APPROXIMATION OF THE STATIONARY STATISTICAL PROPERTIES OF THE DYNAMICAL SYSTEM GENERATED BY THE TWO-DIMENSIONAL RAYLEIGH-BÉNARD CONVECTION PROBLEM

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ABSTRACT. In this article we consider a temporal linear semi-implicit approximation of the two-dimensional Rayleigh-Bénard convection problem. We prove that the stationary statistical properties of this linear semi-implicit scheme converge to those of the 2D Rayleigh-Bénard problem as the time step approaches zero.

CONTENTS

1. INTRODUCTION

In this article we consider temporal approximation of the equations that govern the two-dimensional Rayleigh-Bénard convection problem. We show that the stationary statistical properties of a linear semi-implicit numerical scheme converge to those of the Rayleigh-Bénard problem at vanishing step size following a general framework proposed

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in [26] for temporal approximations of stationary statistical properties for dissipative dynamical systems. For convenience we recall the following result, proven in [26]:

Theorem 1 (Convergence of Stationary Statistical Properties)**.** Let $\{S(t), t > 0\}$ be a continuous semigroup on a separable Hilbert space H which generates a continuous dissipative dynamical system (in the sense of possessing a compact global attractor $\mathcal{A}$) on H. Let $\{S_k, 0 < k < k_0\}$ be a family of continuous maps on H which generates a family of discrete dissipative dynamical systems (with global attractor $\mathcal{A}_k$) on H. Suppose that the following three conditions are satisfied:

$H1:$ [Uniform dissipativity] There exists $k_1 \in (0, k_0)$ such that $\{S_k, 0 < k < k_1\}$ is uniformly dissipative in the sense that

$$K = \cup_{0<k\leq k_1} \mathcal{A}_k \tag{1.1}$$

is pre-compact in H.

$H2:$ [Uniform convergence on the unit time interval] S_k uniformly converges to S on the unit time interval (modulo an initial layer) and uniformly for initial data from the global attractor of S_k in the sense that for any $t_0 \in (0, 1)$

$$\lim_{k\to 0} \sup_{\mathbf{u}\in\mathcal{A}_k,\, nk\in[t_0,1]} \|S_k^n \mathbf{u} - S(nk)\mathbf{u}\| = 0. \tag{1.2}$$

$H3:$ [Uniform continuity of the continuous system] $\{S(t), t > 0\}$ is uniformly continuous on K on the unit time interval in the sense that for any $T^* \in [0, 1]$

$$\lim_{t\to T^*} \sup_{\mathbf{u}\in K} \|S(t)\mathbf{u} - S(T^*)\mathbf{u}\| = 0. \tag{1.3}$$

Then the stationary statistical properties of the discrete dynamical system $\{S_k, 0 < k < k_1\}$ converge to the stationary statistical properties of the continuous dynamical system S.

Our aim in this article is to verify the three conditions stipulated in the theorem above on a specific semi-implicit linear approximation of the 2D Boussinesq system for Rayleigh-Bénard convection. The verification of these three conditions then leads to the desired convergence of stationary statistical properties associated with the numerical schemes to that of the 2D Boussinesq system.

Statistical properties for systems like the Boussinesq equations for Rayleigh-Bénard convection are of great importance. For systems with chaotic and/or turbulent behaviour, it is imperative to study the statistical behaviour of the system instead of single trajectories alone

[17, 16, 11]. Indeed, much of the classical turbulence theories are formulated in statistical forms (via spatial and temporal averages), for instance the famous Kolmogorov $\frac{U^3}{L}$ scaling law of the energy dissipation rate per unit mass as well as the Kolmogorov $k^{-\frac{5}{3}}$ energy spectrum in the inertial range in three dimensional homogeneous isotropic turbulence [7, 13, 17, 5].

For a given abstract autonomous continuous in time dynamical system determined by a semigroup $\{S(t), t \geq 0\}$ on a separable metric space H, we recall that if the system reaches a statistical equilibrium in the sense that the statistics are time independent (stationary statistical properties), the probability measure μ on H that describes the stationary statistical properties can be characterized via either the strong (pull-back) or weak (push-forward) formulation [5, 14, 16].

Let $\{S(t), t \geq 0\}$ be a continuous semigroup on a metric space H which generates a dynamical system on H. A Borel probability measure μ on H is called an **Invariant Measure** (Stationary Statistical Solution) of the dynamical system if

$$\mu(E) = \mu(S^{-1}(t)(E)), \ \forall t \geq 0, \forall E \in \mathcal{B}(H), \tag{1.4}$$

where $\mathcal{B}(H)$ represents the σ-algebra of all Borel sets on H. Equivalently, the invariant measure μ can be characterized through the following *push-forward* weak invariance formulation

$$\int_H \Phi(\mathbf{u})\, d\mu(\mathbf{u}) = \int_H \Phi(S(t)\mathbf{u})\, d\mu(\mathbf{u}), \ \forall t \geq 0, \tag{1.5}$$

for all bounded continuous test functionals Φ.

Invariant measure (stationary statistical solution) for a discrete dynamical system generated by a map $S_{discrete}$ on a metric space H is defined in a similar fashion with the continuous time t replaced by discrete time $n = 0, 1, 2, \ldots$.

We say that the stationary statistical properties of the discrete dynamical system converge to those of the continuous dynamical system if the invariant measures converge in the weak sense.

We are usually interested in $\int_H \Phi(\mathbf{u})\, d\mu(\mathbf{u})$ (statistical average) for various test functionals Φ. These averaged quantities are also called observables in physics literatures. Due to the presumed complexity of the dynamics, the physically interesting stationary statistical properties need to be calculated using numerical methods in generic case. Even under the ergodicity assumption, it is not at all clear that classical numerical schemes which provide accurate approximations on finite time intervals will remain meaningful for stationary statistical properties (long time properties), since small errors will be amplified and accumulated over long time, except in the case that the underlying dynamics is asymptotically stable, where statistical approach is not necessary since there is no chaos. Addressing issues like this is of great importance in many real life applications such as numerical study of climate change, since the climate is the long time statistical property of the underlying system. Therefore, it is central and a challenge to search for numerical methods that are able to capture stationary statistical properties of infinite dimensional complex dynamical system. In a series of recent works, one of the authors of this manuscript, together with a collaborator, proposed a general framework for constructing temporal approximations of dissipative systems such as the 2D Rayleigh-Bénard convection system, so that the stationary statistical properties of the numerical scheme converge to those of the underlying Boussinesq system [26, 25, 2, 1]. The main contribution of this article is the application of the general theory proposed in [26] to the Boussinesq system for Rayleigh-Bénard convection in the 2D case.

One of the main themes in constructing temporal approximations that guarantee the convergence of the stationary statistical properties is the preservation of the dissipativity in some appropriate sense. Similar ideas of preservation of dissipativity have been proposed and investigated by many authors (see [19, 20, 3, 4, 9, 21, 10, 24, 23] among many others). All these previous works emphasized different aspects of dissipativity (uniform boundedness of solutions, global attractors) without referencing to the statistical properties.

In this article, we are going to discretize the equations that model the two-dimensional Rayleigh-Bénard convection problem using a temporal semi-implicit Euler scheme. One of the technical difficulties we encounter is related to the specific treatment of the temperature, for which the utilization of the maximum principle is needed (see Lemma 2 below).

The article is organized as follows: in section 2 we introduce the Rayleigh-Bénard convection problem and the semi-implicit Euler scheme that approximates the solution to the equations that model the problem, in section 3 we prove condition $H1$, the uniform dissipativity of the scheme, in section 4 we prove condition $H2$, the finite time uniform convergence, and in section 5 we first prove condition $H3$, the finite time uniform continuity and then we conclude with the main theorem (see Theorem 2 below) and another important result on convergence of attractors (see Theorem 4 below).

2. The Rayleigh-Bénard convection problem

Let $\Omega = (0,1) \times (0,1)$ be the domain occupied by the fluid and let e_2 be the unit upward vertical vector. The Rayleigh-Bénard convection problem can be modeled by the Boussinesq approximation and they read (see, e.g., [6], [22]):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = -e_2(T - T_1), \tag{2.1}$$

$$\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T - \kappa \Delta T = 0, \tag{2.2}$$

$$\operatorname{div} \mathbf{v} = 0; \tag{2.3}$$

here $\mathbf{v} = (v_1, v_2)$ is the velocity, p is the pressure, T is the temperature, T_1 is the temperature at the top boundary, $x_2 = 1$, and ν, κ are positive constants. We supplement these equations with the initial conditions

$$\mathbf{v}(x,0) = \mathbf{v}_0(x), \tag{2.4}$$

$$T(x,0) = T^0(x), \tag{2.5}$$

where $\mathbf{v}_0 : \Omega \to \mathbb{R}^2$, $T^0 : \Omega \to \mathbb{R}$ are given, and with the boundary conditions

$$\mathbf{v} = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = 1, \tag{2.6}$$

$$T = T_0 = T_1 + 1 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad T = T_1 \quad \text{at} \quad x_2 = 1, \tag{2.7}$$

and

$$\begin{gathered} p, \mathbf{v}, T \text{ and the first derivatives of } \mathbf{v} \text{ and } T \text{ are periodic} \\ \text{of period 1 in the direction } x_1, \end{gathered} \tag{2.8}$$

meaning that $\phi|_{x_1=0} = \phi|_{x_1=1}$ for the corresponding functions ϕ.

Letting

$$\theta = T - T_0 + x_2, \tag{2.9}$$

and changing p to

$$p - \left(x_2 - \frac{x_2^2}{2} \right), \tag{2.10}$$

equations (2.1)–(2.3) together with the boundary conditions (2.6)–(2.8) become

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = -e_2\theta, \tag{2.11}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{v}\cdot\nabla)\theta - v_2 - \kappa\Delta\theta = 0, \tag{2.12}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{2.13}$$

$$\mathbf{v} = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = 1, \tag{2.14}$$

$$\theta = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = 1, \tag{2.15}$$

$$\text{(2.8) holds with } T \text{ replaced by } \theta. \tag{2.16}$$

These equations are supplemented with the initial conditions

$$\mathbf{v}(x,0) = \mathbf{v}_0(x), \tag{2.17}$$

$$\theta(x,0) = T^0(x) - T_0 + x_2 =: \theta_0(x). \tag{2.18}$$

For the mathematical setting of the problem we define the space $H = H_1 \times H_2$, where

(2.19)

$$H_1 = \left\{\mathbf{v} \in L^2(\Omega)^2,\ v_2|_{x_2=0} = v_2|_{x_2=1} = 0, v_1|_{x_1=0} = v_1|_{x_1=1}, \operatorname{div}\mathbf{v} = 0, \right\},$$

$$H_2 = L^2(\Omega), \tag{2.20}$$

and we denote the scalar products and norms in H_1, H_2 and H by $(\cdot,\cdot)$ and $|\cdot|$.

