Well-posedness of the Hele-Shaw-Cahn-Hilliard system

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Abstract

We study the well-posedness of the Hele-Shaw-Cahn-Hilliard system modeling binary fluid flow in porous media with arbitrary viscosity contrast but matched density between the components. Well-posedness that is global in time in the two dimensional case and local in time in the three dimensional case are established. Several blow-up criterions in the three dimensional case are provided as well.

1 Introduction

The modeling and analysis of multi-phase fluid flow is a fascinating, challenging and important problem [4]. Displacement of oil by water in oil reservoir (usually porous media) is one of the well-known examples of two component (phase) flows [5].

A common approach to two phase fluids that are macroscopically immiscible is the sharp interface approach where the two phases are separated by a sharp interface $\Gamma(t)$. The dynamics of the system in porous media is then governed by the following two phase Hele-Shaw (Darcy) system (Muskat problem) [21, 19, 26]:

$$\begin{cases} \nabla \cdot u_i = 0, \text{ in } \Omega_i, \\ u_i = -\frac{1}{12\eta_i}\big(\nabla p_i - \tilde{G}\rho_i e\big), \text{ in } \Omega_i, \\ (u_1 - u_2)\cdot n = 0, \text{ on } \Gamma, \\ p_2 - p_1 = \tau\kappa \text{ on } \Gamma, \end{cases} \tag{1.1}$$

where u_i denotes the fluid velocity of the ith fluid which occupies the region Ω_i, η_i is the viscosity, p_i is the pressure, ρ_i the density, $\tilde{G}$ is the gravitational force in the direction of the unit vector $e = (0, 1)$ in two dimensional case or $e = (0, 0, 1)$ in the three dimensional case, n is the unit normal to the interface Γ, κ is the (mean) curvature and τ is the dimensionless surface tension coefficient. The local in time well-posedness of (1.1) with or without surface tension is known [2, 3, 14]. Global in time well-posedness with surface tension[15, 11] and 2D without surface tension [28] is also known under the assumption that the initial data is a small perturbation of a flat interface or a sphere.

A difficulty commonly associated with the sharp interface approach is the topological change of the interface, especially in terms of pinchoff and reconnection that are important in applications [4, 21]. As an alternative approach, one could consider the so-called phase field models (or diffuse interface models) where an order parameter c is introduced and a capillary stress tensor is used to model the interface between the two fluids and the forces associated [4].

In this paper, we will consider phase field approach to two phase fluid flow with matched density in a Hele-Shaw cell or porous media. The dynamical equations are given by the following Hele-Shaw-Cahn-Hilliard system [21, 13]:

$$\begin{cases} \nabla \cdot u = 0, \\ u = -\frac{1}{12\eta(c)}\big(\nabla p - \frac{1}{\mathbf{M}}\mu\nabla c\big), \\ c_t + u \cdot \nabla c = \frac{1}{\mathbf{Pe}}\Delta\mu, \\ c(0,x) = c_0(x), \end{cases} \tag{1.2}$$

where u is the fluid velocity, c is the order parameter which is related to the concentration of the fluid, the chemical potential μ depends on the order parameter c and is given by

$$\mu(c) = f_0'(c) - \mathbf{C}\Delta c, \tag{1.3}$$

and $\mathbf{Pe}$ is the diffusion Péclet number, $\mathbf{C}$ is the Cahn number, and $\mathbf{M}$ is a Mach number. Furthermore, $\eta(c)$ is the kinematic viscosity coefficient satisfying

$$\eta \in C^\infty(\mathbf{R}), \quad 0 < \lambda \leq \eta(c) \leq \Lambda < \infty, \tag{1.4}$$

the Helmholtz free energy $f_0(c)$ is given by the classical double well potential

$$f_0(c) = (c^2 - 1)^2. \tag{1.5}$$

In the above system (1.2), p is not the physical pressure but the combination of certain generalized Gibbs free energy and the gravitational potential (see [21] for more details). We will assume that the fluid occupies the two or three dimensional torus $\mathbf{T}^d, d = 2, 3$ for simplicity.

One may formally derive the sharp interface model (1.1) by taking appropriate limit within the Hele-Shaw-Cahn-Hilliard system (1.2) [21].

Besides applications in two phase flow in porous media and Hele-Shaw cell, certain simplified versions of this HSCH model has been also used in tumor growth study [31]. Moreover, unconditionally stable schemes has been developed [30] and the existence of certain type of weak solution (without uniqueness) is also derived [16] for the case with matched density and viscosity.

The goal of this manuscript is to study the well-posedness of the matched density Hele-Shaw-Cahn-Hilliard system (1.2) with arbitrary viscosity contrast. The mathematics is non-trivial since the energy is critical in the two dimensional case and super critical in the three dimensional case (see section 5).

The Hele-Shaw-Cahn-Hilliard system can be formally viewed as an appropriate limit of the classical Navier-Stokes-Cahn-Hilliard system [4, 21, 18] which is a popular phase field model for two phase flow. There are a lot of works on the Navier-Stokes-Cahn-Hilliard

system including local in time well-posedness in 2 and 3 dimensional and global in time well-posedness in 2D under various assumptions [1, 7]. We refer to [22, 23, 24, 4] and references therein for more related works.

The rest of the paper is organized as follows. We prove a key estimate on the "pressure" in the second section. This estimate is nontrivial due to the variable coefficient introduced with the mismatched viscosity. New estimates on certain commutator operators in fractional derivative spaces are needed and they are derived in the Appendix. In section three we present the local in time well-posedness based on certain modified Galerkin approximation of the HSCH system and the "pressure" estimate from section 2. In section 4 we provide a Beale-Kao-Majda type blow-up criterion and prove that the system is global in time well-posed in the two dimensional case. We provide a refined blow-up criterion in the 3D case in section 5.

2 The estimate of the pressure

In this section, we present the estimate of the modified pressure p. The variable coefficient necessitates treatment involving fractional derivatives and associated commutator estimates.

Proposition 2.1 *Let $s \geq 0$. Assume that $c \in H^{s+2}(\mathbf{T}^d)$, and p is a smooth solution of the elliptic equation*

$$div\Big(\frac{1}{\eta(c)}\nabla p\Big) = div\Big(\frac{1}{\eta(c)}\mu(c)\nabla c\Big). \tag{2.1}$$

If $s \in (\frac{k-1}{2}, \frac{k}{2}]$ for some $k \in \mathbf{N}$, then the solution p satisfies

$$\|\nabla p\|_{H^s} \leq \mathcal{F}(\|c\|_{L^\infty})\big(1 + \|\nabla c\|_{L^\infty}\big)\big(1 + \|c\|_{H^2}\big)^k \|c\|_{H^{s+2}}. \tag{2.2}$$

Here $\mathcal{F}$ is an increasing function on $\mathbf{R}^+$.

