Applicable Analysis, Vol. 51, pp. 35-40
Reprints available directly from the publisher
Photocopying permitted by license only

A Remark on the Characterization of the Gradient of A Distribution

Communicated by R. Temam

XIAOMING WANG

The Institute for Scientific Computing & Applied Mathematics
and Department of Mathematics
Indiana University, Bloomington, IN 47405

AMS: 26B20, 26B05

ABSTRACT. It is shown in this paper that a scalar test function in $\mathbf{R}^n$ is the divergence of a vector valued test function if and if its integral over the domain vanishes. Using this result, we are able to give an elementary proof of the characterization of the gradient of a distribution. Also an application on characterization of the curl of a distribution in $\mathbf{R}^2$ is presented.

KEY WORDS: Test function, distribution

(Received for Publication 22 February 1993)

One of the basic results about distributions that has been frequently used in fluid mechanics for the theory of Navier-Stokes equations is the characterization of the gradient of a distribution, which says

Theorem *Let Ω be an open set in $\mathbf{R}^n$ and $\mathbf{f} = (f_1, \cdots, f_n) \in (\mathcal{D}'(\Omega))^n$. A necessary and sufficient condition that $\mathbf{f} = grad(p)$ for some p in $\mathcal{D}'(\Omega\ )$ is that*

$$< \mathbf{f}, \mathbf{v} >_{((\mathcal{D}'(\Omega))^n,\ \mathcal{D}(\Omega)^n)} = 0, \quad \forall \mathbf{v} \in \mathcal{V},$$

where

$$\mathcal{V}=\{\ \mathbf{v} \in (\mathcal{D}(\Omega))^n, \ \ div\ \mathbf{v}=0\ \}$$

Also related, in the theory of mathematical physics, it is often important to know whether a given vector field is conservative, that is whether it is the gradient of a scalar field. However, to the author's knowledge, this important property was only proved by using a profound result on currents of de Rham (Lions[2], Témam[4]). Here we present an elementary argument using only integration by parts. Instead of working on distributions, we work on test functions and prove a decomposition theorem for them and then use duality to construct the distribution p stated in the theorem. The decomposition result on test functions seems new and seems to be of interest on its own.

Definition. An open set Ω in $\mathbf{R}^n$ is said to have property (P), if for any $\phi \in \mathcal{D}(\Omega)$ with $\int_\Omega \phi(x)\,dx = 1$ and for any $u \in \mathcal{D}(\Omega)$, there exists $\mathbf{v} \in (\mathcal{D}(\Omega))^n$, such that

$$u = div\,\mathbf{v} + \phi \int_\Omega u\,dx \tag{1}$$

Proposition *Any open connected set in $\mathbf{R}^n$ has property (P). Moreover, the* $\mathbf{v}$ *in the definition can be chosen in such a way that it depends continously on u and ϕ for the usual topology of $\mathcal{D}(\Omega)$(Schwartz[3]).*

Proof. We prove this proposition in three steps.

Step 1. Any n-dimensional rectangle has property (P).

Without loss of generality, we can assume that $\Omega=(0,1)^n$. Indeed, given u and ϕ, we choose $\phi_i \in \mathcal{D}(0,1)$, i=1, $\cdots$, n with $\int_0^1 \phi_i(t)\,dt = 1$ and let

$$w = u - \phi \int_\Omega u \tag{2}$$

then

$$\int_\Omega w = 0 \tag{3}$$

We can write $w = u - \phi \int_\Omega u = div\,\mathbf{v}$ and in this case we have an explicit form for $\mathbf{v}$(Adams[1]). Indeed

$$v_1(x_1, x') = \int_0^{x_1} [w(t, x') - \phi_1(t) \int_0^1 w(s_1, x')\,ds_1]\,dt$$

$$v_2(x_1, x_2, x') = \int_0^{x_2} \phi_1(x_1)[\int_0^1 w(s_1, t, x'') - \phi_2(t) \int_0^1 \int_0^1 w(s_1, s_2, x'')\,ds_1\,ds_2]\,dt$$

.

.