We also define the space $V = V_1 \times V_2$, where

(2.21)

$$V_1 = \left\{\mathbf{v} \in H^1(\Omega)^2,\ \mathbf{v}|_{x_2=0} = \mathbf{v}|_{x_2=1} = 0, \mathbf{v} \text{ periodic in } x_1 \text{ with period } 1,\ \operatorname{div}\mathbf{v} = 0\right\},$$

(2.22)

$$V_2 = \left\{\theta \in H^1(\Omega),\ \theta|_{x_2=0} = \theta|_{x_2=1} = 0, \theta \text{ periodic in } x_1 \text{ with period } 1\right\}.$$

The space V_2 is a Hilbert space with the scalar product and the norm

$$((\phi,\psi)) = \int \nabla\phi\cdot\nabla\psi\,dx, \qquad \|\phi\| = \sqrt{((\phi,\phi))}, \tag{2.23}$$

and we have the Poincaré inequality

$$|\phi| \le \|\phi\|, \quad \forall\,\phi \in V_1 \text{ or } V_2. \tag{2.24}$$

We denote both scalar products and norms in V_1 and V by $((\cdot,\cdot))$ and $\|\cdot\|$.

Let $D(A) = D(A_1) \times D(A_2)$, where

$$D(A_1) = \left\{\mathbf{v} \in V_1 \cap H^2(\Omega)^2, \mathbf{v} \text{ periodic in } x_1 \text{ with period } 1\right\}, \tag{2.25}$$

$$D(A_2) = \left\{\theta \in V_2 \cap H^2(\Omega), \theta \text{ periodic in } x_1 \text{ with period } 1\right\}, \tag{2.26}$$

and let A be the linear operator from $D(A)$ into H and from V into V' defined by

$$(A\mathbf{u_1},\mathbf{u_2}) = a(\mathbf{u_1},\mathbf{u_2}),\ \forall\, \mathbf{u_i} = \{\mathbf{v}_i,\theta_i\} \in D(A),\ i=1,2, \tag{2.27}$$

with

$$a(\mathbf{u_1},\mathbf{u_2}) = \nu((\mathbf{v}_1,\mathbf{v}_2)) + \kappa((\theta_1,\theta_2)). \tag{2.28}$$

We consider the trilinear continuous form b on V, defined by

$$b(\mathbf{u_1},\mathbf{u_2},\mathbf{u_3}) = b_1(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) + b_2(\mathbf{v}_1,\theta_2,\theta_3), \forall \mathbf{u_i} = \{\mathbf{v}_i,\theta_i\} \in V, \tag{2.29}$$

where

$$b_1(\mathbf{y},\mathbf{w},\mathbf{z}) = \sum_{i,j=1,2} \int_\Omega y_i \frac{\partial w_j}{\partial x_i} z_j \,\mathrm{d}x, \forall\, \mathbf{y},\mathbf{w},\mathbf{z} \in H^1(\Omega)^2, \tag{2.30}$$

$$b_2(\mathbf{y},\phi,\psi) = \sum_{i=1}^{2} \int_\Omega y_i \frac{\partial \phi}{\partial x_i} \psi \,\mathrm{d}x, \forall\, \mathbf{y} \in H^1(\Omega)^2, \phi,\psi \in H^1(\Omega). \tag{2.31}$$

The form b_1 is trilinear continuous on $V_1 \times V_1 \times V_1$ and enjoys the following properties:

$$|b_1(\mathbf{y},\mathbf{w},\mathbf{z})| \le c_b |\mathbf{y}|^{1/2} \|\mathbf{y}\|^{1/2} \|\mathbf{w}\| |\mathbf{z}|^{1/2} \|\mathbf{z}\|^{1/2}, \quad \forall\, \mathbf{y},\mathbf{w},\mathbf{z} \in V_1, \tag{2.32}$$

$$\begin{gathered} |b_1(\mathbf{y},\mathbf{w},\mathbf{z})| \le c_b |\mathbf{y}|^{1/2} |\Delta \mathbf{y}|^{1/2} \|\mathbf{w}\| |\mathbf{z}|, \\ \forall\, \mathbf{y} \in D(A_1),\ \mathbf{w} \in V_1,\ \mathbf{z} \in H_1, \end{gathered} \tag{2.33}$$

$$\begin{gathered} |b_1(\mathbf{y},\mathbf{w},\mathbf{z})| \le c_b |\mathbf{y}|^{1/2} \|\mathbf{y}\|^{1/2} \|\mathbf{w}\|^{1/2} |\Delta \mathbf{w}|^{1/2} |\mathbf{z}|, \\ \forall\, \mathbf{y} \in V_1, \mathbf{w} \in D(A_1), \mathbf{z} \in H_1, \end{gathered} \tag{2.34}$$

$$b_1(\mathbf{y},\mathbf{w},\mathbf{w}) = 0, \quad \forall\, \mathbf{y},\mathbf{w} \in V_1, \tag{2.35}$$

the last equation implying

$$b_1(\mathbf{y},\mathbf{w},\mathbf{z}) = -b_1(\mathbf{y},\mathbf{z},\mathbf{w}), \quad \forall\, \mathbf{y},\mathbf{w},\mathbf{z} \in V_1. \tag{2.36}$$

The form b_2 is trilinear continuous on $V_1 \times V_2 \times V_2$ and enjoys the following properties, similar to (2.32)–(2.36):

$$|b_2(\mathbf{y},\phi,\psi)| \le c_b |\mathbf{y}|^{1/2} \|\mathbf{y}\|^{1/2} \|\phi\| |\psi|^{1/2} \|\psi\|^{1/2}, \quad \forall\, \mathbf{y} \in V_1, \phi,\psi \in V_2, \tag{2.37}$$

$$\begin{gathered} |b_2(\mathbf{y},\phi,\psi)| \le c_b |\mathbf{y}|^{1/2} |\Delta \mathbf{y}|^{1/2} \|\phi\| |\psi|, \\ \forall\, \mathbf{y} \in D(A_1),\ \phi \in V_2,\ \psi \in H_2, \end{gathered} \tag{2.38}$$

$$\begin{gathered} |b_2(\mathbf{y},\phi,\psi)| \le c_b |\mathbf{y}|^{1/2} \|\mathbf{y}\|^{1/2} \|\phi\|^{1/2} |\Delta \phi|^{1/2} |\psi|, \\ \forall\, \mathbf{y} \in V_1,\ \phi \in D(A_2),\ \psi \in H_2, \end{gathered} \tag{2.39}$$

$$b_2(\mathbf{y}, \phi, \phi) = 0, \quad \forall\, \mathbf{y} \in V_1,\ \phi \in V_2, \tag{2.40}$$

the last equation implying

$$b_2(\mathbf{y}, \phi, \psi) = -b_2(\mathbf{y}, \psi, \phi), \quad \forall\, \mathbf{y} \in V_1,\ \phi, \psi \in V_2. \tag{2.41}$$

We associate with b the bilinear continuous operator B from $V \times V$ into V' and from $D(A) \times D(A)$ into H, such that

$$\langle B(\mathbf{u_1}, \mathbf{u_2}), \mathbf{u_3} \rangle_{V', V} = b(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}), \quad \forall\, \mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3} \in V. \tag{2.42}$$

We also define the continuous operator in H

$$R\mathbf{u} = \{e_2\theta, -v_2\},\ \mathbf{u} = \{\mathbf{v}, \theta\}. \tag{2.43}$$

For more details about the function spaces $D(A)$, V and H, as well as the operators A, B, R and b, the reader is referred to, e.g., [22].

In the above notation, the system (2.11)–(2.13) can be written as the functional evolution equation

$$\mathbf{u}_t + A\mathbf{u} + B(\mathbf{u}) + R(\mathbf{u}) = 0, \qquad \mathbf{u}(0) = \mathbf{u}_0 = \{\mathbf{v}_0, \theta_0\}. \tag{2.44}$$

In the two-dimensional case under consideration, the solution to the Rayleigh-Bénard convection problem is known to be smooth for all time (cf. [22]). Using the maximum principle for parabolic equations, one can show that $\theta \in L^\infty(\mathbb{R}_+; L^2(\Omega))$ and the velocity $\mathbf{v}$ is bounded uniformly for all time by

$$|\mathbf{v}(t)|^2_{L^2(\Omega)^2} \leq \mathrm{e}^{-\nu t} |\mathbf{v}_0|^2_{L^2(\Omega)^2} + \frac{\theta_\infty^2}{\nu^2}\big(1 - \mathrm{e}^{-\nu t}\big), \tag{2.45}$$

where $\theta_\infty = |\theta|_{L^\infty(\mathbb{R}_+; L^2(\Omega))}$. Furthermore, using techniques based on the uniform Gronwall lemma (cf. [22]), one can bound the solution $\mathbf{u}$ of (2.44) uniformly in V for all $t \geq 0$.

In this article we discretize (2.44) in time using the semi-implicit Euler scheme,

$$\frac{\mathbf{v}^n - \mathbf{v}^{n-1}}{\Delta t} + (\mathbf{v}^{n-1} \cdot \nabla)\mathbf{v}^n - \nu \Delta \mathbf{v}^n + \nabla p^n = -e_2 \theta^n,\ n \geq 1, \tag{2.46}$$

$$\frac{\theta^n - \theta^{n-1}}{\Delta t} + (\mathbf{v}^{n-1} \cdot \nabla)\theta^n - v_2^{n-1} - \kappa \Delta \theta^n = 0,\ n \geq 1, \tag{2.47}$$

where $\mathbf{v}^0(x) = \mathbf{v}_0(x)$, and $\theta^0(x) = \theta_0(x) = T^0(x) - T_0 + x_2$ are given, and we prove that the stationary statistical properties of the numerical scheme converge to those of the continuous dynamical system as the time step approaches zero.

Remark 2.1. Using the Lax-Milgram theorem (see, e.g., [15], [22]), one can prove that the solution to (2.46)–(2.47) exists and is unique provided that $\Delta t \leq 1$. We therefore can define, for each $k = \Delta t > 0$, the discrete semigroup $S_k : H \to H$ that associates with any $(\mathbf{v}^{n-1}, \theta^{n-1}) \in$

H the unique solution, $(\mathbf{v}^n, \theta^n)$, to (2.46)–(2.47). Moreover, the discrete semigroup is regularizing in the sense that $S_k\mathbf{u} \in V, \forall \mathbf{u} \in H$.

For more information on semigroups and dynamical systems generated by semigroups, the interested reader is referred to, e.g., [22], [21], [18], [12], [8].