Proof. Thanks to (1.4), a straightforward energy estimate yields that

$$\|\nabla p\|_{L^2} \leq C\|\mu(c)\|_{L^2}\|\nabla c\|_{L^\infty} \leq C(1 + \|c\|^2_{L^\infty})\|\nabla c\|_{L^\infty}\|c\|_{H^2}. \tag{2.3}$$

Taking the operator $\langle D\rangle^s$ to (2.1) to obtain

$$\begin{aligned}\mathrm{div}\Big(\frac{1}{\eta(c)}\nabla\langle D\rangle^s p\Big) &= \mathrm{div}\langle D\rangle^s\Big(\frac{1}{\eta(c)}\mu(c)\nabla c\Big) - \mathrm{div}\Big(\langle D\rangle^s\Big(\frac{1}{\eta(c)}\nabla p\Big) - \Big(\frac{1}{\eta(c)}\nabla\langle D\rangle^s p\Big)\Big)\\ &= \mathrm{div}(A + B),\end{aligned}$$

from which and the energy estimate, we infer that

$$\|\nabla p\|_{H^s} \leq C\big(\|A\|_{L^2} + \|B\|_{L^2}\big).$$

Due to the definition of $\mu(c)$, we have

$$\frac{1}{\eta(c)}\mu(c)\nabla c = \frac{1}{\eta(c)}f_0'(c)\nabla c - \mathbf{C}\frac{1}{\eta(c)}\Delta c\nabla c = \nabla g_1(c) - \Delta c\nabla g_2(c),$$

for some g_1, g_2 with $g_1(0) = g_2(0) = 0$. We have by Lemma 6.3 that

$$\|\langle D\rangle^s \nabla g_1(c)\|_{L^2} \leq \mathcal{F}(\|c\|_{L^\infty})\|c\|_{H^{s+1}},$$

and using Bony's decomposition to write

$$\begin{aligned}\Delta c \nabla g_2(c) &= T_{\Delta c}\nabla g_2(c) + \widetilde{R}(\Delta c, \nabla g_2(c)) \\ &= \mathrm{div} T_{\nabla c}\nabla g_2(c) - T_{\nabla c}\cdot\nabla\nabla g_2(c) + \widetilde{R}(\Delta c, \nabla g_2(c)),\end{aligned}$$

then from the proof of Lemma 6.2, it is easy to see that

$$\|\langle D\rangle^s \Delta c \nabla g_2(c)\|_{L^2} \leq \mathcal{F}(\|c\|_{L^\infty})\|\nabla c\|_{L^\infty}\|c\|_{H^{s+2}}.$$

Thus we obtain

$$\|A\|_{L^2} \leq \mathcal{F}(\|c\|_{L^\infty})\big(1 + \|\nabla c\|_{L^\infty}\big)\|c\|_{H^{s+2}}.$$

and by Lemma 6.4-6.3 and (2.3), for $s \in (0, 1]$,

$$\|B\|_{L^2} \leq \mathcal{F}(\|c\|_{L^\infty})\|c\|_{H^{s+2}}\|\nabla p\|_{L^2} \leq \mathcal{F}(\|c\|_{L^\infty})\|\nabla c\|_{L^\infty}\|c\|_{H^2}\|c\|_{H^{s+2}}.$$

Thus we obtain that for $s \in (0, 1]$,

$$\|\nabla p\|_{H^s} \leq \mathcal{F}(\|c\|_{L^\infty})\big(1 + \|\nabla c\|_{L^\infty}\big)\big(1 + \|c\|_{H^2}\big)\|c\|_{H^{s+2}}, \tag{2.4}$$

For general s, we will prove it by the induction argument. Let us assume that for $s \in (\frac{k-1}{2}, \frac{k}{2}]$, we have

$$\|\nabla p\|_{H^s} \leq \mathcal{F}(\|c\|_{L^\infty})\big(1 + \|\nabla c\|_{L^\infty}\big)\big(1 + \|c\|_{H^2}\big)^k\|c\|_{H^{s+2}}.$$

Note that (2.4) means that the cases of $k = 1, 2$ hold. Now let us assume $s \in (\frac{k}{2}, \frac{k+1}{2}]$. We infer from Lemma 6.4 and Lemma 6.3 that

$$\|B\|_{L^2} \leq \mathcal{F}(\|c\|_{L^\infty})\big(\|c\|_{H^{s+2}}\|\nabla p\|_{L^2} + \|c\|_{H^2}\|\nabla p\|_{H^{s-\frac{1}{2}}}\big).$$

Then from (2.3) and the induction assumption, it follows that

$$\|B\|_{L^2} \leq \mathcal{F}(\|c\|_{L^\infty})\big(1 + \|\nabla c\|_{L^\infty}\big)\big(1 + \|c\|_{H^2}\big)^{k+1}\|c\|_{H^{s+2}}.$$

Thus for $s \in (\frac{k}{2}, \frac{k+1}{2}]$, we have

$$\|\nabla p\|_{H^s} \leq \mathcal{F}(\|c\|_{L^\infty})\big(1 + \|\nabla c\|_{L^\infty}\big)\big(1 + \|c\|_{H^2}\big)^{k+1}\|c\|_{H^{s+2}}.$$

This completes the proof of Lemma 2.1. ■

3 Local well-posedness

In this section we prove the local well-posedness of the Hele-Shaw-Cahn-Hilliard system. The procedure is mostly standard except the pressure estimate.

Theorem 3.1 *Let $c_0(x) \in H^s(\mathbf{T}^d)$ for $s > \frac{d}{2} + 1$. Then there exists $T > 0$ such that the system (1.2) has a unique solution (c, u) in $[0, T]$ with*

$$c \in C([0,T]; H^s(\mathbf{T}^d)) \cap L^2(0,T; H^{s+2}(\mathbf{T}^d)), \quad u \in C([0,T]; H^{s-2}(\mathbf{T}^d)) \cap L^2(0,T; H^s(\mathbf{T}^d));$$

and satisfying the following energy estimate

$$\|c(t)\|_{H^s}^2 + \int_0^t \|c(\tau)\|_{H^{s+2}}^2 d\tau \le \|c_0\|_{H^s} \exp\big(\int_0^t G(\tau) d\tau\big). \tag{3.1}$$

for $t \in [0, T]$, where

$$G(t) = \mathcal{F}(\|c\|_{L^\infty})\big(1 + \|\nabla c\|_{L^\infty}\big)^2 \big(\|\nabla c\|_{L^\infty} + \|c\|_{H^3}^{\frac{d-2}{2}}\big)^2 \big(1 + \|c\|_{H^2}\big)^{2([2s]+1)}.$$

Proof. We will use the energy method to prove Theorem 3.1.

Step 1. Construction of an approximate solution sequence.

The construction of the approximate solutions is based on Galerkin method. Let us define the operator P_n by

$$P_n f(x) = \sum_{|k| \le n} f_k e^{2\pi i k \cdot x}, \quad f_k = \int_{\mathbf{T}^d} f(x) e^{-2\pi i k \cdot x} dx.$$

Then we consider the following approximate system of (1.2):

$$\begin{cases} \nabla \cdot u_n = 0, \\ u_n = -\frac{1}{12\eta(P_n c_n)}\big(\nabla p_n - \frac{1}{\mathbf{M}}\mu(P_n c_n)\nabla P_n c_n\big), \\ \partial_t c_n + P_n(u_n \cdot \nabla P_n c_n) = \frac{1}{\mathbf{Pe}} \Delta P_n \mu(P_n c_n), \\ c_n(0, x) = P_n c_0(x). \end{cases} \tag{3.2}$$

It is easy to see that

$$\|\Delta P_n \mu(P_n c_n^1) - \Delta P_n \mu(P_n c_n^2)\|_{L^2} \le C(n, \|c_n^1\|_{L^2}, \|c_n^2\|_{L^2}) \|c_n^1 - c_n^2\|_{L^2}.$$

Taking the divergence to the second equation in (3.2) gives

$$\operatorname{div}\Big(\frac{1}{12\eta(P_n c_n)} \nabla p_n\Big) = \frac{1}{\mathbf{M}} \operatorname{div}\Big(\frac{1}{12\eta(P_n c_n)} \mu(P_n c_n) \nabla P_n c_n\Big).$$

Thanks to (1.4), straightforward energy estimate yields that

$$\|\nabla p_n\|_{L^2} \le C(n, \|c_n\|_{L^2}) \|c_n\|_{L^2},$$

thus we infer from the second equation of (3.2) that

$$\|u_n\|_{L^2} \le C(n, \|c_n\|_{L^2}) \|c_n\|_{L^2}.$$

Therefore, we have

$$\|P_n(u_n^1 \cdot \nabla P_n c_n^1) - P_n(u_n^2 \cdot \nabla P_n c_n^2)\|_{L^2} \leq C(n, \|c_n^1\|_{L^2}, \|c_n^2\|_{L^2})\|c_n^1 - c_n^2\|_{L^2}.$$

Thus, the Cauchy-Lipschtiz theorem ensures that there exists $T_n > 0$ such that the approximate system (3.2) has a unique solution $c_n \in C([0, T_n]; L^2(\mathbf{T}^d))$. Note that $P_n^2 = P_n$, $P_n c_n$ is also a solution of (3.2). So the uniqueness implies that $P_n c_n = c_n$. Thus, the approximate system (3.2) reduces to

$$\begin{cases} \nabla \cdot u_n = 0, \\ u_n = -\frac{1}{12\eta(c_n)}\big(\nabla p_n - \frac{1}{\mathbf{M}}\mu(c_n)\nabla c_n\big), \\ \partial_t c_n + P_n(u_n \cdot \nabla c_n) = \frac{1}{\mathbf{Pe}}\Delta P_n \mu(c_n), \\ c_n(0, x) = P_n c_0(x). \end{cases} \tag{3.3}$$

In what follows, we denote T_n^* by the maximal existence time of the solution c_n. Due to $P_n c_n = c_n$, the solution c_n is in fact smooth.

