BOUNDARY LAYER ASSOCIATED WITH A CLASS OF 3D NONLINEAR PLANE PARALLEL CHANNEL FLOWS

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1. INTRODUCTION

The dynamics of the viscous incompressible flow is governed by the classical incompressible Navier-Stokes equations (NSE) for Newtonian fluids [19, 11]:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \epsilon\Delta\mathbf{v} + \nabla p = \mathbf{F}, \nabla\cdot\mathbf{v} = 0, \tag{1.1}$$

where $\mathbf{v}$ is the Eulerian fluid velocity, p is the kinematic pressure, ϵ is the kinematic viscosity and $\mathbf{F}$ is a (given) applied external body force. The system is equipped with an initial condition $\mathbf{v}_0$ and the no-slip no-penetration boundary condition

$$\mathbf{v}\big|_{\partial\Omega} = 0. \tag{1.2}$$

If the kinematic viscosity is small (or the Reynolds number is large) such as in air and water, we may formally set the viscosity to zero in the Navier-Stokes system and we arrive at the Euler system for incompressible inviscid flows:

$$\frac{\partial \mathbf{v}^0}{\partial t} + (\mathbf{v}^0\cdot\nabla)\mathbf{v}^0 + \nabla p^0 = \mathbf{F}, \nabla\cdot\mathbf{v}^0 = 0. \tag{1.3}$$

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The Euler system is equipped with the same initial condition but a different, no-penetration, boundary condition

$$\mathbf{v}\big|_{\partial\Omega} = 0, \tag{1.4}$$

since we have dropped the highest order spatial derivative.

This heuristic limit can not be valid uniformly over space due to the disparity of boundary conditions. This disparity usually leads to the emergence of a thin layer near the boundary of the domain, the so-called boundary layer, where the viscous flow makes a sharp transition from the almost inviscid flow in the interior of the domain to the zero value at the boundary [18].

Boundary layers are of great importance since it is the vorticity generated in the boundary layer and later advected into the main stream that drives the flow in many physical applications. On the other hand, the existence of a boundary layer renders the inviscid limit problem a particularly singular one and hence its analysis a challenge.

Standard classical approach to the boundary layer problem is by approximating the viscous (NSE) solution within the boundary layer via the so-called Prandtl equation [16]. Here we take a slightly different approach and derive Prandtl-type effective equation for the *corrector* $\boldsymbol{\theta}$ that approximates $\mathbf{v} - \mathbf{v}^0$ [23, 6]. This alternative approach has the advantage that the matching procedure is conceptually simple: the sum of the inviscid solution and the corrector is a natural candidate for an approximation to the viscous solution. The analysis of the boundary layer problem then consists of the study of the Prandtl-type equation, and proof of convergence of the approximate solution to the exact solution of the Navier-Stokes system.

There exists an abundant literature on boundary layer theory [15, 18] and the vanishing viscosity limit (we refer in particular to [1, 2, 3, 4, 7, 8, 9, 10, 12, 13] and references therein). However, few examples of nonlinear boundary layers exist in the literature (see [17] for the case of half space in the analytical setting, [21, 22] for the case of channel flow with uniform injection and suction at the boundary among others). The main purpose of this manuscript is to investigate the boundary layer associated with a class of nonlinear plane parallel flows. This family of nonlinear solutions to the Navier-Stokes system was introduced in [24] where the vanishing viscosity limit was established. More recently, the boundary layer for this class of flows was studied in [14] using semiclassical expansions for heat equations with drifts without referring to the Prandtl theory, and certain $L^\infty(L^\infty)$ convergence results derived. In both cases, Couette-type flows are considered, with nonhomogeneous characteristic data. The current manuscript follows the Prandtl type approach, and presents a more detailed analysis of the boundary associated with this class of exact solutions and hence will go beyond the results derived in those two previous works. Our main result (Theorem 5.2) provides error bounds for the approximation of the NSE solution given by the Euler solution plus the corrector, and hence convergence rates in the vanishing viscosity limit (Corollary 5.3 and 5.4). We consider both L^∞ and Sobolev H^1 bounds, which give information on the possible growth of normal derivatives in the boundary layer. We establish these results under some compatibility conditions on the initial and boundary data and body force.

The manuscript is organized as follows. In section 2 we recall the family of nonlinear 3D plane parallel channel flow. Section 3 is devoted to formal asymptotic expansion of this class of flows at small viscosity utilizing the Prandtl type (*corrector*) approach. We construct an approximate solution to the Navier-Stokes system utilizing the solution to the Prandtl type system (corrector) and the solution to the Euler system in Section 4. The main convergence result in provided in Section 5. Improved estimates with higher order approximations are provided in Section 6. Decay estimates of the correctors are furnished in Appendix A.

Throughout the paper, we use C to denote generic constants that may vary line by line, but is independent of the kinematic viscosity ϵ.

2. NONLINEAR PLANE-PARALLEL CHANNEL FLOWS

We start by recalling the ansatz for a plane-parallel channel flow introduced in [24]. That is, we look for solutions of the fluid equations of the form:

$$\mathbf{v}(t,x,y,z) = (u_1(t,z), u_2(t,x,z), 0) \tag{2.1}$$

in an infinitely long horizontal channel, but we impose periodicity in the horizontal coordinates x, y. We hence reduce to work in the spatial domain $Q := [0,L] \times [0,L] \times [0,1]$, where L is the horizontal period. This assumption ensures uniqueness of the solution to the fluid equations. Flows of the form (2.1) are automatically divergence free.

The symmetry of the solution is preserved by both the Navier-Stokes and Euler evolution if the initial condition $\mathbf{v}_0$ and body force $\mathbf{f}$ satisfy the same ansatz, that is:

$$\mathbf{v}^\epsilon|_{t=0} = \mathbf{v}_0(x,y,z) = (a(z), b(x,z), 0), \tag{2.2}$$

$$\mathbf{F} = (f_1(t,x), f_2(t,x,z), 0). \tag{2.3}$$

We will denote by $\mathbf{v}^\epsilon$ the solution of the Navier-Stokes system (1.1) with viscosity ϵ and by $\mathbf{v}^0$ the solution of the Euler system (1.3). Periodicity in the horizontal directions is complemented by boundary conditions in the vertical variable z. For NSE, we prescribed the fluid velocity at the channel walls together with a non-penetration condition:

$$\mathbf{v}^\epsilon|_{z=0} = \boldsymbol{\alpha}^0(t,x,y),\ \mathbf{v}^\epsilon|_{z=1} = \boldsymbol{\alpha}^1(t,x,y),$$

where $\boldsymbol{\alpha}^i(t,x,y) := (\beta_1^i(t), \beta_2^i(t,x), 0), i = 0, 1$.

It is not difficult to show that imposing the symmetry (2.1) on the solution system reduces NSE to the following weakly non-linear system:

$$\begin{aligned} &\partial_t u_1^\epsilon - \epsilon \partial_{zz} u_1^\epsilon = f_1, \\ &\partial_t u_2^\epsilon + u_1^\epsilon \partial_x u_2^\epsilon - \epsilon \partial_{xx} u_2^\epsilon - \epsilon \partial_{zz} u_2^\epsilon = f_2, \end{aligned} \tag{2.4}$$

on $\Omega := [0,L] \times [0,1]$. We remark that although the above system is a 2 by 2 system, the plane-parallel flows we study are not two-dimensional.