3. Uniform Dissipativity

In proving the convergence of the stationary statistical properties of the numerical scheme to those of the continuous dynamical system as the time step approaches zero, we first show that condition $H1$ of Theorem 1 is satisfied, that is, we show the uniform dissipativity of the scheme. In order to do that, we prove the existence of an absorbing ball in V and the uniform dissipativity of the numerical scheme will then be guaranteed by the Rellich compactness theorem.

3.1. **L^2-Uniform Boundedness of $\mathbf{v}^n$ and θ^n.** In order to prove the L^2-uniform boundedness of $\mathbf{v}^n$ and θ^n, we recall the classical truncation operators, that associate with the function φ, the functions φ_+ and φ_-, given by

$$\varphi_+(x) = \max(\varphi(x), 0), \quad \varphi_-(x) = \max(-\varphi(x), 0). \tag{3.1}$$

Note that, with this notation, we have $\varphi = \varphi_+ - \varphi_-$, $|\varphi| = \varphi_+ + \varphi_-$ and $\varphi_+\varphi_- = 0$. Using these operators, we can prove the following preliminary lemma

Lemma 1. If $\varphi, \psi \in L^2(\Omega)$, then

$$2(\varphi - \psi, \varphi_+) \geq |\varphi_+|^2 - |\psi_+|^2 + |\varphi_+ - \psi_+|^2, \tag{3.2}$$

$$-2(\varphi - \psi, \varphi_-) \geq |\varphi_-|^2 - |\psi_-|^2 + |\varphi_- - \psi_-|^2. \tag{3.3}$$

Proof. We have

$$\begin{aligned} 2(\varphi - \psi, \varphi_+) &= 2(\varphi_+ - \varphi_- - \psi_+ + \psi_-, \varphi_+) \\ &= 2(\varphi_+ - \psi_+, \varphi_+) - 2(\varphi_- - \psi_-, \varphi_+) \\ &= |\varphi_+|^2 - |\psi_+|^2 + |\varphi_+ - \psi_+|^2 + 2\int_\Omega \psi_-\varphi_+\, dx \\ &\geq |\varphi_+|^2 - |\psi_+|^2 + |\varphi_+ - \psi_+|^2, \end{aligned} \tag{3.4}$$

since $\psi_-\varphi_+ \geq 0$. The proof is similar for (3.3) and the lemma is proved. □

We are now able to prove the L^2-uniform boundedness of θ^n:

Lemma 2. If $\mathbf{v}^n$ and θ^n satisfy (2.46) and (2.47), then

$$|\theta^n| \leq |\Omega|^{1/2} + \left(|\theta_+^0| + |\theta_-^0|\right)(1+2\kappa\Delta t)^{-\frac{n}{2}}, \forall n \geq 1. \tag{3.5}$$

Moreover, there exists $M_1 = M_1(|\theta_0|)$, given in (3.18) below, such that

$$|\theta^n| \leq M_1, \forall n \geq 1. \tag{3.6}$$

Proof. Rewriting (2.47) as

$$\frac{\theta^n - \theta^{n-1}}{\Delta t} + (\mathbf{v}^{n-1}\cdot\nabla)(\theta^n - x_2) - \kappa\Delta\theta^n = 0, \tag{3.7}$$

multiplying the above equation by $2\Delta t(\theta^n - x_2)_+$ in H_2, and using (3.2), we obtain:

$$\begin{aligned} &|(\theta^n - x_2)_+|^2 - |(\theta^{n-1} - x_2)_+|^2 \\ &+|(\theta^n - x_2)_+ - (\theta^{n-1} - x_2)_+|^2 + 2\Delta t\kappa\|(\theta^n - x_2)_+\|^2 \leq 0. \end{aligned} \tag{3.8}$$

Using the Poincaré inequality (2.24), we find

$$|(\theta^n - x_2)_+|^2 \leq \frac{1}{\alpha}|(\theta^{n-1} - x_2)_+|^2, \tag{3.9}$$

where

$$\alpha = 1 + 2\kappa\Delta t. \tag{3.10}$$

Using recursively (3.9), we find

$$|(\theta^n - x_2)_+|^2 \leq (1+2\kappa\Delta t)^{-n}|(\theta^0 - x_2)_+|^2. \tag{3.11}$$

Similarly, using (3.3), we obtain

$$|(\theta^n - x_2 + 1)_-|^2 \leq (1+2\kappa\Delta t)^{-n}|(\theta^0 - x_2 + 1)_-|^2. \tag{3.12}$$

Setting

$$\bar{\theta}^n = (\theta^n - x_2)_+ - (\theta^n - x_2 + 1)_-, \tag{3.13}$$

$$\tilde{\theta}^n = \theta^n - \bar{\theta}^n, \tag{3.14}$$

we have

$$\theta^n = \tilde{\theta}^n + \bar{\theta}^n, \tag{3.15}$$

and recalling (3.1), we note that

$$x_2 - 1 \leq \tilde{\theta}^n \leq x_2. \tag{3.16}$$

By (3.13), (3.11) and (3.12), we derive

$$\begin{aligned} |\bar{\theta}^n| &\leq |(\theta^n - x_2)_+| + |(\theta^n - x_2 + 1)_-| \\ &\leq (1+2\kappa\Delta t)^{-\frac{n}{2}}(|\theta_+^0| + |\theta_-^0|). \end{aligned} \tag{3.17}$$

The conclusion of the lemma follows right away with

$$M_1(|\theta_0|) = |\Omega|^{1/2} + |\theta_+^0| + |\theta_-^0|. \tag{3.18}$$

This concludes the proof of Lemma 2. □

Corollary 3.1. $B_{L^2}(0, 2|\Omega|^{1/2})$, the ball in L^2 centered at 0 and radius $2|\Omega|^{1/2}$, is an absorbing ball for θ^n in L^2.

Proof. Indeed, let $\mathcal{B}$ be any bounded set in L^2 and assume that it is included in a ball $B(0,R)$ of L^2. It is easy to deduce from (3.5) that for any $\theta_0 \in B(0,R)$, there exists $N_0^1(R,\Delta t) := \frac{\ln\left(\frac{2R}{|\Omega|^{1/2}}\right)}{\kappa\Delta t}$ such that $\theta^n \in B_{L^2}(0, 2|\Omega|^{1/2}), \forall n \geq N_0^1$. □

We are now able to prove the L^2-uniform boundedness of $\mathbf{v}^n$. More precisely, we have the following:

Lemma 3. Let $(\mathbf{v}^n, \theta^n)$ be the solution of the numerical scheme (2.46)–(2.47). Then for every $\Delta t > 0$, we have

$$|\mathbf{v}^n|^2 \leq (1+\nu\Delta t)^{-n}\, |\mathbf{v}_0|^2 + \frac{M_1^2}{\nu^2}\left[1-(1+\nu\Delta t)^{-n}\right],\ \forall\, n \geq 0. \tag{3.19}$$

Moreover, there exists $K_1 = K_1(|\mathbf{v}_0|, |\theta_0|)$, such that

$$|\mathbf{v}^n| \leq K_1, \quad \forall\, n \geq 0, \tag{3.20}$$

and

$$\nu\Delta t \sum_{j=i}^{m} \|\mathbf{v}^j\|^2 \leq |\mathbf{v}^{i-1}|^2 + \frac{1}{\nu}\Delta t \sum_{j=i}^{m} |\theta^j|^2, \quad \forall\, i = 1, \cdots, n, \tag{3.21}$$

$$\kappa\Delta t \sum_{j=i}^{m} \|\theta^j\|^2 \leq |\theta^{i-1}|^2 + \frac{1}{\kappa}\Delta t \sum_{j=i}^{m} |\mathbf{v}^{j-1}|^2, \quad \forall\, i = 1, \cdots, n. \tag{3.22}$$

Proof. Taking the scalar product of (2.46) with $2\Delta t\mathbf{v}^n$ in H_1 and using the relation

$$2(\varphi-\psi, \varphi) = |\varphi|^2 - |\psi|^2 + |\varphi-\psi|^2, \quad \forall \varphi, \psi \in H_1, \tag{3.23}$$

as well as the skew property (2.35), we obtain

$$|\mathbf{v}^n|^2 - |\mathbf{v}^{n-1}|^2 + |\mathbf{v}^n - \mathbf{v}^{n-1}|^2 + 2\nu\Delta t\, \|\mathbf{v}^n\|^2 = -2\Delta t(e_2\theta^n, \mathbf{v}^n). \tag{3.24}$$

Using the Cauchy–Schwarz inequality and the Poincaré inequality (2.24), we majorize the right-hand side of (3.24) by

$$\begin{aligned} -2\Delta t(e_2\theta^n, \mathbf{v}^n) &\leq 2\Delta t|e_2\theta^n||\mathbf{v}^n| \leq 2\Delta t|\theta^n||\mathbf{v}^n| \\ &\leq 2\Delta t|\theta^n|\|\mathbf{v}^n\| \leq \nu\Delta t\|\mathbf{v}^n\|^2 + \frac{1}{\nu}\Delta t\, |\theta^n|^2. \end{aligned} \tag{3.25}$$

Relations (3.24) and (3.25) imply

$$|\mathbf{v}^n|^2 - |\mathbf{v}^{n-1}|^2 + |\mathbf{v}^n - \mathbf{v}^{n-1}|^2 + \nu\Delta t\,\|\mathbf{v}^n\|^2 \le \frac{1}{\nu}\Delta t\,|\theta^n|^2. \tag{3.26}$$

Using again the Poincaré inequality (2.24), we find

$$|\mathbf{v}^n|^2 \le \frac{1}{\alpha}|\mathbf{v}^{n-1}|^2 + \frac{1}{\alpha\nu}\Delta t\,|\theta^n|^2, \tag{3.27}$$

where

$$\alpha = 1 + \nu\Delta t. \tag{3.28}$$

Using recursively (3.27), we find

$$\begin{aligned} |\mathbf{v}^n|^2 &\le \frac{1}{\alpha^n}|\mathbf{v}^0|^2 + \frac{1}{\nu}\Delta t\sum_{i=1}^{n}\frac{1}{\alpha^i}|\theta^{n+1-i}|^2 \\ &\le (1+\nu\Delta t)^{-n}\,|\mathbf{v}^0|^2 + \frac{M_1^2}{\nu^2}\left[1 - (1+\nu\Delta t)^{-n}\right], \end{aligned} \tag{3.29}$$

which proves (3.19).

Taking $K_1^2 = |\mathbf{v}^0|^2 + \frac{M_1^2}{\nu^2}$ relation (3.20) follows right away.

Adding inequalities (3.26) with n from i to m we obtain (3.21) with n in place of m.