Step 2. Energy estimates.

Although the HSCH model (1.2) has a natural energy (which is somewhat equivalent to H^1 estimate, see [21, 30] and section 4 below), it is not sufficient for the strong solution. Therefore we have to derive estimates in Sobolev spaces with higher derivatives.

For this purpose we take the $H^s(\mathbf{T}^d)$ inner product of the third equation (3.3) with c_n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|c_n\|_{H^s}^2 - \frac{1}{\mathrm{Pe}}\big(\Delta P_n \mu(c_n), c_n\big)_{H^s} = -\big(u_n \cdot \nabla c_n, c_n\big)_{H^s}. \tag{3.4}$$

Due to (1.3), we see that

$$-\big(\Delta P_n \mu(c_n), c_n\big)_{H^s} = \mathrm{C}\|\Delta c_n\|_{H^s}^2 - \big(\Delta f_0'(c_n), c_n\big)_{H^s}.$$

We deduce, thanks to Lemma 6.2 that

$$\big|\big(\Delta f_0'(c_n), c_n\big)_{H^s}\big| \leq \|f_0'(c_n)\|_{H^s}\|\Delta c_n\|_{H^s} \leq C\big(1 + \|c_n\|_{L^\infty}^2\big)\|c_n\|_{H^s}\|\Delta c_n\|_{H^s}. \tag{3.5}$$

and by Lemma 6.2 with $\sigma = 1$,

$$\begin{aligned} \big|\big(u_n \cdot \nabla c_n, c_n\big)_{H^s}\big| &\leq \|u_n \cdot \nabla c_n\|_{H^s}\|c_n\|_{H^s} \\ &\leq C\big(\|u_n\|_{H^s}\|\nabla c_n\|_{L^\infty} + \|u_n\|_{H^{\frac{d}{2}-1}}\|\nabla c_n\|_{H^{s+1}}\big)\|c_n\|_{H^s}. \end{aligned} \tag{3.6}$$

Thanks to (3.3), we find that

$$\|u_n\|_{H^s} \leq C\big(\|\frac{1}{\eta(c_n)}\nabla p\|_{H^s} + \|\frac{1}{\eta(c_n)}\mu(c_n)\nabla c_n\|_{H^s}\big). \tag{3.7}$$

By Lemma 6.2, Lemma 6.3 and Proposition 2.1, the first term on the right hand side of (3.7) is bounded by

$$\begin{aligned} &\mathcal{F}(\|c_n\|_{L^\infty})\big(\|c_n\|_{H^{s+\frac{d}{2}}}\|\nabla p\|_{L^2} + \|\nabla p\|_{H^s}\big) \\ &\leq \mathcal{F}(\|c_n\|_{L^\infty})\big(1 + \|\nabla c_n\|_{L^\infty}\big)\big(1 + \|c_n\|_{H^2}\big)^{[2s]+1}\|c_n\|_{H^{s+2}}, \end{aligned}$$

and the second term is bounded by

$$\mathcal{F}(\|c_n\|_{L^\infty})\big(1+\|\nabla c_n\|_{L^\infty}\big)\|c_n\|_{H^{s+2}}.$$

Thus we obtain

$$\|u_n\|_{H^s}\le \mathcal{F}(\|c_n\|_{L^\infty})\big(1+\|\nabla c_n\|_{L^\infty}\big)\big(1+\|c_n\|_{H^2}\big)^{[2s]+1}\|c_n\|_{H^{s+2}},$$

and similarly,

$$\|u_n\|_{H^{\frac{d}{2}-1}}\le \mathcal{F}(\|c_n\|_{L^\infty})\big(1+\|\nabla c_n\|_{L^\infty}\big)\big(1+\|c_n\|_{H^2}\big)^{d-1}\|c_n\|_{H^{\frac{d}{2}+1}},$$

from which and (3.6), we infer that

$$\begin{aligned}\big|\big(u_n\cdot\nabla c_n,c_n\big)_{H^s}\big| \;\le\;& \mathcal{F}(\|c_n\|_{L^\infty})\big(1+\|\nabla c_n\|_{L^\infty}\big)\\ &\times\big(\|\nabla c_n\|_{L^\infty}+\|c_n\|_{H^3}^{\frac{d-2}{2}}\big)\big(1+\|c_n\|_{H^2}\big)^{[2s]+1}\|c_n\|_{H^{s+2}}\|c_n\|_{H^s}. \quad (3.8)\end{aligned}$$

Here we used the following interpolation inequality:

$$\|c_n\|_{H^{\frac{d}{2}+1}}\le \|c_n\|_{H^2}^{2-\frac{d}{2}}\|c_n\|_{H^3}^{\frac{d}{2}-1}.$$

Plugging (3.5) and (3.8) into (3.4) yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|c_n\|_{H^s}^2+\frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta c_n\|_{H^s}^2\\ &\le \mathcal{F}(\|c_n\|_{L^\infty})\big(1+\|\nabla c_n\|_{L^\infty}\big)\big(\|\nabla c_n\|_{L^\infty}+\|c_n\|_{H^3}^{\frac{d-2}{2}}\big)\big(1+\|c_n\|_{H^2}\big)^{[2s]+1}\|c_n\|_{H^{s+2}}\|c_n\|_{H^s},\end{aligned}$$

which along with Young's inequality implies that

$$\begin{aligned}&\frac{d}{dt}\|c_n\|_{H^s}^2+\|c_n\|_{H^{s+2}}^2\\ &\le \mathcal{F}(\|c_n\|_{L^\infty})\big(1+\|\nabla c_n\|_{L^\infty}\big)^2\big(\|\nabla c_n\|_{L^\infty}+\|c_n\|_{H^3}^{\frac{d-2}{2}}\big)^2\big(1+\|c_n\|_{H^2}\big)^{2([2s]+1)}\|c_n\|_{H^s}^2.\end{aligned}$$

Then Gronwall's inequality applied gives

$$E_n^s(t)\stackrel{\text{def}}{=}\|c_n(t)\|_{H^s}^2+\int_0^t\|c_n(\tau)\|_{H^{s+2}}^2d\tau\le\|c_0\|_{H^s}\exp\big(\int_0^t G_n(\tau)d\tau\big) \quad (3.9)$$

for $t\in[0,T_n^*)$, where

$$G_n(t)=\mathcal{F}(\|c_n\|_{L^\infty})\big(1+\|\nabla c_n\|_{L^\infty}\big)^2\big(\|\nabla c_n\|_{L^\infty}+\|c_n\|_{H^3}^{\frac{d-2}{2}}\big)^2\big(1+\|c_n\|_{H^2}\big)^{2([2s]+1)}.$$

Step 3. Uniform estimates and existence of the solution.
Let us define

$$\widetilde{T}_n^*\stackrel{\text{def}}{=}\sup\big\{t\in[0,T_n^*):E_n^s(\tau)\le 2\|c_0\|_{H^s}^2\text{ for }\tau\in[0,t]\big\}.$$