In addition, we will always assume that the initial data, boundary data and forcing satisfy certain compatibility conditions. We recall the zero-order compatibility condition which takes the form

$$\boldsymbol{\alpha}^i(0,x,y) = \mathbf{v}_0(x,y,i), \quad i = 0,1, \tag{2.5}$$

and the first-order compatibility condition

$$\begin{aligned} &\partial_t \beta_1^i(0) = \epsilon \partial_{zz} a(i) + f_1(0,i) \\ &\partial_t \beta_2^i(0,x) + a(i)\partial_x b(x,i) = \epsilon \partial_{xx} b(x,i) + \epsilon \partial_{zz} b(x,i) + f_2(0,x,i), \end{aligned} \tag{2.6}$$

where $i = 0, 1$. These compatibility conditions prevent the formation of an initial layer in the NSE evolution due to the difference in boundary values between the initial data and the fluid velocity at any positive time. In [10, 14], the boundary layer is analyzed without assuming any compatibility condition. In this case, extra vorticity is produced at the boundary in the limit of vanishing viscosity, and in general only L^∞ bounds can be readily obtained for the correctors. While the zero-order condition is uniform in ϵ, the first-order conditions in general imply that the boundary data $\boldsymbol{\alpha}_i$, $i = 0, 1$, is dependent on ϵ for $t > 0$ (we assume the forcing is given and independent of viscosity). We notice however that this undesirable ϵ dependence can be eliminated if the second derivatives of the initial data (a and b) vanish at the channel walls.

Since we are working in a domain that is periodic in the horizontal directions, we will employ the Sobolev spaces, for $m \in \mathbb{Z}_+$,

$$\begin{aligned} & H^m(Q) = H^m_{per}(Q) \\ := \ & \{f : Q \to \mathbb{R} \ | \ \partial^\alpha f \in L^2(Q),\ |\alpha| \le m, f \text{ periodic in the horizontal directions}\}. \end{aligned}$$

We denote with $H^m(\Omega)$ the subspace of functions in $H^m(Q)$ that are constant in y. With abuse of notation, we write $(H^m(Q))^2 \equiv H^m(Q)$ for spaces of vector fields. As customary, H^1_0 is the space of functions in H^1 that vanish at the boundary in trace sense.

Due to the weak coupling in the system (2.4), well-posedness is easily established. For instance, $\mathbf{v}^\epsilon \in L^\infty(H^1(Q))^3$ under the assumption that $\mathbf{v}_0$ and $\boldsymbol{\alpha}^i$ belong to $H^1(Q)$ and $\mathbf{f} \in L^\infty(0,\infty; H^1(Q))^3$. We do not address this point in detail here, and refer for example to [14, 24] for further discussion.

By formally taking the limit $\epsilon \to 0$, NSE become the Euler system (1.3). We continue to assume periodicity in the horizontal directions x, y, but impose the no-penetration condition (1.4) at the channel walls. We observe that solutions of the form (2.1) automatically satisfy the no-penetration condition. Under the plane-parallel symmetry, the Euler system reduces to the following weakly non-linear system

$$\begin{aligned} &\partial_t u_1^0 = f_1, \\ &\partial_t u_2^0 + u_1^0 \partial_x u_2^0 = f_2, \end{aligned} \tag{2.7}$$

in Ω. We take the same initial condition (2.2) for both Euler and Navier-Stokes:

$$\mathbf{v}^0|_{t=0} = \mathbf{v}_0. \tag{2.8}$$

The solution of (2.7) is obtained by solving an ordinary differential equation and a transport equation. Therefore, the solution is regular provided the initial data is regular enough. For example, if $\mathbf{v}_0 \in H^m(Q)$, and $\mathbf{f} \in L^\infty(0,T; H^m(Q))$, $m > 5$ as we will assume throughout, then $\mathbf{v}^0 \in C(0,T; H^m(Q))$.

In the rest of the paper, we will focus on the analysis of the reduced systems (2.4) and (2.7) on Ω. For this purpose, we set

$$\begin{aligned} &\mathbf{u}^\epsilon(t,x,z) := (u_1^\epsilon(t,z), u_2^\epsilon(t,x,z)), \\ &\mathbf{u}^0(t,x,z) := (u_1^0(t,z), u_2^0(t,x,z)), \\ &\mathbf{f}(t,x,z) := (f_1(t,z), f_2(t,x,z)), \\ &\boldsymbol{\beta}^i(t,x) := (\beta_1^i(t), \beta_2^i(t,x)), i = 0, 1, \\ &\boldsymbol{u}_0(t,x,z) = (a(x), b(x,z)). \end{aligned}$$

We assume tacitly throughout the rest of the paper that all functions are periodic in the x variable, so that boundary conditions will be given only at the channel walls, that is, for $z = 0$ and $z = 1$.

3. PRANDTL-TYPE EQUATIONS FOR CORRECTORS

The approach to a rigorous boundary layer analysis that we take is to derive effective Prandtl-type equations for a *corrector* that approximates the difference between the NSE solution $(\mathbf{u}^\epsilon, 0)$ and the Euler solution $(\mathbf{u}^0, 0)$.

We assume that the NSE solution is well approximated by

$$\mathbf{u}^{\mathbf{app}}(t,x,z) := \mathbf{u}^{\mathbf{ou}}(t,x,z) + \boldsymbol{\theta}^{\mathbf{0}}(t,x,\frac{z}{\sqrt{\epsilon}}) + \boldsymbol{\theta}^{\boldsymbol{u},\mathbf{0}}(t,x,\frac{1-z}{\sqrt{\epsilon}}), \tag{3.1}$$

where $\mathbf{u}^{\mathbf{ou}}$ is the so-called *outer solution*, that is, the vector field expected to represent the fluid velocity outside of the boundary layers, while $\boldsymbol{\theta}^{\mathbf{0}}$ and $\boldsymbol{\theta}^{\boldsymbol{u},\mathbf{0}}$ are the correctors respectively near the lower ($z=0$) and upper ($z=1$) wall of the channel. We make the ansatz that the correctors take the following form

$$\begin{aligned} \boldsymbol{\theta}^{\mathbf{0}}(t,x,\frac{z}{\sqrt{\epsilon}}) &= (\theta_1^0(t,x,\frac{z}{\sqrt{\epsilon}}), \theta_2^0(t,x,\frac{z}{\sqrt{\epsilon}})), \\ \boldsymbol{\theta}^{\boldsymbol{u},\mathbf{0}}(t,x,\frac{1-z}{\sqrt{\epsilon}}) &= (\theta_1^{u,0}(t,x,\frac{1-z}{\sqrt{\epsilon}}),\ \theta_2^{u,0}(t,x,\frac{1-z}{\sqrt{\epsilon}})). \end{aligned} \tag{3.2}$$

This form corresponds to the zero order in a formal asymptotic expansion in powers of $\sqrt{\epsilon}$ of the difference between the NSE and Euler solutions in each boundary layer. Introducing the stretched variables $Z = \frac{z}{\sqrt{\epsilon}}$ and $Z^u = \frac{1-z}{\sqrt{\epsilon}}$ we see that the correctors must satisfy the following matching conditions

$$\theta_j^0 \to 0 \text{ as } Z \to \infty;\ \theta_j^{u,0} \to 0 \text{ as } \ Z^u \to \infty, j=1,2, \tag{3.3}$$

in order for the vanishing viscosity limit to hold.

Inserting (3.1) into (2.4) and (2.4) and dropping lower-order terms in ϵ, we obtain the systems of equations that $\mathbf{u}^{\mathbf{ou}}$ and the correctors must satisfy respectively.

(1) The outer solution $\mathbf{u}^{\mathbf{ou}}$ satisfies the reduced Euler equations eqrefEuler with the initial data

$$\mathbf{u}^0|_{t=0} = \mathbf{u}_0. \tag{3.4}$$

consequently, by uniqueness of the solution to the reduced system, we can identify $\mathbf{u}^{\mathbf{ou}} \equiv \mathbf{u}^0$.