Now, taking the scalar product of (2.47) with $2\Delta t\theta^n$ in H_2 and using the skew property (2.40), we obtain

$$|\theta^n|^2 - |\theta^{n-1}|^2 + |\theta^n - \theta^{n-1}|^2 + 2\kappa\Delta t\,\|\theta^n\|^2 = 2\Delta t(v_2^{n-1}, \theta^n). \tag{3.30}$$

Using again the Cauchy–Schwarz inequality and the Poincaré inequality (2.24), we majorize the right-hand side of (3.30) by

$$\begin{aligned} 2\Delta t(v_2^{n-1}, \theta^n) &\le 2\Delta t|v_2^{n-1}||\theta^n| \le 2\Delta t|\mathbf{v}^{n-1}|\|\theta^n\| \\ &\le \kappa\Delta t\|\theta^n\|^2 + \frac{1}{\kappa}\Delta t\,|\mathbf{v}^{n-1}|^2. \end{aligned} \tag{3.31}$$

Relations (3.30) and (3.31) imply

$$|\theta^n|^2 - |\theta^{n-1}|^2 + |\theta^n - \theta^{n-1}|^2 + \kappa\Delta t\,\|\theta^n\|^2 \le \frac{1}{\kappa}\Delta t\,|\mathbf{v}^{n-1}|^2. \tag{3.32}$$

Summing inequalities (3.32) with n from i to m we obtain (3.22) with n in place of m. □

Corollary 3.2. Let $\rho_0 = 2|\Omega|^{1/2} + \frac{\sqrt{5}|\Omega|^{1/2}}{\nu}$. Then $B_H(0, \rho_0)$, the ball in H centered at 0 and radius ρ_0, is an absorbing ball for $(\mathbf{v}^n, \theta^n)$ in H.

Proof. Indeed, let $\mathcal{B}$ be any bounded set in H and assume that it is included in a ball $B(0,R)$ of H. For any initial data $(\mathbf{v}^0,\theta^0)\in\mathcal{B}$, Corollary 3.1 implies that

$$|\theta^n| < 2|\Omega|^{1/2}, \forall n \geq N_0^1(R,\Delta t), \tag{3.33}$$

and then (3.27) becomes

$$|\mathbf{v}^n|^2 \leq \frac{1}{\alpha}|\mathbf{v}^{n-1}|^2 + \frac{4}{\alpha\nu}|\Omega|\Delta t,\ \forall n \geq N_0^1(R,\Delta t), \tag{3.34}$$

where

$$\alpha = 1+\nu\Delta t. \tag{3.35}$$

Iterating the above inequality, we find (for any $n \geq N_0^1(R,\Delta t)$)

(3.36)

$$\begin{aligned}|\mathbf{v}^n|^2 &\leq \frac{1}{\alpha^{(n-N_0^1)}}|\mathbf{v}^{N_0^1}|^2 + \frac{4}{\nu}|\Omega|\Delta t\sum_{i=1}^{n-N_0^1}\frac{1}{\alpha^i}\\ &= (1+\nu\Delta t)^{-(n-N_0^1)}\,|\mathbf{v}^{N_0^1}|^2 + \frac{4}{\nu^2}|\Omega|\left[1-(1+\nu\Delta t)^{-(n-N_0^1)}\right],\\ &\leq (1+\nu\Delta t)^{-(n-N_0^1)}\left[R^2+\frac{4}{\nu^2}(|\Omega|+2R^2)\right]+\frac{4}{\nu^2}|\Omega|\\ &\qquad \text{(by (3.19) and (3.18))},\end{aligned}$$

and one can see that

$$|\mathbf{v}^n|^2 \leq \frac{5}{\nu^2}|\Omega|, \forall n \geq N_0(R,\Delta t) := N_0^1 + N_0^2, \tag{3.37}$$

where

$$N_0^2(R,\Delta t) = \frac{\ln\frac{\nu^2\left[R^2+\frac{4}{\nu^2}(|\Omega|+2R^2)\right]}{|\Omega|}}{\nu\Delta t}. \tag{3.38}$$

We, therefore, have that $(\mathbf{v}^n,\theta^n)\in B_H(0,\rho_0)$, for all $n \geq N_0(R,\Delta t)$, which completes the proof of the corollary. □

3.2. **H^1-Uniform Boundedness of $\mathbf{v}^n$ and θ^n.** In order to prove the H^1-uniform boundedness of $\mathbf{v}^n$ and θ^n, we need the following two lemmas, whose proofs can be found in [20]:

Lemma 4. Given $\Delta t > 0$ and positive sequences ξ_n, η_n and ζ_n such that

$$\xi_n \leq \xi_{n-1}(1+\Delta t\eta_{n-1}) + \Delta t\zeta_n, \qquad \text{for } n \geq 1, \tag{3.39}$$

we have, for any $n \geq 2$,

$$\xi_n \leq \left(\xi_0 + \sum_{i=1}^{n} \Delta t \zeta_i\right) \exp\left(\sum_{i=0}^{n-1} \Delta t \eta_i\right). \tag{3.40}$$

Lemma 5. Given $\Delta t > 0$, a positive integer n_0, positive sequences ξ_n, η_n and ζ_n such that

$$\xi_n \leq \xi_{n-1}(1 + \Delta t \eta_{n-1}) + \Delta t \zeta_n, \qquad \text{for } n \geq n_0, \tag{3.41}$$

and given the bounds

$$\begin{aligned} &\sum_{n=k_0}^{N+k_0} \Delta t \eta_n \leq a_1, \qquad \sum_{n=k_0}^{N+k_0} \Delta t \zeta_n \leq a_2, \\ &\sum_{n=k_0}^{N+k_0} \Delta t \xi_n \leq a_3, \end{aligned} \tag{3.42}$$

for any $k_0 \geq n_0$, we have,

$$\xi_n \leq \left(\frac{a_3}{N\Delta t} + a_2\right) e^{a_1}, \qquad \forall n \geq N + n_0. \tag{3.43}$$

Proposition 1. Let $T > 0$ be arbitrarily fixed and let $(\mathbf{v}^n, \theta^n)$ be the solution of the numerical scheme (2.46)–(2.47). Then there exists $K_4 = K_4(\|\mathbf{v}_0\|, |\theta_0|, T)$, such that for every $\Delta t > 0$, we have

$$\|\mathbf{v}^n\| \leq K_4, \quad \forall n \geq 0, \tag{3.44}$$

$$\begin{aligned} \sum_{n=i}^{m} \|\mathbf{v}^n - \mathbf{v}^{n-1}\|^2 \leq & K_4^2 + \frac{27c_b^4}{2\nu^3} K_1^2 K_4^4 (m - i + 1)\Delta t \\ & + \frac{2}{\nu} M_1^2 (m - i + 1)\Delta t, \quad \forall\, i = 1, \cdots, m. \end{aligned} \tag{3.45}$$

Moreover, for any initial data from H, there exists $K_3(T)$ such that

$$\|\mathbf{v}^n\| \leq K_3, \quad \forall n \geq N + N_0 + 1, \tag{3.46}$$

where $N := \lfloor T/\Delta t \rfloor$ and $T_0 = N_0 \Delta t$ is the time the approximate solution $(\mathbf{v}^n, \theta^n)$ enters the absorbing ball $B(0, \rho_0)$ in H.

Proof. Taking the scalar product of (2.46) with $-2\Delta t \Delta \mathbf{v}^n$ in H_1, we obtain

$$\begin{aligned} \|\mathbf{v}^n\|^2 - \|\mathbf{v}^{n-1}\|^2 + \|\mathbf{v}^n - \mathbf{v}^{n-1}\|^2 - 2\Delta t b_1(\mathbf{v}^{n-1}, \mathbf{v}^n, \Delta \mathbf{v}^n) \\ + 2\nu \Delta t |\Delta \mathbf{v}^n|^2 = 2\Delta t (e_2 \theta^n, \Delta \mathbf{v}^n). \end{aligned} \tag{3.47}$$

Using property (2.34) of the trilinear form b_1 we have the following bound of the nonlinear term,

(3.48)
$$\begin{aligned}2\Delta t b_1(\mathbf{v}^{n-1},\mathbf{v}^n,\Delta\mathbf{v}^n) &\le 2\,c_b\,\Delta t\,|\mathbf{v}^{n-1}|^{1/2}\|\mathbf{v}^{n-1}\|^{1/2}\|\mathbf{v}^n\|^{1/2}|\Delta\mathbf{v}^n|^{3/2}\\ &\le \frac{\nu}{2}\Delta t|\Delta\mathbf{v}^n|^2+\frac{27c_b^4}{2\nu^3}\Delta t|\mathbf{v}^{n-1}|^2\|\mathbf{v}^n\|^2\|\mathbf{v}^{n-1}\|^2.\end{aligned}$$

Using the Cauchy–Schwarz inequality we bound the right-hand side of (3.47) by

$$\begin{aligned}2\Delta t(e_2\theta^n,\Delta\mathbf{v}^n) &\le 2\Delta t|\theta^n||\Delta\mathbf{v}^n|\\ &\le \frac{\nu}{2}\Delta t|\Delta\mathbf{v}^n|^2+\frac{2}{\nu}\Delta t|\theta^n|^2.\end{aligned}\tag{3.49}$$

Relations (3.47)–(3.49) imply

$$\begin{aligned}&\|\mathbf{v}^n\|^2-\|\mathbf{v}^{n-1}\|^2+\|\mathbf{v}^n-\mathbf{v}^{n-1}\|^2+\nu\Delta t|\Delta\mathbf{v}^n|^2\\ &\qquad\le \frac{27c_b^4}{2\nu^3}\Delta t|\mathbf{v}^{n-1}|^2\|\mathbf{v}^n\|^2\|\mathbf{v}^{n-1}\|^2+\frac{2}{\nu}\Delta t|\theta^n|^2,\end{aligned}\tag{3.50}$$

from which we obtain

$$\|\mathbf{v}^n\|^2\le\left(1+\frac{27c_b^4}{2\nu^3}\Delta t|\mathbf{v}^{n-1}|^2\|\mathbf{v}^n\|^2\right)\|\mathbf{v}^{n-1}\|^2+\frac{2}{\nu}\Delta t|\theta^n|^2.\tag{3.51}$$