From (3.9) and Sobolev embedding, we find that

$$\begin{aligned} E_s^n(t) &\leq \|c_0\|_{H^s} \exp\big(\mathcal{A}(\|c_0\|_{H^s}) \int_0^t (1+\|c(\tau)\|_{H^3}^{d-2}) d\tau\big) \\ &\leq \|c_0\|_{H^s} \exp\big(\mathcal{A}(\|c_0\|_{H^s})(t+t^{\frac{1}{2}})\big), \quad t \in [0, \widetilde{T}_n^*). \end{aligned}$$

Here $\mathcal{A}(\cdot)$ is some increasing function. Take T be small enough such that

$$\exp\big(\mathcal{A}(\|c_0\|_{H^s})(T+T^{\frac{1}{2}})\big) \leq \frac{3}{2}.$$

Now we can conclude that $\widetilde{T}_n^* \geq T$. Otherwise, we have

$$E_s^n(t) \leq \frac{3}{2}\|c_0\|_{H^s}^2 \quad \text{for} \quad t \in [0, \widetilde{T}_n^*],$$

which contradicts with the definition of $\widetilde{T}_n^*$. Thus the approximate solution (c_n, u_n) exists on $[0,T]$ and satisfies the following uniform estimate

$$\|c_n(t)\|_{H^s}^2 + \int_0^t \|c_n(\tau)\|_{H^{s+2}}^2 d\tau \leq 2\|c_0\|_{H^s} \tag{3.10}$$

for $t \in [0,T]$. On the other hand, it is easy to verify from the third equation of (3.3) that $\partial_t c_n$ is uniformly bounded in $L^2(0,T;H^{s-2}(\mathbf{T}^d))$. Thus, Lions-Aubin's compactness theorem ensures that there exist a subsequence $(c_{n_k}, u_{n_k})_k$ of $(c_n, u_n)_n$ and a function $c \in L^\infty(0,T;H^s(\mathbf{T}^d)) \cap L^2(0,T;H^{s+2}(\mathbf{T}^d))$ and $u \in L^\infty(0,T;H^{s-2}(\mathbf{T}^d)) \cap L^2(0,T;H^s(\mathbf{T}^d))$ such that

$$\begin{aligned} c_{n_k} &\longrightarrow c, \qquad \text{in} \quad L^2(0,T;H^{s'+2}(\mathbf{T}^d)), \\ u_{n_k} &\longrightarrow u, \qquad \text{in} \quad L^2(0,T;H^{s'}(\mathbf{T}^d)), \end{aligned}$$

as $k \to +\infty$, for any $s' < s$. Then passing to limit in (3.3), it is easy to see that (c,u) satisfies (1.2) in the weak sense and (c,u) satisfies (3.1).

Step 4. Continuity in time of the solution.

Revisiting the proof of (3.9), we can in fact obtain better estimate for c_n (thus for c):

$$\|c\|_{\widetilde{L}^\infty(0,T;H^s(\mathbf{T}^d))}^2 \stackrel{\text{def}}{=} \sum_{j\geq -1} 2^{2js}\|\Delta_j c\|_{L^\infty(0,T;L^2)}^2 \leq C,$$

which will imply $c \in C([0,T];H^s(\mathbf{T}^d))$. In fact, for any $\varepsilon > 0$, take N big enough such that

$$\sum_{j>N} 2^{2js}\|\Delta_j c\|_{L^\infty(0,T;L^2)}^2 \leq \frac{\varepsilon}{4}.$$

For any $t \in (0,T)$ and δ such that $t+\delta \in [0,T]$, we have

$$\begin{aligned} \|c(t+\delta)-c(t)\|_{H^s}^2 &\leq \sum_{j=-1}^N 2^{2js}\|\Delta_j c(t+\delta) - \Delta_j c(t)\|_{L^2}^2 + \frac{\varepsilon}{2} \\ &\leq \sum_{j=-1}^N 2^{2js}|\delta|\|\partial_t c\|_{L^2(0,T;L^2)}^2 + \frac{\varepsilon}{2} \\ &\leq 2N2^{2N}\|\partial_t c\|_{L^2(0,T;L^2)}^2|\delta| + \frac{\varepsilon}{2}. \end{aligned}$$

Thus for $|\delta|$ small enough, we have

$$\|c(t+\delta)-c(t)\|^2 \le \varepsilon.$$

That is, $c(t)$ is continuous in $H^s(\mathbf{T}^d)$ at the time t, thus so does u.

Step 5. Uniqueness of the solution

Assume that (c_1, u_1) and (c_2, u_2) are two solutions of (1.2) with the same initial data. We introduce the difference of two solutions:

$$\delta_c = c_1 - c_2, \quad \delta_u = u_1 - u_2.$$

Then (δ_c, δ_u) satisfies

$$\begin{cases} \partial_t \delta_c + u_1 \cdot \nabla \delta_c + \delta_u \cdot \nabla c_2 = \frac{1}{\mathbf{Pe}} \Delta\big(\mu(c_1) - \mu(c_2)\big), \\ \delta_u = \frac{\eta(c_1)-\eta(c_2)}{12\eta(c_1)\eta(c_2)}\big(\nabla p_1 - \frac{1}{\mathbf{M}}\mu(c_1)\nabla c_1\big) - \frac{1}{12\eta(c_2)}\big(\nabla(p_1-p_2) - \frac{1}{\mathbf{M}}(\mu(c_1)\nabla c_1 - \mu(c_2)\nabla c_2)\big), \\ \delta_c(0) = 0. \end{cases}$$

Making $L^2(\mathbf{T}^d)$ energy estimate yields that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\delta_c\|_{L^2}^2 + \frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta\delta_c\|_{L^2}^2 &\le \frac{1}{\mathbf{Pe}}\big(\Delta(f_0'(c_1) - f_0'(c_2), \delta_c)\big)_{L^2} - \big(\delta_u \cdot \nabla c_2, \delta_c\big)_{L^2} \\ &\le C\big(\|\Delta\delta_c\|_{L^2} + \|\delta_u\|_{L^2}\big)\|\delta_c\|_{L^2}. \end{aligned}$$

On the other hand, we can deduce from the equation of δ_u that

$$\begin{aligned} \|\delta_u\|_{L^2} &\le C\big(\|\delta_c\|_{L^2} + \|\nabla(p_1-p_2)\|_{L^2} + \|\Delta\delta_c\|_{L^2}\big) \\ &\le C\big(\|\delta_c\|_{L^2} + \|\Delta\delta_c\|_{L^2}\big). \end{aligned}$$

Thus we obtain

$$\frac{d}{dt}\|\delta_c\|_{L^2}^2 \le C\|\delta_c\|_{L^2}^2, \quad \|\delta_c(0)\| = 0,$$

which along with Gronwall's inequality implies $\delta_c = 0$, and the uniqueness follows. ■

4 Blow-up criterion and global existence in 2D

In this section we prove a Beale-Kato-Majda type blow-up criterion [25] for the Hele-Shaw-Cahn-Hilliard system. As an application, we obtain the global well-posedness in 2D.

Theorem 4.1 *Let $c_0(x) \in H^s(\mathbf{T}^d)$ for $s > \frac{d}{2} + 1$, and (c, u) be a solution of (1.2) stated in Theorem 3.1. Let T^* be the maximal existence time of the solution. If $T^* < +\infty$, then*

$$\int_0^{T^*} \|\nabla c(t)\|_{L^\infty}^4 dt = +\infty. \tag{4.11}$$

In particular, this implies $T^* = +\infty$ for $d = 2$. That is, the system (1.2) is globally well-posed in 2D.

