L.)} = \text{P(S.S.)} \times \text{P(B.L., given S.S.)}$$

$$= .63 \times \frac{40}{63} = .40$$

We see that the multiplication rule yields the correct answer when we take into account the dependence of the two events.

EXAMPLE 3.6.14
A local sports talk radio station offers the following contest: each Thursday during the football season, listeners are invited to call the station and make four predictions "against the spread." The caller may choose any four college or professional games he/she desires, as long as they are games for which the odds makers have issued a betting line.
Any caller who turns out to be correct on all four predictions will win a $10 bar tab from a local sports bar. If we assume that each week 25 callers will get on the air with their predictions, what will be the expected weekly cost in bar tabs to the bar that sponsors the program? (In order to answer this question, we need to make a reasonable assumption about the significance of "beating the point spread.")

EXAMPLE 3.6.15
THE WORLD FAMOUS CAR AND GOATS PROBLEM
You are a contestant on the extinct TV game show *Let's Make a Deal*.
On the stage, there are three large doors. Behind one door is a new car; behind the other two doors are goats. You are asked to pick one of the doors. You win whatever prize is behind the door that you pick.
You choose a door. Before he reveals your prize, the host opens one of the doors that you didn't pick, and shows you that there was a goat behind that door.
There are still two unopened doors. You have chosen one of them. You now know for sure that behind one of the two doors is a car, and behind the other door is a goat. The host asks you if you want to change your choice of doors.
Is there any mathematical reason why you should switch?

To decide whether or not it is advantegeous to switch, answer the following questions:
What is the probability that you will win a car by switching?
What is the probability that you won't win a car by switching?
These two questions are easy to answer, if you think carefully about the underlying conditions:
How could you win (a car) by switching?
How could you lose by switching?

WORLD WIDE WEB NOTE
For probability practice problems visit the companion website and try THE INFINITE IMPROBABILITY DRIVER.

PRACTICE EXERCISES

Table A below shows the distribution of undergraduate students at Normal University according to the number of credit hours for which they are registered this semester. Table B below shows the distribution of students at Normal University according to cumulative G.P.A.

TABLE A

# of credit hours	% of students
11 or fewer	12%
12	31%
13	6%
14	8%
15	21%
16	9%
17	2%
18 or more	11%

TABLE B

cumulative G.P.A.	% of students
0.00 - 0.80	14%
0.81 - 1.60	16%
1.61 - 2.40	38%
2.41 - 3.20	17%
3.21 - 4.00	15%

1 - 8: Refer to the appropriate table to determine the probability that a randomly selected student:

1. has a G.P.A. less than 0.81, given that the G.P.A is less than 2.41.
A. .259 B. .14 C. .095 D. .206

2. is enrolled for 17 credit hours, given that he/she is enrolled for more than 15 credit hours.
A. .0909 B. .0952 C. .9090 D. .2222

3. has a G.P.A. greater than 3.20, given that the G.P.A is greater than 2.40.
A. .882 B. .048 C. .469 D. .144

4. is enrolled for 12 credit hours, given that he/she is enrolled for 12 or 13 hours.
A. .25 B. .8378 C. .1147 D. .3407

5. …is enrolled for 12 credit hours and has a G.P.A. in the 1.61 - 2.40 range (assume that # credit hours enrolled and cumulative G.P.A are INDEPENDENT of one another).
A. .69 B. .1178 C. .8158 D. .1209

6. …is enrolled for 18 or more credit hours and has a G.P.A. greater than 3.20.
A. .7333 B. .26 C. .0165 D. .24

7. …is enrolled for 11 or fewer credit hours or has a G.P.A. in the 2.41 - 3.20 range.
A. .2696 B. .29 C. .0204 D. .7059

8. …is enrolled for 16 credit hours or has a G.P.A less than 1.61.
A. .363 B. .027 C. .39 D. .3448