We rewrite (3.51) in the form

$$\xi_n\le\xi_{n-1}(1+\Delta t\eta_{n-1})+\Delta t\zeta_n,\tag{3.52}$$

with

$$\xi_n=\|\mathbf{v}^n\|^2,\quad \eta_n=\frac{27c_b^4}{2\nu^3}|\mathbf{v}^{n-1}|^2\|\mathbf{v}^n\|^2,\quad \zeta_n=\frac{2}{\nu}|\theta^n|^2,\tag{3.53}$$

and recalling (3.6) and (3.20), we compute the following:

$$\sum_{i=1}^{n}\Delta t\zeta_i\le\sum_{i=1}^{n}\frac{2}{\nu}M_1^2\Delta t=\frac{2}{\nu}M_1^2 n\Delta t,\tag{3.54}$$

$$\begin{aligned}\sum_{i=0}^{n-1}\Delta t\eta_i &= \frac{27c_b^4}{2\nu^3}K_1^2\sum_{i=0}^{n-1}\Delta t\,\|\mathbf{v}^i\|^2\\ &\le \frac{27c_b^4}{2\nu^4}K_1^2\left[K_1^2+\frac{M_1^2}{\nu}(n-1)\Delta t\right]+\frac{27c_b^4}{2\nu^3}K_1^2\Delta t\|\mathbf{v}^0\|^2\\ &\qquad\text{(by (3.21))}.\end{aligned}\tag{3.55}$$

Then conclusion (3.40) of Lemma 4 yields

(3.56)

$$\|\mathbf{v}^n\|^2 \le \left(\|\mathbf{v}^0\|^2 + \frac{2}{\nu}M_1^2 n\Delta t\right) \exp\left\{\frac{27c_b^4}{2\nu^4}K_1^2\left[K_1^2 + \frac{M_1^2}{\nu}(n-1)\Delta t\right]\right\}$$
$$\exp\left\{\frac{27c_b^4}{2\nu^3}K_1^2\Delta t\|\mathbf{v}^0\|^2\right\} =: K_2^2(\|\mathbf{v}_0\|, |\theta_0|, n\Delta t),$$

and thus

$$\|\mathbf{v}^n\|^2 \le K_2^2(\|\mathbf{v}_0\|, |\theta_0|, T+T_0), \forall n = 0, \cdots, N+N_0. \tag{3.57}$$

In order to derive a bound on $\|\mathbf{v}^n\|^2$ valid for $n \ge N+N_0+1$, we will apply (the discrete uniform Gronwall) Lemma 5. In order to do so, we recall that $|\mathbf{v}^n| < \rho_0$, $|\theta^n| < \rho_0$, for $n \ge N_0$, and we compute the following (for $k_0 \ge N_0+1$):

$$\begin{aligned}\sum_{n=k_0}^{N+k_0} \Delta t \eta_n &= \frac{27c_b^4}{2\nu^3}\Delta t \sum_{n=k_0}^{N+k_0} |\mathbf{v}^{n-1}|^2\|\mathbf{v}^n\|^2 \\ &\le \frac{27c_b^4}{2\nu^4}\rho_0^2\left[\rho_0^2 + \frac{\rho_0^2}{\nu}(N+1)\Delta t\right] \text{(by (3.21))},\end{aligned} \tag{3.58}$$

$$\sum_{n=k_0}^{N+k_0} \Delta t \zeta_n = \sum_{n=k_0}^{N+k_0} \frac{2}{\nu}|\theta^n|^2 \le \frac{2}{\nu}\rho_0^2(N+1)\Delta t, \tag{3.59}$$

$$\begin{aligned}\sum_{n=k_0}^{N+k_0} \Delta t \xi_n &= \sum_{n=k_0}^{N+k_0} \Delta t\|\mathbf{v}^n\|^2 \\ &\le \frac{1}{\nu}\left[\rho_0^2 + \frac{\rho_0^2}{\nu}(N+1)\Delta t\right] \text{(by (3.21))}.\end{aligned} \tag{3.60}$$

Then conclusion (3.43) of Lemma 5 yields

(3.61)

$$\begin{aligned}\|\mathbf{v}^n\|^2 \le &\left\{\frac{1}{\nu N\Delta t}\left[\rho_0^2 + \frac{\rho_0^2}{\nu}(N+1)\Delta t\right] + \frac{2}{\nu}\rho_0^2(N+1)\Delta t\right\} \\ &\exp\left\{\frac{27c_b^4}{2\nu^4}\rho_0^2\left[\rho_0^2 + \frac{\rho_0^2}{\nu}(N+1)\Delta t\right]\right\} \\ &\le \left\{\frac{1}{\nu T}\left[\rho_0^2 + \frac{2\rho_0^2}{\nu}T\right] + \frac{4}{\nu}\rho_0^2 T\right\} \\ &\exp\left\{\frac{27c_b^4}{2\nu^4}\rho_0^2\left[\rho_0^2 + \frac{2\rho_0^2}{\nu}T\right]\right\} =: K_3^2(T), \quad \forall n \ge N+N_0+1.\end{aligned}$$

Combining the above bound with (3.57), we obtain both conclusion (3.44) and conclusion (3.46) of the lemma.

Taking the sum of (3.50) with n from i to m and using (3.44), as well as (3.6) and (3.20), gives (3.45) and thus the proof of Proposition 1 is complete. □

We are now going to prove the H^1-uniform boundedness of θ^n, for all $n \geq 0$. In order to do so, we will first use (the discrete Gronwall) Lemma 4 to derive an upper bound on $\|\theta^n\|$, $n \leq N$, for some $N > 0$, and then we will use another version of the discrete uniform Gronwall lemma (see Lemma 6 below) to obtain an upper bound on $\|\theta^n\|$, $n \geq N$.

Lemma 6. We are given $\Delta t > 0$, positive integers n_0, N and positive sequences ξ_n, η_n, ζ_n such that

$$\Delta t \eta_n < \frac{1}{2}, \qquad \text{for } n \geq n_0, \tag{3.62}$$

$$(1 - \Delta t \eta_n)\xi_n \leq \xi_{n-1} + \Delta t \zeta_n, \qquad \text{for } n \geq n_0. \tag{3.63}$$

Assume also that

$$\begin{aligned} &\Delta t \sum_{n=k_0}^{N+k_0} \eta_n \leq a_1, \qquad \Delta t \sum_{n=k_0}^{N+k_0} \zeta_n \leq a_2, \\ &\Delta t \sum_{n=k_0}^{N+k_0} \xi_n \leq a_3, \end{aligned} \tag{3.64}$$

for any $k_0 \geq n_0$. We then have,

$$\xi_n \leq \left(\frac{a_3}{N \Delta t} + a_2 \right) \mathrm{e}^{4a_1}, \tag{3.65}$$

for any $n \geq N + n_0$.

Proof. Let n_1 and n_2 be such that $n_0 \leq n_1 < n_2 \leq n_1 + N$. Using recursively (3.63), we derive

(3.66)

$$\xi_{n_1+N} \leq \frac{1}{\prod_{n=n_2}^{n_1+N}(1 - \Delta t \eta_n)} \xi_{n_2 - 1} + \Delta t \sum_{n=n_2}^{n_1+N} \frac{1}{\prod_{j=n}^{n_1+N}(1 - \Delta t \eta_j)} \zeta_n.$$

Using the fact that $1 - x \geq \mathrm{e}^{-4x}$, $\forall x \in \left(0, \frac{1}{2}\right)$, and recalling assumptions $(3.64)_1$ and $(3.64)_2$, we obtain

$$\xi_{n_1+N} \leq (\xi_{n_2-1} + a_2) e^{4a_1}.$$

Multiplying this inequality by Δt, summing n_2 from $n_1 + 1$ to $n_1 + N$ and using assumption $(3.64)_3$ gives conclusion (3.65) of the lemma. □

Proposition 2. Let $(\mathbf{v}_0, \theta_0) \in V$ and let $(\mathbf{v}^n, \theta^n)$ be the solution of the numerical scheme (2.46)–(2.47). Also, let $T > 0$ be arbitrarily fixed and let Δt be such that

$$\Delta t \leq \frac{\kappa^3}{27 c_b^4 K_1^2 K_4^2} =: k_1, \tag{3.67}$$

where $K_1(\cdot,\cdot)$ is given in Lemma 3 and $K_4(\cdot,\cdot)$ is given in Proposition 1. Then there exists $K_7(\|\mathbf{v}_0\|, \|\theta_0\|, T)$, such that

$$\|\theta^n\| \leq K_7, \quad \forall n \geq 1, \tag{3.68}$$

$$\begin{aligned} \sum_{n=i}^{m} \|\theta^n - \theta^{n-1}\|^2 \leq & K_7^2 + \frac{27 c_b^4}{2\kappa^3} K_1^2 K_4^2 K_7^2 (m-i+1)\Delta t \\ & + \frac{2}{\kappa} K_1^2 (m-i+1)\Delta t, \quad \forall\, i = 1, \cdots, m. \end{aligned} \tag{3.69}$$

Moreover, for any initial data from H, there exists $K_6(T)$ such that

$$\|\theta^n\| \leq K_6, \quad \forall n \geq N + N_0 + 1, \tag{3.70}$$

where $N := \lfloor T/\Delta t \rfloor$ and $T_0 = N_0 \Delta t$ is the time the approximate solution $(\mathbf{v}^n, \theta^n)$ enters the absorbing ball $B(0, \rho_0)$ in H.

Proof. Taking the scalar product of (2.47) with $-2\Delta t \Delta\theta^n$ in H_2, we obtain

$$\begin{aligned} \|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \|\theta^n - \theta^{n-1}\|^2 - 2\Delta t b_2(\mathbf{v}^{n-1}, \theta^n, \Delta\theta^n) \\ + 2\Delta t (v_2^n, \Delta\theta^n) + 2\kappa \Delta t |\Delta\theta^n|^2 = 0. \end{aligned} \tag{3.71}$$

Using property (2.39) of the trilinear form b_2, we have the following bound of the nonlinear term,

$$\begin{aligned} 2\Delta t b_2(\mathbf{v}^{n-1}, \theta^n, \Delta\theta^n) &\leq 2\, c_b\, \Delta t\, |\mathbf{v}^{n-1}|^{1/2} \|\mathbf{v}^{n-1}\|^{1/2} \|\theta^n\|^{1/2} |\Delta\theta^n|^{3/2} \\ &\leq \frac{\kappa}{2} \Delta t |\Delta\theta^n|^2 + \frac{27 c_b^4}{2\kappa^3} \Delta t |\mathbf{v}^{n-1}|^2 \|\mathbf{v}^{n-1}\|^2 \|\theta^n\|^2. \end{aligned} \tag{3.72}$$

Using the Cauchy–Schwarz inequality, we also have

$$\begin{aligned} 2\Delta t (v_2^n, \Delta\theta^n) &\leq 2\Delta t |\mathbf{v}^n| |\Delta\theta^n| \\ &\leq \frac{\kappa}{2} \Delta t |\Delta\theta^n|^2 + \frac{2}{\kappa} \Delta t |\mathbf{v}^n|^2. \end{aligned} \tag{3.73}$$