Proof. First of all, we derive the basic energy law of the system. Multiplying by μ on both sides of the third equation of (1.2), we get by integration by parts that

$$\int_{\mathbf{T}^d} c_t \mu dx + \int_{\mathbf{T}^d} u \cdot \nabla c \mu dx = -\frac{1}{\mathbf{Pe}} \int_{\mathbf{T}^d} |\nabla \mu|^2 dx.$$

Due to the definition of μ, we have

$$\int_{\mathbf{T}^d} c_t \mu dx = \frac{d}{dt}\Big(\int_{\mathbf{T}^d} f_0(c) dx + \frac{\mathbf{C}}{2} \int_{\mathbf{T}^d} |\nabla c|^2 dx \Big),$$

and due to $\nabla \cdot u = 0$,

$$\int_{\mathbf{T}^d} u \cdot \nabla c \mu dx = -\mathbf{M} \int_{\mathbf{T}^d} u \cdot \big(\nabla p - \frac{1}{\mathbf{M}} \mu \nabla c\big) dx = 12\mathbf{M} \int_{\mathbf{T}^d} \eta(c) |u|^2 dx.$$

Thus we obtain the following classical energy equality [21]

$$\frac{d}{dt}\Big(\int_{\mathbf{T}^d} f_0(c) dx + \frac{\mathbf{C}}{2} \int_{\mathbf{T}^d} |\nabla c|^2 dx \Big) + \frac{1}{\mathbf{Pe}} \int_{\mathbf{T}^d} |\nabla \mu|^2 dx + 12\mathbf{M} \int_{\mathbf{T}^d} \eta(c) |u|^2 dx = 0.$$

That is,

$$E(t) + \frac{1}{\mathbf{Pe}} \int_0^t \int_{\mathbf{T}^d} |\nabla \mu(\tau)|^2 dx d\tau + 12\mathbf{M} \int_0^t \int_{\mathbf{T}^d} \eta(c) |u(\tau)|^2 dx d\tau = E(0), \tag{4.12}$$

where

$$E(t) \stackrel{\text{def}}{=} \int_{\mathbf{T}^d} f_0(c(t,x)) dx + \frac{\mathbf{C}}{2} \int_{\mathbf{T}^d} |\nabla c(t,x)|^2 dx.$$

From the energy equality (4.12), it follows that

$$\|c(t)\|_{H^1}^2 + \frac{1}{\mathbf{Pe}} \int_0^t \|\nabla \mu\|_{L^2}^2 d\tau \le E(0).$$

On the other hand, we have

$$\|\nabla \Delta c\|_{L^2} \le C\big(\|\nabla \mu\|_{L^2} + \|\nabla c\|_{L^2} + \|c^2 \nabla c\|_{L^2} \big).$$

and by Sobolev inequality,

$$\begin{aligned} \|c^2 \nabla c\|_{L^2} &\le C\|c\|_{L^6}^2 \|\nabla c\|_{L^6} \le C\|c\|_{H^1}^2 \|c\|_{H^2} \\ &\le C\|c\|_{H^1}^{\frac{5}{2}} \|c\|_{H^3}^{\frac{1}{2}} \le C\|c\|_{H^1}^5 + \frac{1}{2}\|c\|_{H^3}, \end{aligned}$$

which implies that

$$\|c\|_{H^3} \le C\big(\|\nabla \mu\|_{L^2} + \|c\|_{H^1} + \|c\|_{H^1}^5 \big).$$

Therefore we conclude that

$$\|c\|_{L^\infty(0,T;H^1)} + \|c\|_{L^2(0,T;H^3)} \le C\big(T, \|c_0\|_{H^1}\big). \tag{4.13}$$

Next, we derive H^2 energy estimate of the solution. We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta c\|_{L^2}^2+\frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta^2 c\|_{L^2}^2 &= -\big(u\cdot\nabla c,\Delta^2 c\big)_{L^2}+\frac{1}{\mathbf{Pe}}\big(\Delta f_0'(c),\Delta^2 c\big)\\ &\leq \|u\|_{L^2}\|\nabla c\|_{L^\infty}\|\Delta^2 c\|_{L^2}+\frac{1}{\mathbf{Pe}}\|\Delta f_0'(c)\|_{L^2}\|\Delta^2 c\|_{L^2}.\end{aligned}\tag{4.14}$$

It is easy to verify that

$$\begin{aligned}\|u\|_{L^2} &\leq C\big(\|\nabla p\|_{L^2}+\|\mu(c)\nabla c\|_{L^2}\big)\\ &\leq C\big(\|\nabla c\|_{L^\infty}\|\Delta c\|_{L^2}+(\|c\|_{L^3}+\|c\|_{L^9}^3)\|\nabla c\|_{L^6}\big)\\ &\leq C\big(\|\nabla c\|_{L^\infty}+\|c\|_{L^3}+\|c\|_{L^9}^3\big)\|c\|_{H^2},\end{aligned}$$

and

$$\|\Delta f_0'(c)\|_{L^2}\leq C\big(1+\|c\|_{L^\infty}^2\big)\|c\|_{H^2}.$$

Plugging them into (4.14) yields that

$$\frac{d}{dt}\|\Delta c\|_{L^2}^2+\|\Delta^2 c\|_{L^2}^2\leq C\big(1+\|\nabla c\|_{L^\infty}^4+\|c\|_{L^\infty}^4+\|c\|_{L^3}^4+\|c\|_{L^9}^{12}\big)\|c\|_{H^2}^2,$$

which along with Gronwall's inequality leads to

$$\|c\|_{H^2}\leq\|c_0\|_{H^2}\exp\big(C\int_0^t H(\tau)d\tau\big),\tag{4.15}$$

where $H(t)=1+\|\nabla c\|_{L^\infty}^4+\|c\|_{L^\infty}^4+\|c\|_{L^3}^4+\|c\|_{L^9}^{12}$.

Now we are in position to prove the blow-up criterion. We will prove it by way of contradiction argument. Assume that $T^*<+\infty$ and

$$\int_0^{T^*}\|\nabla c(t)\|_{L^\infty}^4 dt<+\infty,$$

which together with (4.13) and Sobolev's inequality implies that

$$\int_0^{T^*}H(\tau)d\tau<+\infty,$$

for example,

$$\int_0^{T^*}\|c(t)\|_{L^9}^{12}dt\leq C\int_0^{T^*}\|c(t)\|_{H^1}^{11}\|c(t)\|_{H^3}dt<+\infty.$$

Then we infer from (4.15) that

$$\|c\|_{L^\infty(0,T^*;H^2)}<+\infty,$$

which implies that

$$\int_0^{T^*}G(t)dt<+\infty,\quad G(t)\quad\text{be as in Theorem 3.1.}$$

Then the energy inequality (3.1) ensures that

$$\sup_{t\in[0,T^*]}\|c(t)\|_{H^s}^2+\int_0^{T^*}\|c(\tau)\|_{H^{s+2}}^2d\tau<+\infty,$$

which means that the solution can be continued after $t=T^*$, and thus contradicts with the definition of T^*.

As an application of blow-up criterion, we can deduce the global existence in 2D. Indeed, in two dimensional case, we get by Gagliardo-Nirenberg inequality and (4.13) that

$$\int_0^{T^*}\|\nabla c(t)\|_{L^\infty}^4dt\le C\int_0^{T^*}\|c(t)\|_{H^1}^2\|c(t)\|_{H^3}^2dt<+\infty,$$

which implies $T^*=+\infty$ by the blow-up criterion. ■

5 A refined blow-up criterion in 3D

We first turn to a simple model relating to the Hele-Shaw-Cahn-Hilliard system:

$$\begin{cases} u=-\nabla p+\Delta c\nabla c, \quad \nabla\cdot u=0,\\ c_t+u\cdot\nabla c+\Delta^2c=0.\end{cases}\tag{5.16}$$

For this system, we still have the energy equality:

$$\|\nabla c(t)\|_{L^2}^2+2\int_0^t\|\nabla\Delta c(\tau)\|_{L^2}^2+\|u(\tau)\|_{L^2}^2d\tau=\|\nabla c_0\|_{L^2}.$$