.

$$v_n(x_1, \cdots, x_n)$$

$$= \int_0^{x_n} \phi_1(x_1)\cdots\phi_{n-1}(x_{n-1})[\int_0^1 \cdots \int_0^1 w(s_1, \cdots, s_{n-1}, t)\,ds_1 \cdots ds_{n-1} - \phi_n(t) \int_\Omega w]\,dt$$

$$= \int_0^{x_n} \phi_1(x_1)\cdots\phi_{n-1}(x_{n-1}) \int_0^1 \cdots \int_0^1 w(s_1, \cdots, s_{n-1}, t)\,ds_1 \cdots ds_{n-1}\,dt$$

It is obvious that $\mathbf{v}$ depends continously on u and ϕ and it is easily checked that $div\,\mathbf{v}=w$, hence $u = div\,\mathbf{v} + \phi \int_\Omega u$.

Step 2. If Ω_1 and Ω_2 have property (P) and $\Omega_1 \cap \Omega_2$ is nonempty, then $\Omega_1 \cup \Omega_2$ has property (P) too.

Hence we are given $u, \phi \in \mathcal{D}(\Omega_1 \cup \Omega_2)$, with $\int_{\Omega_1 \cup \Omega_2} \phi = 1$ and we look for $\mathbf{v}$ such that (1) holds.

Choose $\psi \in \mathcal{D}(\mathbf{R}^n)$ such that $\text{supp}(\psi) \Subset \Omega_1 \cap \Omega_2$ and $\int \psi = 1$. Choose a partition of unity of $\Omega_1 \cup \Omega_2$ subordinated to the covering Ω_1, Ω_2, i.e. choose $\theta_1, \theta_2 \in \mathcal{D}(\mathbf{R}^n)$ such that

(i) $\theta_1 + \theta_2 = 1$ on $\text{supp}(u) \cup \text{supp}(\phi)$

(ii) $\text{supp}(\theta_i) \Subset \Omega_i$, i=1,2.

Hence

$$\theta_i u \in \mathcal{D}(\Omega_i), \ \theta_i \phi \in \mathcal{D}(\Omega_i), \text{ i=1,2.}$$

Since Ω_i have property (P) and $\psi \in \mathcal{D}(\mathbf{R}^n)$ with $\text{supp}(\psi) \subset \Omega_1 \cap \Omega_2$, $\int \psi=1$, there exist $\mathbf{v}_i, \mathbf{w}_i \in \mathcal{D}(\Omega_i)$ such that

$$\theta_i u = div\ \mathbf{v}_i + \psi \int \theta_i u \tag{4}$$

$$\theta_i \phi = div\ \mathbf{w}_i + \psi \int \theta_i \phi \tag{5}$$

Adding (4) and (5) for i=1,2 respectively, we find

$$u = div\ (\mathbf{v}_1 + \mathbf{v}_2) + \psi \int u \tag{6}$$

$$\phi = div\ (\mathbf{w}_1 + \mathbf{w}_2) + \psi \int \phi = div\ (\mathbf{w}_1 + \mathbf{w}_2) + \psi \tag{7}$$

Hence

$$u = div((\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{w}_1 + \mathbf{w}_2) \int u) + \phi \int u \tag{8}$$

Step 3. Any open connected set in $\mathbf{R}^n$ has property (P).

Let ϕ, $u \in \mathcal{D}(\Omega)$ be given with $\int \phi\, dx = 1$, then $\text{supp}(\phi) \cup \text{supp}(u)$ is a compact subset of Ω. Thus there exist finitely many n-dimensional rectangles, say $R_j, j = 1, \cdots, m$ such that

(i) $supp(\phi) \cup supp(u) \subset \bigcup_{j=1}^m (R_j)$,

(ii) $\bigcup_{j=1}^m (R_j)$ is connected,

(iii) $\bigcup_{j=1}^m (R_j) \subset \Omega$.

Claim $\bigcup_{j=1}^m (R_j)$ has property (P).

Each R_j has property (P) by step1. $R_1 \cup (\cup_{j=2}^m (R_j))$ is connected, thus there exists a j_2, such that $R_1 \cap R_{j_2}$ is non-empty. Thanks to step2, $R_{j_1} \cup R_{j_2}$ has property (P), where $j_1 = 1$. Now that $(R_{j_1} \cup R_{j_2}) \cup (\cup_{j \neq j_1, j_2} R_j)$ is connected, hence there exists a j_3, $j_3 \neq j_1$ *or* j_2, such that $R_{j_3} \cap (R_{j_1} \cup R_{j_2})$ is non-empty. Hence $R_{j_1} \cup R_{j_2} \cup R_{j_3}$ has property (P). We proceed in this way and within finitely many steps, we will exhaust all R_j. Therefore, the claim is proved.