Relations (3.71)–(3.73) imply

$$\begin{aligned} \|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \|\theta^n - \theta^{n-1}\|^2 + \kappa \Delta t |\Delta\theta^n|^2 \\ \leq \frac{27 c_b^4}{2\kappa^3} \Delta t |\mathbf{v}^{n-1}|^2 \|\mathbf{v}^{n-1}\|^2 \|\theta^n\|^2 + \frac{2}{\kappa} \Delta t |\mathbf{v}^n|^2, \end{aligned} \tag{3.74}$$

from which, recalling (3.20) and (3.44), we obtain

$$\|\theta^n\|^2 \leq \frac{1}{\alpha}\|\theta^{n-1}\|^2 + \frac{2}{\alpha\kappa}\Delta t\, K_1^2, \tag{3.75}$$

where

$$\alpha = 1 - \frac{27c_b^4}{2\kappa^3}K_1^2K_4^2\Delta t \quad (> 0 \text{ by } (3.67)). \tag{3.76}$$

Using recursively (3.75), we find

$$\begin{aligned}\|\theta^n\|^2 &\leq \left(1 - \frac{27c_b^4}{2\kappa^3}K_1^2K_4^2\Delta t\right)^{-n}\left(\|\theta^0\|^2 + \frac{4\kappa^2}{27c_b^4K_4^2}\right)\\ &\leq 4^{\frac{27c_b^4}{2\kappa^3}K_1^2K_4^2 n\Delta t}\left(\|\theta^0\|^2 + \frac{4\kappa^2}{27c_b^4K_4^2}\right)\\ &\left(\text{since } 1 - x \geq 4^{-x}, x \in \left(0, \frac{1}{2}\right)\right),\end{aligned} \tag{3.77}$$

and thus

$$\begin{aligned}\|\theta^n\|^2 \leq& 4^{\frac{27c_b^4}{2\kappa^3}K_1^2K_4^2(T+T_0)}\left(\|\theta^0\|^2 + \frac{4\kappa^2}{27c_b^4K_4^2}\right)\\ &=: K_5^2(\|\mathbf{v}_0\|, \|\theta_0\|, T+T_0),\ \forall n = 0, \cdots, N+N_0.\end{aligned} \tag{3.78}$$

In order to derive a bound on $\|\theta^n\|^2$ valid for $n \geq N + N_0 + 1$, we rewrite (3.74) in the form (3.63), with

$$\xi_n = \|\theta^n\|^2, \quad \eta_n = \frac{27c_b^4}{2\kappa^3}|\mathbf{v}^{n-1}|^2\|\mathbf{v}^{n-1}\|^2 \quad \text{and} \quad \zeta = \frac{2}{\kappa}|\mathbf{v}^n|^2, \tag{3.79}$$

and for $k_0 \geq N_0 + 1$, we compute (recalling that $|\mathbf{v}^n| < \rho_0$, $|\theta^n| < \rho_0$, for $n \geq N_0$):

$$\begin{aligned}\Delta t\sum_{n=k_0}^{N+k_0}\eta_n =& \frac{27c_b^4}{2\nu^3}\Delta t\sum_{n=k_0}^{N+k_0}|\mathbf{v}^{n-1}|^2\|\mathbf{v}^n\|^2\\ &\leq \frac{27c_b^4}{2\nu^4}\rho_0^2\left[\rho_0^2 + \frac{\rho_0^2}{\nu}(N+1)\Delta t\right] \text{ (by (3.21))},\end{aligned} \tag{3.80}$$

$$\Delta t\sum_{n=k_0}^{N+k_0}\zeta_n = \Delta t\sum_{n=k_0}^{N+k_0}\frac{2}{\kappa}|\mathbf{v}^n|^2 \leq \frac{2}{\kappa}\rho_0^2(N+1)\Delta t, \tag{3.81}$$

$$\begin{aligned}\Delta t\sum_{n=k_0}^{N+k_0}\xi_n =& \Delta t\sum_{n=k_0}^{N+k_0}\|\theta^n\|^2\\ &\leq \frac{1}{\kappa}\left[\rho_0^2 + \frac{\rho_0^2}{\kappa}(N+1)\Delta t\right] \text{ (by (3.22))}.\end{aligned} \tag{3.82}$$

Then conclusion (3.65) of Lemma 6 yields

$$\begin{aligned}\|\theta^n\|^2 &\le \frac{\rho_0^2}{\kappa}\left(\frac{1}{T}+\frac{2}{\kappa}+4T\right)\exp\left\{\frac{54c_b^4}{\nu^4}\rho_0^4\left(1+\frac{2}{\nu}T\right)\right\},\\ &=: K_6^2(T), \quad \forall n \ge N+N_0+1.\end{aligned} \tag{3.83}$$

Combining the above inequality with (3.78) we obtain conclusions (3.68) and (3.70) of the lemma.

Summing (3.74) with n from i to m and using (3.68), as well as (3.20) and (3.44), we obtain (3.69), and this completes the proof of Proposition 2. □

Remark 3.1. The uniform bounds (3.46) and (3.70) imply the existence of an absorbing ball of radius $\rho_1 := \max\{K_3, K_6\}$ in V that absorbs any bounded set in H. Since by the Rellich compactness theorem V is compactly imbedded in H, the existence of the absorbing set in V guarantees the uniform dissipativity of the numerical scheme.

Relations (3.46) and (3.70) also imply the existence of a compact global attractor, $\mathcal{A}_k$, for each time step $k = \Delta t$ satisfying (3.67) and we have the following

Proposition 3 (Uniform dissipativity)**.** The scheme (2.46)– (2.47) possesses an absorbing ball in V and

$$\sup_{(\mathbf{v},\theta)\in K} \|(\mathbf{v},\theta)\| \le \rho_1, \tag{3.84}$$

where $\rho_1 > 0$ is the radius of the absorbing ball and $K = \cup_{0<k\le k_1}\mathcal{A}_k$.

4. Finite Time Uniform Convergence

We now show that condition $H2$ of Theorem 1 is satisfied, that is, the solutions of the numerical scheme converge uniformly (with respect to the initial data from the union of the global attractors) to the solution of the continuous system on the interval $[0,1]$.

For any function ψ and for any $k = \Delta t > 0$, we define the following:

$$\psi_k(t) = \psi^n, \quad t \in [(n-1)k, nk), \tag{4.1}$$

$$\tilde{\psi}_k(t) = \psi^n + \frac{t-nk}{k}(\psi^n - \psi^{n-1}), \quad t \in [(n-1)k, nk). \tag{4.2}$$

With the above notations, equations (2.46) and (2.47) can be rewritten as (for $t \in [(n-1)k, nk)$):

$$\frac{\partial \tilde{\mathbf{v}}_k(t)}{\partial t} + (\mathbf{v}_k(t-k)\cdot\nabla)\mathbf{v}_k(t) - \nu\Delta\mathbf{v}_k(t) + \nabla p_k(t) = e_2\theta_k(t), \tag{4.3}$$

$$\frac{\partial \tilde{\theta}_k(t)}{\partial t} + (\mathbf{v}_k(t-k)\cdot\nabla)\theta_k(t) - (\mathbf{v}_k(t-k))_2 - \kappa\Delta\theta_k(t) = 0, \tag{4.4}$$

or

$$\frac{\partial \tilde{\mathbf{v}}_k(t)}{\partial t} + (\tilde{\mathbf{v}}_k(t)\cdot\nabla)\tilde{\mathbf{v}}_k(t) - \nu\Delta\tilde{\mathbf{v}}_k(t) + \nabla\tilde{p}_k(t) = e_2\tilde{\theta}_k(t) + \mathbf{f}_k(t), \tag{4.5}$$

$$\frac{\partial \tilde{\theta}_k(t)}{\partial t} + (\tilde{\mathbf{v}}_k(t)\cdot\nabla)\tilde{\theta}_k(t) - (\tilde{\mathbf{v}}_k(t))_2 - \kappa\Delta\tilde{\theta}_k(t) = g_k(t), \tag{4.6}$$

where

$$\begin{aligned}\mathbf{f}_k(t) &= (\tilde{\mathbf{v}}_k(t)\cdot\nabla)\tilde{\mathbf{v}}_k(t) - (\mathbf{v}_k(t-k)\cdot\nabla)\mathbf{v}_k(t) - \nu\Delta(\tilde{\mathbf{v}}_k(t) - \mathbf{v}_k(t))\\ &\quad + \nabla(\tilde{p}_k(t) - p_k(t)) - e_2(\tilde{\theta}_k(t) - \theta_k(t)),\end{aligned} \tag{4.7}$$

$$\begin{aligned}g_k(t) &= (\tilde{\mathbf{v}}_k(t)\cdot\nabla)\tilde{\theta}_k(t) - (\mathbf{v}_k(t-k)\cdot\nabla)\theta_k(t)\\ &\qquad - (\tilde{\mathbf{v}}_k(t) - \mathbf{v}_k(t-k))_2 - \kappa\Delta(\tilde{\theta}_k(t) - \theta_k(t)).\end{aligned} \tag{4.8}$$

We now prove that $\mathbf{f}_k$ and g_k are "small" in the following sense:

Lemma 7. For any $T^* > 0$ there exist $K_9(\|\mathbf{v}_0\|, \|\theta_0\|, T^*)$ and $K_{10}(\|\mathbf{v}_0\|, \|\theta_0\|, T^*)$ such that

$$\|\mathbf{f}_k\|^2_{L^2(0,T^*;V_1')} \leq \Delta t K_9, \tag{4.9}$$

and

$$\|g_k\|^2_{L^2(0,T^*;V_2')} \leq \Delta t K_{10}. \tag{4.10}$$

Proof. Let us first note that for any $t \in [(n-1)k, nk)$ we have

$$\tilde{\psi}_k(t) - \psi_k(t-k) = \frac{t-(n+1)k}{k}(\psi^n - \psi^{n-1}), \tag{4.11}$$

$$\tilde{\psi}_k(t) - \psi_k(t) = \frac{t-nk}{k}(\psi^n - \psi^{n-1}). \tag{4.12}$$

Using property (2.32) of the trilinear form b_1, we have

$$\begin{aligned}&\|(\tilde{\mathbf{v}}_k(t)\cdot\nabla)\tilde{\mathbf{v}}_k(t) - (\mathbf{v}_k(t-k)\cdot\nabla)\mathbf{v}_k(t)\|_{V_1'}\\ &\quad \leq c(\|\tilde{\mathbf{v}}_k(t) - \mathbf{v}_k(t-k)\|\|\tilde{\mathbf{v}}_k(t)\| + \|\mathbf{v}_k(t-k)\|\|\tilde{\mathbf{v}}_k(t) - \mathbf{v}_k(t)\|\\ &\quad \leq cK_4\|\mathbf{v}^n - \mathbf{v}^{n-1}\| \quad (\text{by } (4.11), (4.12), (3.44)).\end{aligned} \tag{4.13}$$

We also have

$$\nu\|\Delta(\tilde{\mathbf{v}}_k(t)-\mathbf{v}_k(t))\|_{V_1'} \le \nu\|\mathbf{v}^n-\mathbf{v}^{n-1}\|, \tag{4.14}$$

$$\|e_2(\tilde{\theta}_k(t)-\theta_k(t))\|_{V_1'} \le \|\theta^n-\theta^{n-1}\|. \tag{4.15}$$

Relations (4.13)–(4.15) imply

$$\|\mathbf{f}_k(t)\|_{V_1'} \le cK_8(\|\mathbf{v}^n-\mathbf{v}^{n-1}\|+\|\theta^n-\theta^{n-1}\|), \tag{4.16}$$

and thus

(4.17)

$$\begin{aligned}\|\mathbf{f}_k\|^2_{L^2(0,T^*;V_1')} &= \int_0^{T^*}\|\mathbf{f}_k(t)\|^2_{V_1'}dt = \sum_{n=1}^{N+1}\int_{(n-1)\Delta t}^{n\Delta t}\|\mathbf{f}_k(t)\|^2_{V_1'}dt\\ &\le \Delta t K_9(\|\mathbf{v}_0\|,\|\theta_0\|,T^*) \quad \text{(by (3.45) and (3.69))},\end{aligned}$$

which proves (4.9).