(2) The lower corrector $\boldsymbol{\theta}^{\mathbf{0}} = (\theta_1^0, \theta_2^0)$ satisfies

$$\partial_t\theta_1^0 - \partial_{ZZ}\theta_1^0 = 0, \tag{3.5a}$$

$$\partial_t\theta_2^0 - \partial_{ZZ}\theta_2^0 + \theta_1^0\partial_x\theta_2^0 + u_1^0(t,0)\partial_x\theta_2^0 + \theta_1^0\partial_x u_2^0(t,x,0) = 0 \tag{3.5b}$$

$$(\theta_1^0,\theta_2^0)|_{Z=0} = (\beta_1^0(t) - u_1^0(t,0), \beta_2^0(t,x) - u_2^0(t,x,0)), \tag{3.5c}$$

$$(\theta_1^0,\theta_2^0)|_{Z=\infty} = 0, \tag{3.5d}$$

$$(\theta_1^0,\theta_2^0)|_{t=0} = (0,0). \tag{3.5e}$$

(3) The upper corrector $\boldsymbol{\theta}^{\boldsymbol{u},\mathbf{0}} = (\theta_1^{u,0}, \theta_2^{u,0})$ satisfies

$$\begin{aligned}
&\partial_t \theta_1^{u,0} - \partial_{Z^u Z^u} \theta_1^{u,0} = 0, \\
&\partial_t \theta_2^{u,0} - \partial_{Z^u Z^u} \theta_2^{u,0} + \theta_1^{u,0} \partial_x \theta_2^{u,0} + u_1^0(t,1) \partial_x \theta_2^{u,0} + \theta_1^{u,0} \partial_x u_2^0(t,x,1) = 0, \\
&(\theta_1^{u,0}, \theta_2^{u,0})|_{Z^u=0} = (\beta_1^1(t) - u_1^0(t,1), \beta_2^1(t,x) - u_2^0(t,x,1)), \\
&(\theta_1^{u,0}, \theta_2^{u,0})|_{Z^u=\infty} = 0, \\
&(\theta_1^{u,0}, \theta_2^{u,0})|_{t=0} = 0.
\end{aligned} \tag{3.6}$$

The well-posedness of the above systems is readily established, so that (3.1) gives a well-defined vector field. Furthermore, the correctors exhibit certain rates of decay in the stretched variables, which in turn will be used to establish error bounds for the approximate solution $\mathbf{u}^{\mathbf{app}}$. The solvability of the systems (3.5) and (3.6) along with the decay properties of the correctors are discussed in Appendix A.

4. APPROXIMATE SOLUTIONS

In order to derive error bounds for the approximate solution to NSE introduced in Section 3 above, it is more convenient to modify (3.1) so that the boundary condition (2.4) are met exactly. Such a modification is well-known in the literature (see [5], [20], [21], [22], [24] for instance).

Let $\psi(z)$ be a smooth function defined on $[0,1]$ such that $\psi(z) = 1$ when $z \in [0, \frac{1}{3}]$ and $\psi(z) = 0$ when $z \in [\frac{1}{2}, 1]$. We have $\psi(z)\psi(1-z) \equiv 0$ when $z \in [0,1]$. Next, we define a truncated approximation $\tilde{\mathbf{u}}^{\mathbf{app}}(t,x,z) = (\tilde{u}_1^{app}(t,z), \tilde{u}_2^{app}(t,x,z))$ to the NSE solution, where

$$\tilde{u}_1^{app}(t,z) := u_1^0(t,z) + \psi(z)\theta_1^0(t, \frac{z}{\sqrt{\epsilon}}) + \psi(1-z)\theta_1^{u,0}(t, \frac{1-z}{\sqrt{\epsilon}}), \tag{4.1a}$$

$$\tilde{u}_2^{app}(t,x,z) := u_2^0(t,x,z) + \psi(z)\theta_2^0(t,x, \frac{z}{\sqrt{\epsilon}}) + \psi(1-z)\theta_2^{u,0}(t,x, \frac{1-z}{\sqrt{\epsilon}}). \tag{4.1b}$$

Then $\tilde{\mathbf{u}}^{\mathbf{app}}$ satisfies the following system

$$\begin{aligned}
&\partial_t \tilde{u}_1^{app} - \epsilon \partial_{zz} \tilde{u}_1^{app} = f_1 + A + B, \\
&\partial_t \tilde{u}_2^{app} + \tilde{u}_1^{app} \partial_x \tilde{u}_2^{app} - \epsilon \partial_{xx} \tilde{u}_2^{app} - \epsilon \partial_{zz} \tilde{u}_2^{app} = f_2 + D + E + F,
\end{aligned} \tag{4.2}$$

where

$$A = -2\sqrt{\epsilon}[\psi^{'}(z)\partial_Z\theta_1^0 + \psi^{'}(1-z)\partial_{Z^u}\theta_1^{u,0}], \tag{4.3a}$$

$$B = -\epsilon[\partial_{zz}u_1^0 + \psi^{''}(z)\theta_1^0 + \psi^{''}(1-z)\theta_1^{u,0}], \tag{4.3b}$$

$$D = \psi(z)(\psi(z)-1)\theta_1^0\partial_x\theta_2^0 + \psi(1-z)(\psi(1-z)-1)\theta_1^{u,0}\partial_x\theta_2^{u,0}, \tag{4.3c}$$

$$\begin{aligned} E = \sqrt{\epsilon}[&\psi(z)(Z\theta_1^0\partial_x\partial_z u_2^0(t,x,0) + \partial_z u_1^0(t,0)Z\partial_x\theta_2^0) \\ &- \psi(1-z)(\partial_z u_1^0(t,1)Z^u\partial_x\theta_2^{u,0} + \theta_1^0\partial_x\partial_z u_2^0(t,x,1)Z^u) \\ &- 2\psi^{'}(z)\partial_Z\theta_2^0 - 2\psi^{'}(1-z)\partial_{Z^u}\theta_2^{u,0}], \end{aligned} \tag{4.3d}$$

$$\begin{aligned} F = \epsilon(&-\psi(z)\partial_{xx}\theta_2^0 - \psi(1-z)\partial_{xx}\theta_2^{u,0} - \partial_{xx}u_2^0 - \partial_{zz}u_2^0 \\ &- \psi^{''}\theta_2^0 - \psi^{''}(1-z)\theta_2^{u,0}). \end{aligned} \tag{4.3e}$$

The corresponding initial and boundary conditions are respectively

$$\begin{aligned} &\tilde{\mathbf{u}}^{\mathbf{app}}|_{t=0} = \mathbf{u}_0, \\ &\tilde{\mathbf{u}}^{\mathbf{app}}|_{z=0} = \boldsymbol{\beta}^0(t,x),\ \tilde{\mathbf{u}}^{\mathbf{app}}|_{z=1} = \boldsymbol{\beta}^1(t,x). \end{aligned} \tag{4.4}$$

Both $\mathbf{u}^{\mathbf{app}}$ and the truncated $\tilde{\mathbf{u}}^{\mathbf{app}}$ depend on viscosity ϵ, but for sake of notation we do not explicitly show it.

5. ERROR ESTIMATES AND CONVERGENCE RATES

We are now ready to prove our main result, that is, error bounds for the approximation $\tilde{\mathbf{u}}^{\mathbf{app}}$ of the true NSE solution, which then yield convergence rates as viscosity vanishes. Later, we will improve upon these results by including more terms from an asymptotic expansion in power of $\sqrt{\epsilon}$ in both the outer solution and the correctors.