Now there exists a $\mathbf{v} \in (\mathcal{D}(\cup_{j=1}^m R_j))^n \hookrightarrow (\mathcal{D}(\Omega))^n$

$$u = div\, \mathbf{v} + \phi \int u\, dx$$

Now it is clear that in each step $\mathbf{v}$ depends continuously on u by our construction.
Q.E.D.

Corollary *Let Ω be an open connected set in $\mathbf{R}^n$ and $u \in \mathcal{D}(\Omega)$, then $\int_\Omega u = 0$ if and only if there exists a $\mathbf{v} \in (\mathcal{D}(\Omega))^n$, such that u=$div\, \mathbf{v}$.*

Proof Obvious.

Now we are ready for the proof of the theorem.

Proof of the Theorem Without loss of generality, we assume that Ω is connected. Otherwise we just discuss the result in each connected component.

Choose a $\phi \in \mathcal{D}(\Omega)$ with $\int_\Omega \phi\, dx = 1$. According to the proposition, $\forall u \in \mathcal{D}(\Omega)$, there exists a $\mathbf{v} \in (\mathcal{D}(\Omega))^n$ such that

$$u = div\, \mathbf{v} + \phi \int u\, dx \tag{9}$$

Define a functional p on $\mathcal{D}(\Omega)$ in the following way

$$p(u) = - < \mathbf{f}, \mathbf{v} >_{((\mathcal{D}'(\Omega))^n,\, (\mathcal{D}(\Omega))^n)} \tag{10}$$

Claim 1. p is well-defined, i.e. it is independent of the choice of $\mathbf{v}$ such that (9) holds.

Say u has two decompositions

$$\begin{aligned} u &= div\, \mathbf{v}_1 + \phi \int u \\ &= div\, \mathbf{v}_2 + \phi \int u \end{aligned}$$

then

$$div(\mathbf{v}_1 - \mathbf{v}_2) = 0, \quad \text{i.e. } \mathbf{v}_1 - \mathbf{v}_2 \in \mathcal{V}$$

So

$$\begin{aligned} - < \mathbf{f}, \mathbf{v}_1 > &= - < \mathbf{f}, \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_2 > \\ &= - < \mathbf{f}, \mathbf{v}_1 - \mathbf{v}_2 > - < \mathbf{f}, \mathbf{v}_2 > \\ &= - < \mathbf{f}, \mathbf{v}_2 > \end{aligned}$$

Claim 2. p is linear.

Let

$$u_1 = div\ \mathbf{v}_1 + \phi \int u_1$$
$$u_2 = div\ \mathbf{v}_2 + \phi \int u_2$$

$$\alpha u_1 + \beta u_2 = div(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) + \phi \int (\alpha u_1 + \beta u_2), \qquad \alpha, \beta \in \mathbf{R}^1$$
$$p(\alpha u_1 + \beta u_2) = - < \mathbf{f}, \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 >$$
$$= -\alpha < \mathbf{f}, \mathbf{v}_1 > -\beta < \mathbf{f}, \mathbf{v}_2 >$$
$$= \alpha p(u_1) + \beta p(u_2)$$

Claim 3. $p(u)$ depends continuously on u for the topology of $\mathcal{D}(\Omega)$ i.e. $p \in \mathcal{D}'(\Omega)$.