Now using property (2.37) of the trilinear form b_2, we have

(4.18)

$$\begin{aligned}&\|((\tilde{\mathbf{v}}_k(t)\cdot\nabla)\tilde{\theta}_k(t)-\mathbf{v}_k(t-k)\cdot\nabla)\theta_k(t)\|_{V_2'}\\ &\quad\le c(\|\tilde{\mathbf{v}}_k(t)-\mathbf{v}_k(t-k)\|\|\tilde{\theta}_k(t)\|+\|\mathbf{v}_k(t-k)\|\|\tilde{\theta}_k(t)-\theta_k(t)\|\\ &\quad\le cK_4\|\theta^n-\theta^{n-1}\| \quad \text{(by (4.11), (4.12), (3.44))}.\end{aligned}$$

We also have

$$\|(\tilde{\mathbf{v}}_k(t)-\mathbf{v}_k(t-k))_2\|_{V_2'} \le c\|\mathbf{v}^n-\mathbf{v}^{n-1}\|, \tag{4.19}$$

$$\kappa\|\Delta(\tilde{\theta}_k(t)-\theta_k(t))\|_{V_2'} \le c\|\theta^n-\theta^{n-1}\|. \tag{4.20}$$

Relations (4.18)–(4.20) imply

$$\|g_k(t)\|_{V_2'} \le c(\|\mathbf{v}^n-\mathbf{v}^{n-1}\|+(1+K_4)\|\theta^n-\theta^{n-1}\|), \tag{4.21}$$

and thus

(4.22)

$$\begin{aligned}\|g_k\|^2_{L^2(0,T^*;V_2')} &= \int_0^{T^*}\|g_k(t)\|^2_{V_2'}dt = \sum_{n=1}^{N+1}\int_{(n-1)\Delta t}^{n\Delta t}\|g_k(t)\|^2_{V_2'}dt\\ &\le \Delta t K_{10}(\|\mathbf{v}_0\|,\|\theta_0\|,T^*) \quad \text{(by (3.45) and (3.69))},\end{aligned}$$

which proves (4.10).

Thus, the lemma is proved. □

Proposition 4 (Finite time uniform convergence)**.** For any $T^*>0$ we have

$$\lim_{k\to 0}\sup_{(\mathbf{v}_0,\theta_0)\in\mathcal{A}_k,\, nk\in[0,T^*]}|S_k^n(\mathbf{v}_0,\theta_0)-S(nk)(\mathbf{v}_0,\theta_0)|=0. \tag{4.23}$$

Proof. Let

$$\vec{\xi}_k(t) = \mathbf{v}(t) - \tilde{\mathbf{v}}_k(t), \qquad \eta_k(t) = \theta(t) - \tilde{\theta}_k(t). \tag{4.24}$$

Subtracting (4.5) and (4.6) from (2.11) and (2.12), respectively, we obtain

$$\begin{aligned}\frac{\partial \vec{\xi}_k(t)}{\partial t} &+ (\vec{\xi}_k(t)\cdot\nabla)\mathbf{v}(t) + (\tilde{\mathbf{v}}_k(t)\cdot\nabla)\vec{\xi}_k(t) - \nu\Delta\vec{\xi}_k(t) \\ &+ \nabla(p(t) - \tilde{p}_k(t)) = e_2\eta_k(t) - \mathbf{f}_k(t),\end{aligned} \tag{4.25}$$

$$\begin{aligned}\frac{\partial \eta_k(t)}{\partial t} &+ (\vec{\xi}_k(t)\cdot\nabla)\theta(t) + (\tilde{\mathbf{v}}_k(t)\cdot\nabla)\eta_k(t) - (\vec{\xi}_k(t))_2 \\ &- \kappa\Delta\eta_k(t) = -g_k(t).\end{aligned} \tag{4.26}$$

Multiplying (4.25) by $\vec{\xi}_k(t)$ and integrating over Ω, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\vec{\xi}_k(t)|^2 + b_1(\vec{\xi}_k(t), \mathbf{v}(t), \vec{\xi}_k(t)) + \nu\|\vec{\xi}_k(t)\|^2 \\ &\quad = (e_2\eta_k(t), \vec{\xi}_k(t)) - (\mathbf{f}_k(t), \vec{\xi}_k(t)).\end{aligned} \tag{4.27}$$

Using property (2.32) of the form b_1, we bound the nonlinear term as

$$\begin{aligned}b_1(\vec{\xi}_k(t), \mathbf{v}(t), \vec{\xi}_k(t)) &\le c|\vec{\xi}_k(t)|\|\vec{\xi}_k(t)\|\|\mathbf{v}(t)\| \\ &\le \frac{\nu}{6}\|\vec{\xi}_k(t)\|^2 + \frac{c}{\nu}|\vec{\xi}_k(t)|^2\|\mathbf{v}(t)\|^2.\end{aligned} \tag{4.28}$$

Using the Cauchy–Schwarz inequality, we also have

$$\begin{aligned}|(e_2\eta_k(t), \vec{\xi}_k(t))| &\le |\eta_k(t)||\vec{\xi}_k(t)| \\ &\le |\eta_k(t)|\|\vec{\xi}_k(t)\| \\ &\le \frac{\nu}{6}\|\vec{\xi}_k(t)\|^2 + \frac{c}{\nu}|\eta_k(t)|^2,\end{aligned} \tag{4.29}$$

$$\begin{aligned}|(\mathbf{f}_k(t), \vec{\xi}_k(t))| &\le \|\mathbf{f}_k(t)\|_{V'}\|\vec{\xi}_k(t)\| \\ &\le \frac{\nu}{6}\|\vec{\xi}_k(t)\|^2 + \frac{c}{\nu}\|\mathbf{f}_k(t)\|^2_{V'}.\end{aligned} \tag{4.30}$$

Relations (4.27)–(4.30) imply

$$\begin{aligned}\frac{d}{dt}|\vec{\xi}_k(t)|^2 + \nu\|\vec{\xi}_k(t)\|^2 \le &\frac{c}{\nu}\|\mathbf{v}(t)\|^2|\vec{\xi}_k(t)|^2 \\ &+ \frac{c}{\nu}|\eta_k(t)|^2 + \frac{c}{\nu}\|\mathbf{f}_k(t)\|^2_{V'}.\end{aligned} \tag{4.31}$$

Now multiplying (4.26) by $\eta_k(t)$ and integrating over Ω, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\eta_k(t)|^2 + b_2(\vec{\xi}_k(t), \theta(t), \eta_k(t)) - ((\vec{\xi}_k(t))_2, \eta_k(t)) \\ &\quad + \kappa\|\eta_k(t)\|^2 = -(g_k(t), \eta_k(t)).\end{aligned} \tag{4.32}$$

Using property (2.37) of the form b_2, we bound the nonlinear term as

$$b_2(\vec{\xi}_k(t),\theta(t),\eta_k(t)) \le c|\vec{\xi}_k(t)|^{1/2}\|\vec{\xi}_k(t)\|^{1/2}\|\theta(t)\||\eta_k(t)|^{1/2}\|\eta_k(t)\|^{1/2} \le \frac{\nu}{6}\|\vec{\xi}_k(t)\|^2+\frac{\kappa}{6}\|\eta_k(t)\|^2 + \frac{c}{\nu}\|\theta(t)\|^2|\vec{\xi}_k(t)|^2+\frac{c}{\kappa}\|\theta(t)\|^2|\eta_k(t)|^2. \tag{4.33}$$

Using the Cauchy–Schwarz inequality, we also have the following bounds:

$$|((\vec{\xi}_k(t))_2,\eta_k(t))| \le |\vec{\xi}_k(t)||\eta_k(t)| \le \frac{\kappa}{6}\|\eta_k(t)\|^2+\frac{c}{\kappa}|\vec{\xi}_k(t)|^2, \tag{4.34}$$

$$|(g_k(t),\eta_k(t))| \le \|g_k(t)\|_{V'}\|\eta_k(t)\| \le \frac{\kappa}{6}\|\eta_k(t)\|^2+\frac{c}{\kappa}\|g_k(t)\|_{V'}^2. \tag{4.35}$$

Relations (4.32)–(4.35) imply

$$\frac{d}{dt}|\eta_k(t)|^2+\kappa\|\eta_k(t)\|^2 \le \frac{\nu}{3}\|\vec{\xi}_k(t)\|^2+\frac{c}{\nu}\|\theta(t)\|^2|\vec{\xi}_k(t)|^2 + \frac{c}{\kappa}\|\theta(t)\|^2|\eta_k(t)|^2+\frac{c}{\kappa}|\vec{\xi}_k(t)|^2 + \frac{c}{\kappa}\|g_k(t)\|_{V'}^2. \tag{4.36}$$

Adding (4.31) and (4.36), we obtain

$$\begin{aligned}\frac{d}{dt}(|\vec{\xi}_k(t)|^2+|\eta_k(t)|^2)+\frac{2}{3}\nu\|\vec{\xi}(t)\|^2+\kappa\|\eta(t)\|^2 &\\ \le \frac{c}{\nu}\left(\|\mathbf{v}(t)\|^2+\|\theta(t)\|^2+\frac{\nu}{\kappa}\right)|\vec{\xi}_k(t)|^2 &\\ + c\left(\frac{1}{\nu}+\frac{1}{\kappa}\|\theta(t)\|^2\right)|\eta_k(t)|^2 &\\ + \frac{c}{\nu}\|\mathbf{f}_k(t)\|_{V'}^2+\frac{c}{\kappa}\|g_k(t)\|_{V'}^2.&\end{aligned} \tag{4.37}$$

Since the solution $(\mathbf{v},\theta)$ of the continuous problem is uniformly bounded in V for all $t\ge 0$ (cf. [22]), inequality (4.37) becomes

$$\begin{aligned}\frac{d}{dt}(|\vec{\xi}_k(t)|^2+|\eta_k(t)|^2)+\frac{2}{3}\nu\|\vec{\xi}(t)\|^2+\kappa\|\eta(t)\|^2 &\\ \le K_{11}(|\vec{\xi}_k(t)|^2+|\eta_k(t)|^2)+\frac{c}{\nu}\|\mathbf{f}_k(t)\|_{V'}^2+\frac{c}{\kappa}\|g_k(t)\|_{V'}^2,&\end{aligned} \tag{4.38}$$

where $K_{11}=K_{11}(\|\mathbf{v}_0\|,\|\theta_0\|)$ depends on the initial data, but is independent of t.