The approximation error is given by $\mathbf{u}^{\mathbf{err}}(t,x,z) = \mathbf{u}^{\epsilon}(t,z) - \tilde{\mathbf{u}}^{\mathbf{app}}$, and it satisfies the following system of equations

$$\partial_t u_1^{err} - \epsilon\partial_{zz}u_1^{err} = -(A+B), \tag{5.1}$$

$$\partial_t u_2^{err} + u_1^{err}\partial_x\tilde{u}_2^{app} + u_1^{\epsilon}\partial_x u_2^{err} - \epsilon\partial_{xx}u_2^{err} - \epsilon\partial_{zz}u_2^{err} = -(D+E+F), \tag{5.2}$$

where A through F are given in (4.3), with boundary conditions and initial data

$$\begin{aligned} &\mathbf{u}^{\mathbf{err}}|_{z=0} = 0,\ \mathbf{u}^{\mathbf{err}}|_{z=1} = 0, \\ &\mathbf{u}^{\mathbf{err}}|_{t=0} = 0. \end{aligned} \tag{5.3}$$

A key technical result is an anisotropic Sobolev embedding contained in the following lemma, which is proved in [5] (Corollary 7.3), and [20] (Remark 4.2) for instance.

Lemma 5.1 (Temam&Wang). *For all* $u \in H_0^1(\Omega)$

$$\begin{aligned}\|u\|_{L^\infty(\Omega)} \leq C\,(&\|u\|_{L^2}^{\frac{1}{2}}\|\partial_z u\|_{L^2}^{\frac{1}{2}} \\ &+ \|\partial_z u\|_{L^2}^{\frac{1}{2}}\|\partial_x u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}^{\frac{1}{2}}\|\partial_x\partial_z u\|_{L^2}^{\frac{1}{2}}),\end{aligned} \tag{5.4}$$

where the left-hand or both sides of the inequality could be infinite.

The main results of this paper are contained in the next theorem.

Theorem 5.2. *Let* $\mathbf{u_0} \in H^m(\Omega)$*,* $\boldsymbol{\beta^i} \in H^2(0,T;H^m(\Omega))$*,* $i = 0,1$*, and* $\mathbf{f} \in L^\infty(0,T;H^m(\Omega))$*,* $m > 5$*, satisfy the zero-order compatibility condition* (2.5)*. Then, there exists positive constants* C_i*,* $i = 1,2,3$*, independent of* ϵ *such that for any solution* $\mathbf{u}^\epsilon$ *of the system* (2.4) *with initial condition* $\mathbf{u}_0$ *and boundary data* $\boldsymbol{\beta^i}$*,*

$$\|\mathbf{u}^\epsilon - \tilde{\mathbf{u}}^{\mathbf{app}}\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1\,\epsilon^{\frac{3}{4}}, \tag{5.5}$$

$$\|\mathbf{u}^\epsilon - \tilde{\mathbf{u}}^{\mathbf{app}}\|_{L^\infty(0,T;H^1(\Omega))} \leq C_2\,\epsilon^{\frac{1}{4}}, \tag{5.6}$$

$$\|\mathbf{u}^\epsilon - \tilde{\mathbf{u}}^{\mathbf{app}}\|_{L^\infty((0,T)\times\Omega)} \leq C_3\,\sqrt{\epsilon}, \tag{5.7}$$

where $\tilde{\mathbf{u}}^{\mathbf{app}}$ *is given in equation* (4.1)*.*

We do not concern ourselves with optimizing the regularity imposed on the data in Theorem 5.2, since our aim is to investigate the boundary layer which is present even for smooth data.

Before proceeding with the proof of the theorem, we state some immediate consequences.

Corollary 5.3. *Under the hypotheses of Theorem 5.2, the following optimal convergence rate holds:*

$$C_1\epsilon^{\frac{1}{4}} \leq \|\mathbf{u}^\epsilon - \mathbf{u}^0\|_{L^\infty(0,T;L^2(\Omega))} \leq C_2\epsilon^{\frac{1}{4}}, \tag{5.8}$$

where C_1, C_2 *are constants depending on* $\mathbf{u}_0, \mathbf{f}$ *and* $\boldsymbol{\beta^i}, i = 1,2$*, but independent of* ϵ*.*

Corollary (5.8) is a consequence of the estimate $\|\boldsymbol{\theta}^{\mathbf{0}}\|_{L^\infty(0,T;L^2(\Omega))} \approx \epsilon^{\frac{1}{4}}$.

We can similarly establish convergence rates of the Navier-Stokes to the Euler solution. These rates recover and improve upon some of the results of [14, 24].

Corollary 5.4. *Under the hypotheses of Theorem 5.2, there exist positive constants* C_i*,* $i = 1,2$ *independent of* ϵ *such that for any* $\delta \in (0,1)$ *such that* $\frac{\delta}{\sqrt{\epsilon}} \to \infty$ *as* $\epsilon \to 0$*,*

$$\begin{aligned}\|\mathbf{u}^\epsilon - \mathbf{u^0}\|_{L^\infty(0,T;H^1(\Omega^\delta))} &\leq C_1\,\epsilon^{\frac{1}{4}}), \\ \|\mathbf{u}^\epsilon - \mathbf{u^0}\|_{L^\infty((0,T)\times\Omega^\delta))} &\leq C_2\,\sqrt{\epsilon}),\end{aligned}$$

where $\Omega^\delta = [0,L] \times [\delta, 1-\delta]$*.*

Proof of Theorem 5.2. We first employ the results of the Appendix and standard energy estimates to derive error bounds in $L^\infty([0,T],L^2(\Omega)$ and $L^\infty([0,T],H^1(\Omega)$. We then apply the anisotropic Sobolev inequality (5.4) to obtain bounds in $L^\infty([0,T]\times\Omega)$.

Multiplying (5.1) by u_1^{err} and integrating by parts over Ω, we obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_1^{err}\|_{L^2(0,1)}^2 + \epsilon\|\partial_z u_1^{err}\|_{L^2(0,1)}^2 &= -\int_0^1 (A+B)u_1^{err}dz \\
&\le 2\int_{\frac{1}{3}}^{\frac{2}{3}} |\sqrt{\epsilon}(\partial_{Z^u}\theta_1^{u,0} + \partial_Z\theta_1^0) + \epsilon(\theta_1^0+\theta_1^{u,0})||u_1^{err}|\,dz \\
&\quad + \epsilon\|u_1^0\|_{H^2(0,1)}\|u_1^{err}\|_{L^2(\Omega)} \\
&\le \|u_1^{err}\|_{L^2(0,1)}\big[18\epsilon^{\frac{7}{4}}\,(\|\langle Z^u\rangle^2\partial_{Z^u}\theta_1^{u,0}\|_{L^2(0,\infty)} + \|\langle Z\rangle^2\partial_Z\theta_1^0\|_{L^2(0,\infty)}) \\
&\quad + \epsilon\,(\|\theta_1^{u,0}\|_{L^2(0,\infty)} + \|\theta_1^0\|_{L^2(0,\infty)} + \|u_1^0\|_{H^2(0,1)})\big], \qquad (5.9)
\end{aligned}$$

where some of the terms on the right hand side of the last inequality are estimates as exemplified below (the limits of integration are determined by the support properties of the cut-off function ψ):

$$\begin{aligned}
\int_{\frac{1}{3}}^{\frac{2}{3}} |\partial_Z\theta_1^0(t,\frac{z}{\sqrt{\epsilon}})u_1^{err}(t,z)|\,dz &\le \frac{9\epsilon}{4}\|u_1^{err}\|_{L^2(0,1)}(\int_{\frac{1}{3\sqrt{\epsilon}}}^{\frac{2}{3\sqrt{\epsilon}}}\langle Z\rangle^4|\partial_Z\theta_1^0(t,Z)|^2\sqrt{\epsilon}\,dZ)^{\frac{1}{2}} \\
&\le \frac{9\epsilon^{\frac{5}{4}}}{4}\|u_1^{err}\|_{L^2(0,1)}\|\langle Z\rangle^2\partial_Z\theta_1^0\|_{L^2(0,\infty)}.
\end{aligned}$$