Say $u_j, u \in \mathcal{D}(\Omega), u_j \longrightarrow u$ in $\mathcal{D}'(\Omega)$. Thus $supp(u) \cup (\cup_{j \geq 1} supp(u_j))$ is included in a compact subset of Ω. The constructive procedure of our proposition tells us that we can find $\mathbf{v}$, $\mathbf{v}_j \in (\mathcal{D}(\Omega))^n$ such that

$$u = div\ \mathbf{v} + \phi \int u$$
$$u_j = div\ \mathbf{v}_j + \phi \int u_j$$
$$\mathbf{v}_j \longrightarrow \mathbf{v} \quad \text{in} \quad (\mathcal{D}(\Omega))^n$$

Hence

$$p(u_j) = - <\mathbf{f}, \mathbf{v}_j > \longrightarrow - <\mathbf{f}, \mathbf{v} >= p(u)$$

Claim 4. grad$(p) = \mathbf{f}$

Indeed, for every $\mathbf{w} \in (\mathcal{D}(\Omega))^n$,

$$< \text{grad}(p), \mathbf{w} >_{((\mathcal{D}')^n, \mathcal{D}^n)} = - < p, div\ \mathbf{w} >_{(\mathcal{D}', \mathcal{D})}$$
$$= -p(div\ \mathbf{w}) =< \mathbf{f}, \mathbf{w} > \qquad \text{by (10)}$$

Hence

$$\text{grad}(p) = \mathbf{f}.$$

Q.E.D.

As an immediate application of our proposition on test functions, we present a characterization of the curl of a distribution in $\mathbf{R}^2$.

Proposition′ *Let Ω be an open connected set in $\mathbf{R}^2$ and let $\phi \in \mathcal{D}(\Omega)$ with $\int_\Omega \phi = 1$, then for any $u \in \mathcal{D}(\Omega)$, there exists a $\mathbf{w} \in (\mathcal{D}(\Omega))^2$, such that*

$$u = curl\ \mathbf{w} + \phi \int_\Omega u$$

Moreover, the $\mathbf{w}$ can be chosen in such a way that it depends continuously on u and ϕ for the usual topology of $\mathcal{D}(\Omega)$.

Proof Recall that for $\mathbf{w} \in (\mathcal{D}(\Omega))^2$, $curl\ \mathbf{w} = \dfrac{\partial w_2}{\partial x_1} - \dfrac{\partial w_1}{\partial x_2}$. Let $\mathbf{v}$ be the test function provided by the previous proposition, set $\mathbf{w} = (-v_2, v_1)$. It is easy to see that this $\mathbf{w}$ does the job. Q.E.D.

Theorem′ *Let Ω be an open set in $\mathbf{R}^2$ and $\mathbf{f} = (f_1, f_2) \in (\mathcal{D}'(\Omega))^2$. A necessary and sufficient condition that $\mathbf{f} = curl\ (g)$ for some g in $\mathcal{D}'(\Omega\ )$ is that*

$$< \mathbf{f}, \mathbf{w} >_{((\mathcal{D}'(\Omega))^2,\ \mathcal{D}(\Omega)^2)} = 0, \quad \forall \mathbf{w} \in \mathcal{W},$$

where

$$\mathcal{W} = \{\ \mathbf{w} \in (\mathcal{D}(\Omega))^2,\ \ curl\ \mathbf{w} = 0\ \}$$

Proof The proof is exactly the same as the one for the previous theorem.

Assume that Ω is connected and $\phi \in \mathcal{D}(\Omega)$ with $\int_\Omega \phi\ dx = 1$. According to proposition′, $\forall u \in \mathcal{D}(\Omega)$, there exists a $\mathbf{w} \in (\mathcal{D}(\Omega))^2$ such that

$$u = curl\ \mathbf{w} + \phi \int u\, dx \tag{11}$$

Define a functional g on $\mathcal{D}(\Omega)$ in the following way

$$g(u) = - < \mathbf{f}, \mathbf{w} >_{((\mathcal{D}'(\Omega))^2,\ (\mathcal{D}(\Omega))^2)} \tag{12}$$

Repeat the same procedure as we did in the proof of the previous theorem, we see that g is well defined, linear, continous and satifies *curl (g)=f*. Q.E.D.

Acknowledgments: The author would like to express his sincere thanks to Professors Roger Témam and Ciprian Foias for their encouragement. He especially wishes to thank Prof. Témam who took the painful job reading the hand written manuscript and made a lot of helpful comments.

REFERENCES

[1] R.S.Adams, *Sobolev Spaces*, Academic Press, New York, (1975).
[2] J.L.Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, (1967).
[3] L.Schwartz, *Théorie des Distribution I, II*, (2nd edition, 1957), Hermann, Paris, (1950-1951).
[4] R.Témam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, (1984).