By Gronwall's lemma and using the fact that $\vec{\xi}_k(0) = \eta(0) = 0$, we obtain

$$\begin{aligned} |\vec{\xi}_k(t)|^2 + |\eta_k(t)|^2 &\le K_{12}(\|\mathbf{f}_k\|^2_{L^2(0,T^*;V_1')} + \|g_k\|^2_{L^2(0,T^*;V_2')}) \\ &\le K_{12}\Delta t(K_9 + K_{10}) \quad \text{(by (4.9) and (4.10))}, \end{aligned} \tag{4.39}$$

where K_{12} depends on the initial data. Taking $(\mathbf{v}_0, \theta_0)$ in K and using (3.84), the bound above becomes independent on the initial data and we obtain

$$|\vec{\xi}_k(t)|^2 + |\eta_k(t)|^2 \le c\Delta t, \tag{4.40}$$

for some constant $c > 0$.

Thus,

$$\begin{aligned} &\lim_{k\to 0} \sup_{(\mathbf{v}_0,\theta_0)\in\mathcal{A}_k,\, nk\in[0,T^*]} |S_k^n(\mathbf{v}_0,\theta_0) - S(nk)(\mathbf{v}_0,\theta_0)| \\ &= \lim_{k\to 0} \sup_{(\mathbf{v}_0,\theta_0)\in\mathcal{A}_k,\, nk\in[0,T^*]} |(\mathbf{v}^n,\theta^n) - (\mathbf{v}(nk),\theta(nk))| \\ &= \lim_{k\to 0} \sup_{(\mathbf{v}_0,\theta_0)\in\mathcal{A}_k,\, nk\in[0,T^*]} |(\tilde{\mathbf{v}}_k(nk),\tilde{\theta}_k(nk)) - (\mathbf{v}(nk),\theta(nk))| \\ &= \lim_{k\to 0} \sup_{(\mathbf{v}_0,\theta_0)\in\mathcal{A}_k,\, nk\in[0,T^*]} |(\vec{\xi}_k(nk),\eta_k(nk))| = 0, \end{aligned} \tag{4.41}$$

and the lemma is proved. □

5. Finite Time Uniform Continuity

The last step in proving the convergence of the stationary statistical properties of the numerical scheme to those of the continuous dynamical system is to show that condition $H3$ of Theorem 1 is satisfied, that is, the solution of the continuous system is uniformly (with respect to the initial data from the union of the global attractors) continuous on the unit time interval $[0, 1]$.

In order to do that, we need to note that we have the following bounds on the solution $(\mathbf{v}, \theta)$ to the continuous problem (2.11)–(2.16) (see, e.g., [22]), for any initial data $(\mathbf{v}_0, \theta_0) \in K$:

$$\|(\mathbf{v},\theta)\|_{L^\infty(0,T^*;V)} \le c, \tag{5.1}$$

$$\|(\Delta\mathbf{v},\Delta\theta)\|_{L^2(0,T^*;L^2(\Omega)^3)} \le c. \tag{5.2}$$

Proposition 5 (Finite time uniform continuity)**.** For any $T^* \in [0, 1]$ we have

$$\lim_{t\to T^*} \sup_{(\mathbf{v}_0,\theta_0)\in K} |S(t)(\mathbf{v}_0,\theta_0) - S(T^*)(\mathbf{v}_0,\theta_0)| = 0. \tag{5.3}$$

Proof. For any $(\mathbf{v}_0, \theta_0) \in K$, we have

(5.4)
$$\begin{aligned} |S(t)(\mathbf{v}_0,\theta_0) - S(T^*)(\mathbf{v}_0,\theta_0)| &= |(\mathbf{v}(t) - \mathbf{v}(T^*), \theta(t) - \theta(T^*))| \\ &= \left| \left(\int_t^{T^*} \frac{d\mathbf{v}}{ds}(s)ds, \int_t^{T^*} \frac{d\theta}{ds}(s)ds \right) \right| \end{aligned}$$

Integrating (2.11) from t to T^*, we obtain

(5.5)
$$\begin{aligned} \left| \int_t^{T^*} \frac{d\mathbf{v}}{ds}(s)ds \right| &= \left| \int_t^{T^*} ((\mathbf{v}(s)\cdot\nabla)\mathbf{v}(s) - \nu\Delta\mathbf{v}(s) + \nabla p(s) + e_2\theta(s))ds \right| \\ &\le c \left| \int_t^{T^*} (|\mathbf{v}(s)|^{\frac{1}{2}} |\Delta\mathbf{v}(s)|^{\frac{1}{2}} \|\mathbf{v}(s)\| + |\Delta\mathbf{v}(s)| + |\theta(s)|)ds \right| \\ &\le c\sqrt{|T^* - t|} \quad \text{(by (5.1) and (5.2))}. \end{aligned}$$

Similarly, integrating (2.12) from t to T^*, we obtain

(5.6)
$$\begin{aligned} \left| \int_t^{T^*} \frac{d\theta}{ds}(s)ds \right| &= \left| \int_t^{T^*} ((\mathbf{v}(s)\cdot\nabla)\theta(s) - v_2(s) - \kappa\Delta\theta(s))ds \right| \\ &\le c \left| \int_t^{T^*} (|\mathbf{v}(s)|^{\frac{1}{2}} |\Delta\mathbf{v}(s)|^{\frac{1}{2}} \|\theta(s)\| + |\Delta\mathbf{v}(s)| + |\mathbf{v}(s)| + \kappa|\Delta\theta(s)|)ds \right| \\ &\le c\sqrt{|T^* - t|} \quad \text{(by (5.1) and (5.2))}. \end{aligned}$$

Combining (5.5) and (5.6), we obtain conclusion (5.3) of the proposition. Thus, Proposition 5 is proved. □

Propositions 3, 4 and 5, as well as Theorem 1, enable us to draw the following conclusion, which represents our main result:

Theorem 2. Let $\{S_k, 0 < k < k_1\}$ be the family of continuous semigroups on H that associates with any $(\mathbf{v}^{n-1}, \theta^{n-1}) \in H$ the unique solution, $(\mathbf{v}^n, \theta^n)$, to (2.46)–(2.47) and let $\{S(t), t > 0\}$ be the continuous semigroup on H that associates with any $(\mathbf{v}^0, \theta^0) \in H$ the unique solution, $(\mathbf{v}(t), \theta(t))$, to (2.11)–(2.18). Then the stationary statistical properties of the discrete dynamical system generated by $\{S_k, 0 < k < k_1\}$ converge to those of the continuous dynamical system generated by $\{S(t), t > 0\}$ as the time step approaches zero.

Besides the above result on the convergence of the statistical properties, we have also obtained a result on the convergence of attractors. In order to see that, we recall the following theorem (see, e.g., [26]):

Theorem 3 (Convergence of Global attractors)**.** Let $\{S(t), t > 0\}$ be a continuous semigroup on a Banach space H which generates a continuous dissipative dynamical system (in the sense of possessing a compact global attractor $\mathcal{A}$) on H. Let $\{S_k, 0 < k < k_0\}$ be a family of continuous maps on H which generates a family of discrete dissipative dynamical systems (with global attractor $\mathcal{A}_k$) on H. Suppose that the following two conditions are satisfied:

$H4$: [Uniform boundedness] There exists $k_1 \in (0, k_0]$ such that $\{S_k, 0 < k < k_1\}$ is uniformly bounded in the sense that

$$K = \cup_{0<k\leq k_1} \mathcal{A}_k \tag{5.7}$$

is bounded in H.

$H5$: [Finite time uniform convergence] S_k uniformly converges to S on any finite time interval (modulo an initial layer) and uniformly for initial data from the global attractor of S_k in the sense that there exists $t_0 > 0$ such that for any $T^* > t_0 > 0$

$$\lim_{k\to 0} \sup_{\mathbf{u}\in\mathcal{A}_k,\, nk\in[t_0,T^*]} \|S_k^n \mathbf{u} - S(nk)\mathbf{u}\| = 0. \tag{5.8}$$

Then the global attractors converge in the sense of Hausdorff semi-distance, i.e.,

$$\lim_{k\to 0} dist_H(\mathcal{A}_k, \mathcal{A}) = 0, \tag{5.9}$$

where

$$dist_H(\mathcal{A}_k, \mathcal{A}) = \sup_{x\in\mathcal{A}_k} \inf_{y\in\mathcal{A}} \|x - y\|_H. \tag{5.10}$$

Propositions 3 and 4 guarantee that conditions (5.7) and (5.8) of the above theorem are satisfied, and therefore, we have the following

Theorem 4. Let $\{S_k, 0 < k < k_1\}$ be the family of continuous semigroups on H that associates with any $(\mathbf{v}^{n-1}, \theta^{n-1}) \in H$ the unique solution, $(\mathbf{v}^n, \theta^n)$, to (2.46)–(2.47) and let $\{S(t), t > 0\}$ be the continuous semigroup on H that associates with any $(\mathbf{v}^0, \theta^0) \in H$ the unique solution, $(\mathbf{v}(t), \theta(t))$, to (2.11)–(2.18). Then the global attractors $\mathcal{A}_k$ of the discrete dynamical system generated by $\{S_k, 0 < k < k_1\}$ converge to the global attractor $\mathcal{A}$ of the continuous dynamical system generated by $\{S(t), t > 0\}$, where the convergence is understood in the sense of the semi-distance defined in (5.10).

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