Applying Cauchy's and then Grönwall's inequalities to (5.9) gives:

$$\begin{aligned}
&\|u_1^{err}\|_{L^\infty(0,T;L^2(0,1))} + \sqrt{\epsilon}\,\|\partial_z u_1^{err}\|_{L^2(0,T;L^2(0,1))} \\
&\quad\le 18\epsilon\,(\|\langle Z^u\rangle^2\partial_{Z^u}\theta_1^{u,0}\|_{L^2(0,T;L^2(0,\infty))} + \|\langle Z\rangle^2\partial_Z\theta_1^0\|_{L^2(0,T;L^2(0,\infty))} \\
&\qquad + \|\theta_1^0\|_{L^2(0,T;L^2(0,\infty))} + \|\theta_1^{u,0}\|_{L^2(0,T;L^2(0,\infty))} + \|u_1^0\|_{L^2(0,T;H^2(0,1))}) \\
&\quad\le C_1\,\epsilon, \qquad (5.10)
\end{aligned}$$

where the constant C_1 depends on $\|u_1^0\|_{L^\infty(0,T;H^2(0,1))}$, and T explicitly, but is independent of ϵ by Lemma A.1 and A.2.

Next, we multiply (5.1) by $-\partial_{zz}u_1^{err}$ and integrate over Ω:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_z u_1^{err}\|_{L^2(0,1)}^2 + \epsilon\|\partial_{zz}u_1^{err}\|_{L^2(0,1)}^2 \\
&\le 18\epsilon(\|\langle Z^u\rangle^2\partial_{Z^u}\theta_1^{u,0}\|_{L^2(0,T;L^2(0,\infty))} + \|\langle Z\rangle^2\partial_Z\theta_1^0\|_{L^2(0,T;L^2(0,\infty))} + \|\theta_1^0\|_{L^2(0,T;L^2(0,\infty))} \\
&\quad + \|\theta_1^{u,0}\|_{L^2(0,T;L^2(0,\infty))} + \|u_1^0\|_{L^2(0,T;H^2(0,1))})\|\partial_{zz}u_1^{err}\|_{L^2(0,T;L^2(0,1))} \\
&\le \frac{\epsilon}{8}\|\partial_{zz}u_1^{err}\|_{L^2(0,1)}^2 + 2C_1^2\epsilon, \qquad (5.11)
\end{aligned}$$

by Cauchy's inequality and Lemma A.1 and A.2 again. Integrating (5.11) in time gives

$$\|\partial_z u_1^{err}\|_{L^\infty(0,T;L^2(0,1))} + \frac{\sqrt{\epsilon}}{4}\|\partial_{zz} u_1^{err}\|_{L^2(0,T;L^2(0,1))} \le 2C_1\sqrt{\epsilon}. \tag{5.12}$$

Therefore, we conclude that

$$\begin{aligned}&\|u_1^{err}\|_{L^\infty(0,T;L^2(0,1))} \le C_1\epsilon,\ \|u_1^{err}\|_{L^\infty(0,T;H^1(0,1))} \le 2C_1\sqrt{\epsilon},\\ &\|u_1^{err}\|_{L^\infty((0,T)\times(0,1))} \le \|u_1^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(0,1))}\|u_1^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;H^1(0,1))} \le 2C_1\epsilon^{\frac{3}{4}}.\end{aligned} \tag{5.13}$$

Multiplying (5.2) by u_2^{err}, integrating over Ω, and noticing that u_1^ϵ is independent of x, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_2^{err}\|^2_{L^2(\Omega)} + \epsilon\|\partial_z u_2^{err}\|^2_{L^2(\Omega)} + \epsilon\|\partial_x u_2^{err}\|^2_{L^2(\Omega)}\\ &\qquad = -\int_\Omega u_1^{err}\partial_x \tilde{u}_2^{app} u_2^{err}\,dx - \int_\Omega D u_2^{err}\,dx - \int_\Omega E u_2^{err}\,dx - \int_\Omega F u_2^{err}\,dx\\ &=: J_1 + J_2 + J_3 + J_4.\end{aligned} \tag{5.14}$$

We bound each term on the right-hand side separately:

$$\begin{aligned}J_1 &\le C_1\,\epsilon\,(\|\partial_x u_2^0\|_{L^\infty(\Omega)} + \|\partial_x\theta_2^0\|_{L^\infty(\Omega_\infty)} + \|\partial_x\theta_2^{u,0}\|_{L^\infty(\Omega_\infty)})\|u_2^{err}\|_{L^2(\Omega)}\\ &\le C_1C_2\,\epsilon\,\|u_2^{err}\|_{L^2(\Omega)},\end{aligned} \tag{5.15}$$

with C_2 a constant that depends on $\|u_2^0\|_{L^\infty(0,T;H^{2+s}(\Omega))}$, $s > 0$, but is independent of ϵ. Above, we have used (5.13) in the first inequality and Lemma A.2 in the second.

The support properties of ψ and similar techniques as those used in (5.9) imply the following bound for J_2

$$\begin{aligned}J_2 &\le \int_0^L\Big(\int_{\frac{1}{3}}^{\frac{2}{3}}(|\theta_1^0\partial_x\theta_2^0| + |\theta_1^{u,0}\partial_x\theta_2^{u,0}|)|u_2^{err}|\,dz\,dx\\ &\le 9\,\epsilon\,\|u_2^{err}\|_{L^2(\Omega)}\big(\|\theta_1^0\|_{L^\infty(0,\infty)}\|\langle Z\rangle^2\partial_x\theta_2^0\|_{L^2(\Omega_\infty)}\\ &\qquad + 9\epsilon\|\theta_1^{u,0}\|_{L^\infty(0,\infty)}\|\langle Z^u\rangle^2\partial_x\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\big),\end{aligned} \tag{5.16}$$

Similarly, we deal with J_3, J_4 as follows

$$\begin{aligned}J_3 \le \epsilon^{\frac{3}{4}}\|u_2^{err}\|_{L^2(\Omega)}\big(&\|u_2^0\|_{H^{3+s}\Omega)}\|\langle Z\rangle\theta_1^0\|_{L^2(0,\infty)} + \|u_1^0\|_{H^2(0,1)}\|\langle Z\rangle\partial_x\theta_2^0\|_{L^2(\Omega_\infty)}\\ &+ \|u_2^0\|_{H^{3+s}(\Omega)}\|\langle Z^u\rangle\theta_1^{u,0}\|_{L^2(0,\infty)} + \|u_1^0\|_{H^2(0,1)}\|\langle Z\rangle\partial_x\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\\ &+ \|\partial_Z\theta_2^0\|_{L^2(\Omega_\infty)} + \|\partial_{Z^u}\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\big),\end{aligned} \tag{5.17}$$

where $s > 0$ is arbitrary, and

$$\begin{aligned} J_4 \leq \epsilon \|u_2^{err}\|_{L^2(\Omega)} \big(&\|\partial_{xx}\theta_2^0\|_{L^2(\Omega_\infty)} + \|\partial_{xx}\theta_2^{u,0}\|_{L^2(\Omega_\infty)} + \|u_2^0\|_{H^2(\Omega)} \\ &+ \|\theta_2^0\|_{L^2(\Omega_\infty)} + \|\theta_2^{u,0}\|_{L^2(\Omega_\infty)} \big). \end{aligned} \tag{5.18}$$

By substituting (5.15)-(5.18) into (5.14), applying first Cauchy and then Grönwall's inequalities, we have

$$\|u_2^{err}\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\epsilon}\|\nabla_{x,z} u_2^{err}\|_{L^2(0,T;L^2(\Omega))} \leq C_5 \epsilon^{\frac{3}{4}}, \tag{5.19}$$

with C_5 a constant that depends on T, $\|u_1^0\|_{L^\infty(0,T;H^{2+s}(0,1))}$, and $\|u_2^0\|_{L^\infty(0,T;H^{3+s}(\Omega))}$, but is independent of ϵ, where in the last inequality we have applied the results in Appendix A once again.