Moreover, if c is a solution of (5.16), then $c_\lambda(t,x)\stackrel{\text{def}}{=}c(\lambda^4t,\lambda x)$ is also a solution. It is easy to see that

$$\|\nabla c_\lambda(t,x)\|_{L^2}=\lambda^{\frac{d}{2}-1}\|\nabla c(\lambda^4t,x)\|_{L^2},\quad \int_0^\infty\|\nabla\Delta c_\lambda(\tau)\|_{L^2}^2d\tau=\lambda^{2-d}\int_0^\infty\|\nabla\Delta c(\tau)\|_{L^2}^2d\tau.$$

Thus, the energy is scaling invariance for $d=2$. From this view of point, the 2D system is critical and the 3D system is supercritical like the 3D Navier-Stokes equations. Due to the bi-Laplacian Δ^2, there is no maximum principle for this system, which is the main obstacle to obtain the global existence in 3D case. For the 2D critical QG equation

$$\theta_t+(-\Delta)^{\frac12}\theta+u\cdot\nabla\theta=0,\quad u=\big(-(-\Delta)^{-\frac12}\partial_{x_2}\theta,(-\Delta)^{-\frac12}\partial_{x_1}\theta\big),$$

Caffarelli and Vasseur [8] proved the global regularity of weak solution. The key step of their proof is to prove the Hölder continuity of the solution by using the DeGiorgi method. Note that the QG equation has maximum principle. For the 3D Hele-Shaw-Cahn-Hilliard system, we also show that the Hölder continuity of the solution will control the blow-up of the solution.

Theorem 5.1 *Let $\alpha\in(0,1)$ and $c_0(x)\in H^s(\mathbf{T}^3)$ for $s\ge3$. Assume that (c,u) be a solution of (1.2) stated in Theorem 3.1. Let T^* be the maximal existence time of the solution. If $T^*<+\infty$, then*

$$\int_0^{T^*}\|c(t)\|_{C^\alpha}^{\frac{8}{\alpha}}dt=+\infty.$$

Proof. We will prove it by contradiction argument. Assume that $T^* < +\infty$ and

$$\int_0^{T^*} \|c(t)\|_{C^\alpha}^{\frac{8}{\alpha}} dt < +\infty. \tag{5.17}$$

Taking Δ_j to the third equation of (1.2) to obtain

$$\partial_t \Delta_j c + \frac{\mathbf{C}}{\mathbf{Pe}} \Delta^2 \Delta_j c = -\Delta_j (u \cdot \nabla c) + \frac{1}{\mathbf{Pe}} \Delta \Delta_j f_0'(c).$$

Making $L^2(\mathbf{T}^3)$ energy estimate, we get by Lemma 6.1 that for $j \geq 0$,

$$\frac{d}{dt} \|\Delta_j c\|_{L^2}^2 + c 2^{4j} \|\Delta_j c\|_{L^2}^2 \leq C\big(\|\Delta_j (u \cdot \nabla c)\|_{L^2} + \|\Delta f_0'(c)\|_{L^2}\big) \|\Delta_j c\|_{L^2}.$$

Dividing the above inequality by $\|\Delta_j c\|_{L^2}$ gives

$$\frac{d}{dt} \|\Delta_j c\|_{L^2} + c 2^{4j} \|\Delta_j c\|_{L^2} \leq C\big(\|\Delta_j (u \cdot \nabla c)\|_{L^2} + \|\Delta f_0'(c)\|_{L^2}\big),$$

which implies that

$$\|\Delta_j c(t)\|_{L^2} \leq \|\Delta_j c_0\|_{L^2} + C \int_0^t e^{-c 2^{4j}(t-\tau)} \big(\|\Delta_j (u \cdot \nabla c)(\tau)\|_{L^2} + \|\Delta f_0'(c(\tau))\|_{L^2}\big) d\tau. \tag{5.18}$$

We denote

$$\|c\|_{B^s_{2,\infty}} \stackrel{\text{def}}{=} \sup_{j \geq -1} 2^{js} \|\Delta_j c\|_{L^2}.$$

Using the definition of Sobolev space, it is easy to find that

$$\|c\|_{H^{s-\epsilon}}^2 \leq \sum_{j \geq -1} 2^{-2\varepsilon j} \|c\|_{B^s_{2,\infty}}^2 \leq C \|c\|_{B^s_{2,\infty}}^2, \quad \forall \varepsilon > 0.$$

It follows from (5.18) that

$$\begin{aligned} \|c(t)\|_{B^3_{2,\infty}} &\leq \|c(t)\|_{L^2} + \|c_0\|_{H^3} \\ &+ C \sup_{j \geq 0} 2^{3j} \int_0^t e^{-c 2^{4j}(t-\tau)} \big(\|\Delta_j (u \cdot \nabla c)(\tau)\|_{L^2} + \|\Delta f_0'(c(\tau))\|_{L^2}\big) d\tau. \end{aligned} \tag{5.19}$$

Now we claim that

$$\|\Delta_j (u \cdot \nabla c)\|_{L^2} \leq C 2^{j(1-\alpha)} \|u\|_{L^2} \|c\|_{C^\alpha}. \tag{5.20}$$

Now we have

$$\begin{aligned} \|u\|_{L^2} &\leq C \|\mu(c) \nabla c\|_{L^2} \leq C \|c\|_{H^{3-\alpha}} \|c\|_{C^\alpha} + C\big(\|c\|_{L^3} + \|c\|_{L^6}^2 \|c\|_{L^\infty}\big) \|\nabla c\|_{L^6} \\ &\leq C\big(1 + \|c\|_{H^1} + \|c\|_{H^1}^2\big) \|c\|_{C^\alpha} \|c\|_{B^3_{2,\infty}}. \end{aligned}$$

Here we used the product estimate

$$\|\Delta c \nabla c\|_{L^2} \leq C \|c\|_{H^{3-\alpha}} \|c\|_{C^\alpha} \leq C \|c\|_{B^3_{2,\infty}} \|c\|_{C^\alpha},$$

which can be proved as in Lemma 6.2. And similarly we have

$$\|\Delta f_0'(c)\|_{L^2} \le C\big(1+\|c\|_{C^\alpha}^2\big)\|c\|_{H^2}.$$

Plugging the above estimates into (5.19) yields that

$$\begin{aligned}\|c(t)\|_{B^3_{2,\infty}} &\le \|c(t)\|_{L^2} + \|c_0\|_{H^3}\\ &\quad + C\sup_{j\ge 0} 2^{j(4-\alpha)}\int_0^t e^{-c2^{4j}(t-\tau)}\big(1+\|c\|_{H^1}+\|c\|_{H^1}^2\big)\big(1+\|c\|_{C^\alpha}^2\big)\|c\|_{B^3_{2,\infty}}d\tau,\end{aligned}$$

which along with Hölder inequality gives

$$\begin{aligned}\|c(t)\|_{L^\infty(0,t;B^3_{2,\infty})} &\le \|c(t)\|_{L^\infty(0,t;L^2)} + \|c_0\|_{H^3}\\ &\quad + \big(1+\|c\|_{L^\infty(0,t;H^1)}+\|c\|^2_{L^\infty(0,t;H^1)}\big)\big(t^{\frac{4}{\alpha}}+\|c\|^2_{L^{\frac{8}{\alpha}}(0,t;C^\alpha)}\big)\|c\|_{L^\infty(0,t;B^3_{2,\infty})}.\end{aligned}$$

The above argument is still valid on the interval $[T,T^*)$ for $T<T^*$. Thus we get by using (4.13) that

$$\begin{aligned}\|c(t)\|_{L^\infty(T,T^*;B^3_{2,\infty})} &\le \|c_0\|_{H^1} + \|c_0(T)\|_{H^3}\\ &\quad + C(\|c_0\|_{H^1})\big((T^*-T)^{\frac{4}{\alpha}}+\|c\|^2_{L^{\frac{8}{\alpha}}(T,T^*;C^\alpha)}\big)\|c\|_{L^\infty(T,T^*;B^3_{2,\infty})}.\end{aligned}$$

Due to (5.17), we can choose T such that

$$C(\|c_0\|_{H^1})\big((T^*-T)^{\frac{4}{\alpha}}+\|c\|^2_{L^{\frac{8}{\alpha}}(T,T^*;C^\alpha)}\big) \le \frac{1}{2},$$

Then we obtain

$$\|c(t)\|_{L^\infty(T,T^*;B^3_{2,\infty})} \le 2\big(\|c_0\|_{H^1}+\|c_0(T)\|_{H^3}\big),$$

which implies by $\|\nabla c\|_{L^\infty}\le C\|c\|_{B^3_{2,\infty}}$ that

$$\int_0^{T^*}\|\nabla c(t)\|^4_{L^\infty}dt < +\infty,$$

which is impossible by Theorem 4.1 if $T^*<+\infty$.