Exercises 9 - 11 refer to this situation: Homer has a '68 VW Bus and an '85 Yugo. On 25% of the days of the year, Homer finds that the VW will not start. On 30% of the days of the year, the Yugo will not start. Whether or not a particular vehicle starts seems to be random and independent of the other vehicle. On a given day, what is the probability…

9. …that the VW starts and the Yugo doesn't start?
A. .45 B. .225 C. .55 D. .075

10. …that both vehicles start?
A. .525 B. .55 C. .075 D. 1.35

11. …that at least one of the vehicles doesn't start?
A. .075 B. .895 C. .55 D. .475

12 - 13: Statistics for a certain carnival game reveal that the contestants win a large teddy bear 1% of the time, win a small teddy bear 4% of the time, win a feather attached to an alligator clip 35% of the time, and lose the rest of the time. What is the probability that a randomly selected player…

12. …wins a large teddy bear, given that he/she wins something?
A. .0085 B. .2857 C. .029 D. .025

13. …wins a small teddy bear, given that/he she wins a teddy bear?
A. .8 B. .2 C. .3 D. .03

14. Referring to the carnival game in the previous example: If Bernie and Ernie each play once, what is the probability that Bernie loses and Ernie wins a feather, assuming that Ernie and Bernie are a couple of randomly selected, independent guys?
A. .95 B. .5833 C. .21 D. .15

15. Like #14: what is the probability that at least one of them wins something?
A. .8 B . .16 C. .64 D. .96

16. True fact from medical history: *If a human is bitten by a dog showing symptoms of rabies, and the human does not seek medical treatment, the probability that the human will develop symptoms of rabies (a disease that is nearly always fatal) is about 1/6.*

If two people are bitten by dogs that show symptoms of rabies, what is the probability that neither person will develop symptoms of rabies?
A. 2/6 B. 10/6 C. 1/36 D. 10/36 E. 25/36

17. There are 8 Republicans and 6 Democrats on a congressional committee. The Gomermatic Corporation is going to randomly select two committee members to be recipients of $100,000 campaign contributions. Find the probability that both selectees will be Democrats.
A. .165 B. .813 C. .857 D. .536

18. The table below shows the distribution according to salary of the employees of a large corporation.

annual salary	% of employees
\$0 - 9,999	4%
10,000 - 29,999	38%
30,000 - 59,999	32%
60,000 - 99,999	17%
100,000 or more	9%

If Homerina and Gomerina are a couple of randomly selected, independent persons, what is the probability that at least one of them has salary less than \$30,000?
A. .9324 B. .76 C. .6636 D. .84

19. In a basket, there are 10 ripe peaches, 8 unripe peaches, 12 ripe apples, and 4 unripe apples. If two fruit are chosen, what is the probability that neither are peaches?
A. .2727 B. .2215 C. .2803 D. .2139

ANSWERS TO LINKED EXAMPLES

EXAMPLE 3.6.3	C			
EXAMPLE 3.6.6	1. .36	2. .11	3. .89	
EXAMPLE 3.6.7	1. .49	2. .09	3. .91	
EXAMPLE 3.6.8	1. .004	2. .8464	3. .6724	4. .3276
EXAMPLE 3.6.9	.3844			
EXAMPLE 3.6.11	.067			
EXAMPLE 3.6.12	1. .57	2. .012		
EXAMPLE 3.6.13	.06			
EXAMPLE 3.6.14	We assume that the purpose of the point spread is, on average, to reduce all bets to 50/50 propositions. Thus the probability that a randomly selected person will get all four guesses correct is (.5)(.5)(.5)(.5) = 0625. If there are 25 participants, the expected number who get a four guesses right is (.0625)(25) =1.5625. It would be reasonable to expect that there would be one or two winners per week.			
EXAMPLE 3.6.15	You should switch.			

ANSWERS TO PRACTICE EXERCISES

1. D	**2.** A	**3.** C	**4.** B	**5.** B
6. C	**7.** A	**8.** A	**9.** B	**10.** A
11. D	**12.** D	**13.** A	**14.** C	**15.** C
16. E	**17.** A	**18.** A	**19.** D	