Similarly, multiplying both sides of (5.2) by $-\partial_{xx}u_2^{err}$ and integrating by parts over Ω gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_x u_2^{err}\|_{L^2(\Omega)}^2 + \epsilon\|\nabla_{x,z}\partial_x u_2^{err}\|_{L^2(\Omega)}^2 \\ &\leq \|u_1^{err}\|_{L^2(\Omega)}\|\partial_x u_2^{err}\|_{L^2(\Omega)}\big(\|\partial_{xx}u_2^0\|_{L^\infty(\Omega)} + \|\partial_{xx}\theta_2^0\|_{L^\infty(\Omega_\infty)} \\ &\quad + \|\partial_{xx}\theta_2^{u,0}\|_{L^\infty(\Omega_\infty)}\big) + \epsilon\big(\|\langle Z\rangle^2\theta_1^0\|_{L^\infty(0,\infty)}\|\partial_{xx}\theta_2^0\|_{L^2(\Omega_\infty)} \\ &\quad + \|\langle Z^u\rangle^2\theta_1^{u,0}\|_{L^\infty(0,\infty)}\|\partial_{xx}\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\big) \\ &\quad + \epsilon^{\frac{3}{4}}\|\partial_x u_2^{err}\|_{L^2(\Omega)}\big(\|\langle Z\rangle\theta_1^0\|_{L^2(0,\infty)}\|u_2^0\|_{H^{4+s}(\Omega)} + \|u_1^0\|_{H^2(0,1)}\|\langle Z\rangle\partial_{xx}\theta_2^0\|_{L^2(\Omega_\infty)} \\ &\quad + \|\partial_x\partial_Z\theta_2^0\|_{L^2(\Omega_\infty)} + \|\langle Z^u\rangle\theta_1^{u,0}\|_{L^2(0,\infty)}\|u_2^0\|_{H^{4+s}(\Omega)} + \|\langle Z\rangle\partial_{xx}\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\|u_1^0\|_{H^2(0,1)} \\ &\quad + \|\partial_x\partial_Z\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\big) + \epsilon\|\partial_x u_2^{err}\|_{L^2(\Omega)}\big(\|\partial_{xxx}\theta_2^0\|_{L^2(\Omega_\infty)} + \|\partial_{xxx}\theta_2^{u,0}\|_{L^2(\Omega_\infty)} \\ &\quad + \|u_2^0\|_{H^3(\Omega)} + \|\partial_x\theta_2^0\|_{L^2(\Omega_\infty)} + \|\partial_x\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\big) \end{aligned} \tag{5.20}$$

from which it follows that

$$\|\partial_x u_2^{err}\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\epsilon}\|\nabla_{x,z}(\partial_x u_2^{err})\|_{L^2(0,T;L^2(\Omega))} \leq C_6\epsilon^{\frac{3}{4}}, \tag{5.21}$$

where C_6 depends on T, $\|u_1^0\|_{L^\infty(0,T;H^2(0,1))}$ and $\|u_2^0\|_{L^\infty(0,T;H^{4+s}(\Omega))}$, $s > 0$, but is independent of ϵ.

Analogous calculations give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_z u_2^{err}\|_{L^2(\Omega)}^2 + \epsilon\|\nabla\partial_z u_2^{err}\|_{L^2(\Omega)}^2 \\
&\le \|u_1^{err}\|_{L^2(\Omega)}\big(\|\partial_x u_2^0\|_{L^\infty(\Omega)} + \|\partial_x\theta_2^0\|_{L^\infty(\Omega_\infty)} + \|\partial_x\theta_2^{u,0}\|_{L^\infty(\Omega_\infty)}\big) \\
&\quad + \|u_1^\epsilon\|_{L^\infty(\Omega)}\|\partial_x u_2^{err}\|_{L^\infty(\Omega)}\|\partial_{zz}u_2^{err}\|_{L^2(\Omega)} \\
&\quad + \epsilon\|\partial_{zz}\theta_2^0\|_{L^2(\Omega_\infty)}\big(\|\theta_1^0\|_{L^\infty(0,\infty)}\|\langle Z\rangle^2\partial_x\theta_2^0\|_{L^2(\Omega_\infty)} \\
&\quad + \|\theta_1^{u,0}\|_{L^\infty(0,\infty)}\|\langle Z^u\rangle^2\partial_x\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\big) \\
&+ \epsilon^{\frac{3}{4}}\|\partial_{zz}u_2^{err}\|_{L^2(\Omega)}\big(\|\langle Z\rangle\theta_1^0\|_{L^2(0,\infty)}\|u_2^0\|_{H^{3+s}(\Omega)} \\
&\quad + \|u_1^0\|_{H^2(0,1)}\|\langle Z\rangle\partial_x\theta_2^0\|_{L^2(\Omega_\infty)} + \|\langle Z^u\rangle\theta_1^{u,0}\|_{L^2(0,\infty)}\|u_2^0\|_{H^{3+s}(\Omega)} \\
&\quad + \|u_1^0\|_{H^2(0,1)}\|\langle Z^u\rangle\partial_x\theta_2^{u,0}\|_{L^2(\Omega_\infty)} + \|\partial_Z\theta_2^0\|_{L^2(\Omega_\infty)} + \|\partial_{Z^u}\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\big) \\
&\quad + \epsilon\|\partial_{zz}u_2^{err}\|_{L^2(\Omega)}\big(\|\partial_{xx}\theta_2^0\|_{L^2(\Omega_\infty)} + \|\partial_{xx}\theta_2^{u,0}\|_{L^2(\Omega_\infty)} + \|u_2^0\|_{H^2(\Omega)} \\
&\quad + \|\theta_2^0\|_{L^2(\Omega_\infty)} + \|\theta_2^{u,0}\|_{L^2(\Omega_\infty)}\big),
\end{aligned} \tag{5.22}
$$

which imply, utilizing (5.13) and (5.21),

$$
\|\partial_z u_2^{err}\|_{L^\infty(0,T;L^2)} + \sqrt{\epsilon}\|\nabla_{x,z}\partial_z u_2^{err}\|_{L^2(0,T;L^2(\Omega))} \le C_7\epsilon^{\frac{1}{4}}, \tag{5.23}
$$

where C_7 depends on $\|u_1^\epsilon\|_{L^\infty((0,T)\times\Omega)}$, $\|u_1^0\|_{L^\infty(0,T;H^{2+s}(0,1))}$ and $\|u_2^0\|_{L^\infty(0,T;H^{3+s}(\Omega))}$, independent of ϵ. A uniform bound on $\|u_1^\epsilon\|_{L^\infty((0,T)\times\Omega)}$ in terms of $\|f_1\|_{L^\infty((0,T)\times\Omega)}$, $\|a(z)\|_{L^\infty(0,1)}$, and $\|\beta_1^0(t)\|_{L^\infty(0,T)}$, follows from the maximum principle for the heat equation.

In the above calculations, we cannot improve the a bound of order $\epsilon^{\frac{3}{4}}$, since we cannot perform any integration by parts in the right-hand side involving second or mixed derivatives in z, as $\partial_z u_2^{err}$ may not vanish at the boundary.