It remains to prove (5.20). As in proof of Lemma 6.2, we have

$$\begin{aligned}\Delta_j(u\cdot\nabla c) &= \Delta_j\sum_{|j-k|\le 4} S_{k-1}u\cdot\nabla\Delta_k c + \Delta_j\sum_{|j-k|\le 4}\Delta_k u\cdot\nabla S_{k-1}c\\ &\quad + \Delta_j\sum_{|k-k'|\le 1, k\ge j-3}\Delta_k u\cdot\nabla\Delta_{k'}c = A_1+A_2+A_3.\end{aligned}$$

We get by Lemma 6.1 that

$$\|A_1\|_{L^2}\le C\sum_{|j-k|\le 4}\|S_{k-1}u\|_{L^2}\|\nabla\Delta_k c\|_{L^\infty}\le C2^{j(1-\alpha)}\|u\|_{L^2}\|c\|_{C^\alpha},$$

and for A_2,

$$\begin{aligned}
\|A_2\|_{L^2} &\leq C \sum_{|j-k|\leq 4} \|\Delta_k u\|_{L^2} \|\nabla S_{k-1} c\|_{L^\infty} \\
&\leq C\|u\|_{L^2} \sum_{|j-k|\leq 4} \sum_{\ell \leq k-2} 2^\ell \|\Delta_\ell c\|_{L^\infty} \\
&\leq C\|u\|_{L^2} \|c\|_{C^\alpha} \sum_{|j-k|\leq 4} \sum_{\ell \leq k-2} 2^{\ell(1-\alpha)} \leq C 2^{j(1-\alpha)} \|u\|_{L^2} \|c\|_{C^\alpha},
\end{aligned}$$

and due to $\nabla \cdot u = 0$,

$$\begin{aligned}
\|A_3\|_{L^2} &\leq \|\Delta_j \sum_{|k-k'|\leq 1, k\geq j-3} \nabla \cdot (\Delta_k u \Delta_{k'} c)\|_{L^2} \\
&\leq C 2^j \sum_{|k-k'|\leq 1, k\geq j-3} 2^{-k'\alpha} \|u\|_{L^2} 2^{k'\alpha} \|\Delta_{k'} c\|_{L^\infty} \\
&\leq C 2^{j(1-\alpha)} \|u\|_{L^2} \|c\|_{C^\alpha}.
\end{aligned}$$

Then the inequality (5.20) follows from the estimates of A_1, A_2 and A_3. The proof of Theorem 5.1 is completed. ■

6 Appendix

Let us first recall some basic facts about the Littlewood-Paley theory. Let φ, χ be two functions in $C^\infty(\mathbf{T}^d)$ such that $\text{supp}\widehat{\varphi} \subset \{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, $\text{supp}\widehat{\chi} \subset \{|\xi| \leq \frac{4}{3}\}$ and

$$\widehat{\chi}(\xi) + \sum_{j\geq 0} \widehat{\varphi}(2^{-j}\xi) = 1.$$

Then the Littlewood-Paley operators are defined by

$$\begin{aligned}
&\Delta_j f = \varphi_j * f = \int_{\mathbf{T}^d} \varphi_j(x-y) f(y) dy, \quad \varphi_j(x) = 2^{jd} \varphi(2^j x), \quad j \geq 0, \\
&S_j f = \chi_j * f = \sum_{k=-1}^{j-1} \Delta_k f, \quad \Delta_{-1} f = \chi * f.
\end{aligned}$$

Some classical spaces can be characterized in terms of Δ_j. Let $s \in \mathbf{R}$, the Sobolev space $H^s(\mathbf{T}^d)$ is defined by

$$H^s(\mathbf{T}^d) \stackrel{\text{def}}{=} \big\{u \in \mathcal{D}'(\mathbf{T}^d) : \|u\|_{H^s}^2 \stackrel{\text{def}}{=} \sum_{j\geq -1} 2^{2js} \|\Delta_j u\|_{L^2}^2 < \infty\big\}.$$

We denote by $(u, v)_{H^s}$ the inner product in $H^s(\mathbf{T}^d)$. And for $s \in (0, 1)$, the Hölder space $C^s(\mathbf{T}^d)$ is defined by

$$C^s(\mathbf{T}^d) \stackrel{\text{def}}{=} \big\{u \in \mathcal{D}'(\mathbf{T}^d) : \|u\|_{C^s} \stackrel{\text{def}}{=} \sup_{j\geq -1} 2^{js} \|\Delta_j u\|_{L^\infty}\big\}.$$

We refer to [29] for more details. Let us recall Bony's decomposition from [6]:

$$fg = T_f g + T_g f + R(f,g), \tag{6.1}$$

where

$$T_f g = \sum_{j\geq -1} S_{j-1} f \Delta_j g, \quad R(f,g) = \sum_{|j-j'|\leq 1} \Delta_j f \Delta_{j'} g.$$

We also denote $\widetilde{R}(f,g) = T_g f + R(f,g)$.

Lemma 6.1 *[9] Let $k \in \mathbf{N}, 1 \leq p \leq q \leq \infty$. Then there exists a positive constant C independent of j such that*

$$\begin{aligned}
&\|\partial^\alpha \Delta_j f\|_{L^q} + \|\partial^\alpha S_j f\|_{L^q} \leq C 2^{j|\alpha| + dj(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \\
&\|\Delta_j f\|_{L^p} \leq C 2^{-jk} \sup_{|\alpha|=k} \|\partial^\alpha \Delta_j f\|_{L^p}, \quad j \geq 0.
\end{aligned}$$

Lemma 6.2 *Let $s \geq 0$. Then there holds*

$$\|fg\|_{H^s} \leq C\big(\|f\|_{L^\infty}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{L^\infty}\big). \tag{6.2}$$

If $0 < \sigma \leq \frac{d}{2}$, then there holds

$$\|fg\|_{H^s} \leq C\big(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{H^{\frac{d}{2}-\sigma}}\|g\|_{H^{s+\sigma}}\big). \tag{6.3}$$

Proof. The inequality (6.2) is classical, see [20]. Here we only present the proof of (6.3). Using the Bony's decomposition (6.1) to write

$$\Delta_j(fg) = \Delta_j(T_f g) + \Delta_j(T_g f) + \Delta_j R(f,g).$$

Taking into considering the support of Fourier transform of the term $T_f g$, we have

$$\Delta_j(T_f g) = \sum_{|j'-j|\leq 4} \Delta_j(S_{j'-1} f \Delta_{j'} g).$$

Due to $0 < \sigma \leq \frac{d}{2}$, this gives by Lemma 6.1 that

$$\|S_j f\|_{L^\infty} \leq \begin{cases} C 2^{j\frac{d}{2}} \|f\|_{L^2}, & \text{if } \sigma = \frac{d}{2}, \\ C \displaystyle\sum_{k\leq j-1} 2^{k\frac{d}{2}} \|\Delta_k f\|_{L^2} \leq C 2^{j\sigma} \|f\|_{H^{\frac{d}{2}-\sigma}}, & \text{if } \sigma < \dfrac{d}{2}, \end{cases}$$

which implies that

$$\begin{aligned}
\|\Delta_j(T_f g)\|_{L^2} &\leq C \sum_{|j'-j|\leq 4} \|S_{j'-1} f\|_{L^\infty} \|\Delta_{j'} g\|_{L^2} \\
&\leq C \|f\|_{H^{\frac{d}{2}-\sigma}} \sum_{|j'-j|\leq 4} 2^{j'\sigma} \|\Delta_{j'} g\|_{L^2} \\
&\leq C 2^{-js} c_j \|f\|_{H^{\frac{d}{2}-\sigma}} \|g\|_{H^{s+\sigma}},
\end{aligned} \tag{6.4}$$

here and hereafter $\{c_j\}_{j\geq -1}$ denotes a sequence satisfying $\|\{c_j\}_{j\geq -1}\|_{\ell^2}\leq 1$.