Proceeding in a similar fashion we also have

$$
\begin{aligned}
&\|\partial_{xx}u_2^{err}\|_{L^\infty(0,T;L^2(\Omega))} \le C_8\epsilon^{\frac{3}{4}}, \\
&\|\partial_z\partial_x u_2^{err}\|_{L^\infty(0,T;L^2(\Omega))} \le C_8\epsilon^{\frac{1}{4}},
\end{aligned} \tag{5.24}
$$

with C_8 depending on T, $\|u_2^0\|_{L^\infty(0,T;H^{5+s}(0,1))}$, $\|u_1^0\|_{L^\infty(0,T;H^2(\Omega))}$, and $\|u_1^\epsilon\|_{L^\infty((0,T)\times\Omega)}$, independent of ϵ.

Collecting (5.19), (5.21) and (5.22), we have that

$$
\|u_2^{err}\|_{L^\infty(0,T;H^1(\Omega))} \le C\epsilon^{\frac{1}{4}}. \tag{5.25}
$$

Finally, the anisotropic Sobolev embedding (5.4), together with inequalities (5.19), (5.21) and (5.22), gives

$$\begin{aligned}\|u_2^{err}\|_{L^\infty((0,T)\times\Omega)} &\le C(\|u_2^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\|\partial_z u_2^{err}\|^{\frac{1}{2}}_{L^2(0,T;L^2(\Omega))}\\ &\quad+\|\partial_z u_2^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\|\partial_x u_2^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\\ &\quad+\|u_2^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\|\partial_x\partial_z u_2^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))})\\ &\le C\sqrt{\epsilon}. \end{aligned} \tag{5.26}$$

With this estimate, we conclude the proof of Theorem 5.2. □

6. IMPROVED CONVERGENCE RATE

It is possible to derive convergence rates of higher order in ϵ than those of Section 5 by including more terms in the asymptotic expansion for the outer solution and the correctors. (We recall that we showed the convergence rate is of order $\epsilon^{\frac{3}{4}}, \epsilon^{\frac{1}{4}}, \epsilon^{\frac{1}{4}}$ for L^2, H^1, L^∞ norm respectively.) The higher-order expansions can be also used to show the optimality of the convergence rate.

In this section, we illustrate this idea by presenting the corresponding results up to the first-order expansion.

Accordingly, we replace (3.1) with the following ansatz

$$\mathbf{u}^{\mathbf{app,1}}(t,x,z) := \mathbf{u}^{\mathbf{ou}}(t,x,z) + \mathbf{u}^{\mathbf{lc}}(t,x,\frac{z}{\sqrt{\epsilon}}) + \mathbf{u}^{\mathbf{uc}}(t,x,\frac{1-z}{\sqrt{\epsilon}}), \tag{6.1}$$

where

- $\mathbf{u}^{\mathbf{ou}}(t,x,z) = \mathbf{u}^{\mathbf{0}}(t,x,z) + \sqrt{\epsilon}\mathbf{u}^{\mathbf{1}}(t,x,z)$ is the outer solution, valid in $\Omega = [0,L]\times[0,1]$;
- $\mathbf{u}^{\mathbf{lc}}(t,x,\frac{z}{\sqrt{\epsilon}}) = \boldsymbol{\theta}^{\mathbf{0}}(t,x,\frac{z}{\sqrt{\epsilon}}) + \sqrt{\epsilon}\boldsymbol{\theta}^{\mathbf{1}}(t,x,\frac{z}{\sqrt{\epsilon}})$ is the lower corrector, defined in $\Omega_\infty := [0,L]\times[0,\infty)$;
- $\mathbf{u}^{\mathbf{ub}}(t,x,\frac{1-z}{\sqrt{\epsilon}}) = \boldsymbol{\theta}^{\boldsymbol{u},\mathbf{0}}(t,x,\frac{1-z}{\sqrt{\epsilon}}) + \sqrt{\epsilon}\boldsymbol{\theta}^{\boldsymbol{u},\mathbf{1}}(t,x,\frac{1-z}{\sqrt{\epsilon}})$ is the upper corrector, also defined in Ω_∞.

As before, the corrector much satisfy the matching conditions:

$$\boldsymbol{\theta}^{\boldsymbol{i}} \to 0 \text{ as } Z\to\infty;\ \boldsymbol{\theta}^{\boldsymbol{u},\boldsymbol{i}} \to 0 \text{ as } Z^u\to\infty, \tag{6.2}$$

where $i = 0, 1$ and $Z = \frac{z}{\sqrt{\epsilon}}$ and $Z^u = \frac{1-z}{\sqrt{\epsilon}}$ are the stretched variables.

Next we derive the systems satisfied by the outer solution and the correctors. By consistency, the terms at leading order in both the outer solution and the correctors are given by the Euler solution $\mathbf{u}^{\mathbf{0}}$, and $\boldsymbol{\theta}^{\mathbf{0}}$, $\boldsymbol{\theta}^{\boldsymbol{u},\mathbf{0}}$ constructed in Section 3 respectively. The first-order terms are given below:

(1) The first-order term of the outer solution $\mathbf{u^1}(t,x,z) = (u_1^1(t,z), u_2^1(t,x,z))$ satisfies the following system of transport equations

$$\begin{aligned}
&\partial_t u_1^1 = 0,\\
&\partial_t u_2^1 + u_1^1 \partial_x u_2^0 + u_1^0 \partial_x u_2^1 = 0,\\
&(u_1^1, u_2^1)|_{t=0} = (0,0).
\end{aligned} \tag{6.3}$$

Given the regularity of the Euler solution u^0, it is immediate to see that $(u_1^1, u_2^1) \equiv 0$.

(2) The first-order term in the lower corrector $\boldsymbol{\theta^1}(t,x,Z) = (\theta_1^1, \theta_2^1)$ satisfies the system

$$\begin{aligned}
&\partial_t \theta_1^1 - \partial_{ZZ}\theta_1^1 = 0,\\
&\partial_t \theta_2^1 + u_1^0|z = 0\, \partial_x \theta_2^1 + \theta_1^0\, \partial_x \theta_2^1 + \theta_1^1\, (\partial_x \theta_2^0 + \partial_x u_2^0|_{z=0}) -\\
&\qquad - \partial_{ZZ}\theta_2^1 = -Z(\theta_1^0 \partial_{xz}^2 u_2^0|_{z=0} + \partial_z u_1^0|_{z=0} \partial_x \theta_2^0),\\
&(\theta_1^1, \theta_2^1)|_{Z=0} = (-u_1^1|_{z=0}, -u_2^1|_{z=0}) = 0,\\
&(\theta_1^1, \theta_2^1)|_{Z=\infty} = (0,0),\ (\theta_1^1, \theta_2^1)|_{t=0} = (0,0),
\end{aligned}$$

from which it follows that $\theta_1^1 \equiv 0$. Hence, the equations of $\boldsymbol{\theta^1}$ reduce to

$$\begin{aligned}
&\theta_1^1 = 0,\\
&\partial_t \theta_2^1 + u_1^0(t,0)\partial_x \theta_2^1 + \theta_1^0 \partial_x \theta_2^1 - \partial_{ZZ}\theta_2^1\\
&\qquad = -Z(\theta_1^0 \partial_{xz}^2 u_2^0(t,x,0) + \partial_z u_1^0(t,0)\partial_x \theta_2^0),\\
&\theta_2^1|_{Z=0} = 0,\\
&\theta_2^1|_{Z=\infty} = 0,\ \theta_2^1|_{t=0} = 0.
\end{aligned} \tag{6.4}$$

(3) By symmetry, the first-order term in the upper corrector $\boldsymbol{\theta^{u,1}}(t,x,Z^u) = (\theta_1^{u,1}, \theta_2^{u,1})$ satisfies the following system

$$\begin{aligned}
&\theta_1^{u,1} = 0,\\
&\partial_t \theta_2^{u,1} + u_1^0(t,1)\partial_x \theta_2^{u,1} + \theta_1^0 \partial_x \theta_2^{u,1} - \partial_{Z^u Z^u}\theta_2^{u,1}\\
&\qquad = -Z^u(\theta_1^{u,0} \partial_{xz}^2 u_2^0(t,x,1) + \partial_z u_1^0(t,1)\partial_x \theta_2^{u,0}),\\
&\theta_2^{u,1}|_{Z^u=0,\infty} = 0,\ \theta_2^{u,1}|_{t=0} = 0.
\end{aligned} \tag{6.5}$$

Solvability and regularity of linear parabolic systems such as (6.4) and (6.5) is well known and can be established here following arguments similar to those in the Appendix for the zero-order correction. The first order compatibility condition (2.6) between initial and boundary data and the force is used to improve the regularity of the correctors; for instance, we can show that $\partial_t\theta_1^0 \in L^\infty(0,T;L^2(0,\infty))$ and $\partial_{xxx}\theta_2^1 \in L^2(0,T;H^2(\Omega_\infty))$. We omit the proof and refer to [25] for more details.