Similarly, we have

$$\begin{aligned}\|\Delta_j(T_g f)\|_{L^2} &\leq C\sum_{|j'-j|\leq 4}\|S_{j'-1}g\|_{L^\infty}\|\Delta_{j'}f\|_{L^2}\\ &\leq C\sum_{|j'-j|\leq 4}\|g\|_{L^\infty}\|\Delta_{j'}f\|_{L^2}\\ &\leq C2^{-js}c_j\|g\|_{L^\infty}\|f\|_{H^s}. \end{aligned}\tag{6.5}$$

Noticing that, after taking into account the support of the Fourier transforms,

$$\Delta_j R(f,g)=\sum_{j',j''\geq j-3;|j'-j''|\leq 1}\Delta_j(\Delta_{j'}f\Delta_{j''}g),$$

it follows from Lemma 6.1 that

$$\begin{aligned}\|\Delta_j R(f,g)\|_{L^2} &\leq C\sum_{j',j''\geq j-3;|j'-j''|\leq 1}2^{j\frac{d}{2}}\|\Delta_{j'}f\|_{L^2}\|\Delta_{j''}g\|_{L^2}\\ &\leq C2^{-js}\sum_{j',j''\geq j-3;|j'-j''|\leq 1}2^{(j-j')(\frac{d}{2}+s)}2^{j'(\frac{d}{2}-\sigma)}\|\Delta_{j'}f\|_{L^2}2^{j''(s+\sigma)}\|\Delta_{j''}g\|_{L^2}\\ &\leq C2^{-js}c_j\|f\|_{H^{\frac{d}{2}-\sigma}}\|g\|_{H^{s+\sigma}}. \end{aligned}\tag{6.6}$$

Thanks to the definition of Sobolev space, (6.3) follows from (6.4)-(6.6). ■

Lemma 6.3 *[29] Let $s>0$. Assume that $F(\cdot)$ is a smooth function on $\mathbf{R}$ with $F(0)=0$. Then we have*

$$\|F(f)\|_{H^s}\leq C(1+\|f\|_{L^\infty})^{\lfloor s\rfloor+1}\|f\|_{H^s},$$

where the constant C depends on $\sup\limits_{k\leq\lfloor s\rfloor+2,|t|\leq\|f\|_{L^\infty}}\|F^{(k)}(t)\|_{L^\infty}$.

Lemma 6.4 *Let $s>0$. Then there holds*

$$\|\langle D\rangle^s(fg)-f\langle D\rangle^s g\|_{L^2}\leq C\big(\|f\|_{H^{s+2}}\|g\|_{L^2}+\|f\|_{H^2}\|g\|_{H^{s-\frac{1}{2}}}\big).$$

If $s\in(0,1]$, then we have

$$\|\langle D\rangle^s(fg)-f\langle D\rangle^s g\|_{L^2}\leq C\|f\|_{H^{s+2}}\|g\|_{L^2}.$$

Here the Fourier multiplier $\langle D\rangle^s$ is defined by

$$\langle D\rangle^s f(x)=\sum_{k\in\mathbf{Z}^d}(1+|k|^2)^{\frac{s}{2}}e^{2\pi ik\cdot x}\widehat{f}(k).$$

Proof. Using Bony's decomposition (6.1) to write

$$\begin{aligned}&\langle D\rangle^s(fg)=\langle D\rangle^s(T_f g)+\langle D\rangle^s T_g f+\langle D\rangle^s R(f,g),\\ &f\langle D\rangle^s g=T_f\langle D\rangle^s g+T_{\langle D\rangle^s g}f+R(f,\langle D\rangle^s g).\end{aligned}$$

Thus we have

$$\langle D\rangle^s(fg) - f\langle D\rangle^s g = \langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g + \pi(f,g),$$

where

$$\pi(f,g) = \langle D\rangle^s T_g f + \langle D\rangle^s R(f,g) - T_{\langle D\rangle^s g} f - R(f,\langle D\rangle^s g).$$

As in the proof of (6.3), we can deduce by Lemma 6.1 that

$$\|\pi(f,g)\|_{L^2} \le C\|f\|_{H^{s+2}}\|g\|_{L^2}.$$

We illustrate the process by working out the estimate on the first term. Thanks to Lemma 6.1, we have

$$\begin{aligned}
\|\langle D\rangle^s T_g f\|_{L^2}^2 &= \sum_{j\ge -1}\|\Delta_j\langle D\rangle^s T_g f\|_{L^2}^2 \le C\sum_{j\ge -1} 2^{2js}\|\Delta_j T_g f\|_{L^2}^2 \\
&\le C\sum_{|j-j'|\le 4} 2^{2js}\|S_{j'-1}g\Delta_{j'}f\|_{L^2}^2 \\
&\le C\sum_{|j-j'|\le 4} 2^{2js}\|S_{j'-1}g\|_{L^\infty}^2\|\Delta_{j'}f\|_{L^2}^2 \\
&\le C\sum_{|j-j'|\le 4} 2^{2j(s+\frac{d}{2})}\|g\|_{L^2}^2\|\Delta_{j'}f\|_{L^2}^2 \\
&\le C\|g\|_{L^2}^2\|f\|_{H^{s+\frac{d}{2}}}^2 \le C\|g\|_{L^2}^2\|f\|_{H^{s+2}}^2.
\end{aligned}$$

Let $m(\xi_1,\xi_2)$ be the symbol of the paraproduct operator $T_f g$. Then $\langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g$ has the symbol

$$m(\xi_1,\xi_2)\big(\langle\xi_1+\xi_2\rangle^s - \langle\xi_2\rangle^s\big),$$

which is supported in the region $|\xi_1+\xi_2| \sim |\xi_2|$. By the fundamental theorem of calculus we have

$$m(\xi_1,\xi_2)\big(\langle\xi_1+\xi_2\rangle^s - \langle\xi_2\rangle^s\big) = \int_0^1 \xi_1\cdot m(\xi_1,\xi_2)\nabla h^s(t\xi_1+\xi_2)dt, \quad h^s(\xi) = \langle\xi\rangle^s.$$

It is easy to verify that $\langle\xi_1\rangle^\theta m(\xi_1,\xi_2)\nabla h^s(t\xi_1+\xi_2)\langle\xi_2\rangle^{1-\theta-s}$ with $\theta\in[0,1]$ is a Coifman-Meyer paraproduct uniformly for $t\in[0,1]$. Then we have

$$\|\langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g\|_{L^2} \le C\|\langle D\rangle^{1-\theta}f\|_{L^p}\|\langle D\rangle^{s+\theta-1}g\|_{L^q}$$

for $\theta\in[0,1]$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and $1<q<\infty$, see P. 106 in [32]. Taking $\theta=\frac{1}{2}$,$(p,q)=(\infty,2)$ for $d=2$, and $\theta=0$, $(p,q)=(6,3)$ for $d=3$, we obtain

$$\|\langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g\|_{L^2} \le C\|f\|_{H^2}\|g\|_{H^{s-\frac{1}{2}}}.$$

In case of $s\in(0,1]$, taking $\theta=1-s$ and $(p,q)=(\infty,2)$ to obtain

$$\|\langle D\rangle^s(T_f g) - T_f\langle D\rangle^s g\|_{L^2} \le C\|f\|_{H^{s+2}}\|g\|_{L^2}.$$

This completes the proof of Lemma 6.4. ■

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