As in Section 3, it is convenient for the analysis to modify the definition of the approximate solution $\mathbf{u}^{\mathbf{app},\mathbf{1}}$ so that the boundary conditions are exactly met. We therefore define a truncated approximation $\tilde{\mathbf{u}}^{\mathbf{app},\mathbf{1}}(t,x,z) = (\tilde u_1^{app,1}(t,z), \tilde u_2^{app,1}(t,x,z))$ by

$$\tilde u_1^{app,1}(t,z) := u_1^0(t,z) + \psi(z)\theta_1^0(t,\frac{z}{\sqrt{\epsilon}}) + \psi(1-z)\theta_1^{u,0}(t,\frac{1-z}{\sqrt{\epsilon}}), \tag{6.6a}$$

$$\begin{aligned}\tilde u_2^{app,1}(t,x,z) &:= u_2^0(t,x,z) + \psi(z)(\theta_2^0(t,x,\frac{z}{\sqrt{\epsilon}}) + \sqrt{\epsilon}\theta_2^1(t,x,\frac{z}{\epsilon}))\\ &\quad + \psi(1-z)(\theta_2^{u,0}(t,x,\frac{1-z}{\sqrt{\epsilon}}) + \sqrt{\epsilon}\theta_2^{u,1}(t,x,\frac{1-z}{\sqrt{\epsilon}})),\end{aligned} \tag{6.6b}$$

where ψ is the cut-off function used in Section 4. We note that, since$\theta_1^1 = u_1^1 = 0$, $\tilde u_1^{app,1} \equiv \tilde u_1^{app}$ in equation (4.1a).

Then $\tilde{\mathbf{u}}^{app,1}$ satisfies the following system

$$\begin{aligned}&\partial_t\tilde u_1^{app,1} - \epsilon\partial_{zz}\tilde u_1^{app,1} = f_1 + A + B,\\ &\partial_t\tilde u_2^{app,1} + \tilde u_1^{app,1}\partial_x\tilde u_2^{app,1} - \epsilon\partial_{xx}\tilde u_2^{app,1} - \epsilon\partial_{zz}\tilde u_2^{app,1}\\ &\qquad = f_2 + D + \hat E + \hat F + \hat G,\end{aligned} \tag{6.7}$$

where A, B and D are as in (4.3a), (4.3b) and (4.3c), respectively, and $\hat E$, $\hat F$ and $\hat G$ are given by

$$\begin{aligned}\hat E &= \sqrt{\epsilon}[\psi(z)(\psi(z)-1)\theta_1^0\partial_x\theta_2^1 + \psi(1-z)(\psi(1-z)-1)\theta_1^{u,0}\partial_x\theta_2^{u,1}\\ &\quad - 2\psi^{'}(z)\partial_Z\theta_2^0 - 2\psi^{'}(1-z)\partial_{Z^u}\theta_2^{u,0}],\\ \hat F &= \epsilon[\psi(z)(Z\partial_x\theta_2^1\partial_z u_1^0(t,0) + \frac{1}{2}\partial_{zz}u_1^0(t,0)Z^2\partial_x\theta_2^0 + \frac{1}{2}\theta_1^0\partial_x\partial_{zz}u_2^0(t,x,0)Z^2\\ &\quad - \partial_{xx}\theta_2^0) + \psi(1-z)(-Z^u\partial_z u_1^0(t,1)\partial_x\theta_2^{u,1} + \frac{1}{2}\partial_{zz}u_1^0(t,1)(Z^u)^2\partial_x\theta_2^{u,0}\\ &\quad + \frac{1}{2}\theta_1^{u,0}\partial_x(\partial_{zz}u_2^0(t,x,1))(Z^u)^2 - \partial_{xx}\theta_2^{u,0}) - 2\psi^{'}(z)\partial_Z\theta_2^1\\ &\quad - 2\psi^{'}(1-z)\partial_{Z^u}\theta_2^{u,1} - (\partial_{xx}u_2^0 + \partial_{zz}u_2^0) - \psi^{''}\theta_2^0 - \psi^{''}(1-z)\theta_2^{u,0}],\\ \hat G &= \epsilon^{\frac{3}{2}}[-\psi^{''}(z)\theta_2^1 - \psi^{''}(1-z)\theta_2^{u,1} + \psi(z)(\frac{1}{2}\partial_{zz}u_1^0(t,0)Z^2\partial_x\theta_2^1 - \partial_{xx}\theta_2^1)\\ &\quad + \psi(1-z)(\frac{1}{2}\partial_{zz}u_1^0(t,1)(Z^u)^2\partial_x\theta_2^{u,1} - \partial_{xx}\theta_2^{u,1}),\end{aligned}$$

The system above is complemented by the following boundary and initial conditions:

$$\begin{aligned}&\tilde{\mathbf{u}}^{\mathbf{app,1}}|_{t=0}=(a(z),b(x,z)),\\&\tilde{\mathbf{u}}^{app,1}|_{z=0}=\boldsymbol{\beta}^{\mathbf{0}},\ \tilde{\mathbf{u}}^{\mathbf{app,1}}|_{z=1}=\boldsymbol{\beta}^{\mathbf{1}}.\end{aligned}\tag{6.8}$$

Parallel to the analysis in Section 5, we define the approximation error $\hat{\mathbf{u}}^{err}(t,x,z):=\mathbf{u}^{\epsilon}-\tilde{\mathbf{u}}^{\mathbf{app,1}}$ Then the error satisfies the following system

$$\partial_t\hat{u}_1^{err}-\epsilon\partial_{zz}\hat{u}_1^{err}=-(A+B),\tag{6.9}$$

$$\begin{aligned}&\partial_t\hat{u}_2^{err}+\hat{u}_1^{err}\partial_x\tilde{u}_2^{app,1}+u_1^{\epsilon}\partial_x\hat{u}_2^{err}-\epsilon\partial_{xx}\hat{u}_2^{err}-\epsilon\partial_{zz}\hat{u}_2^{err}\\&\qquad=-(D+\hat{E}+\hat{F}+\hat{G}),\end{aligned}\tag{6.10}$$

together with the following initial and boundary conditions

$$\begin{aligned}&\hat{\mathbf{u}}^{err}|_{z=0}=0,\ \hat{\mathbf{u}}^{err}|_{z=1}=0,\\&\hat{\mathbf{u}}^{err}|_{t=0}=0.\end{aligned}\tag{6.11}$$

Using the expansion (6.1), we can improve the convergence rate of Theorem (5.2) under more regularity and compatibility conditions on the data. Again, we do not optimize the regularity needed to establish the result.

Theorem 6.1. *Let* $\mathbf{u_0}\in H^m(\Omega)$, $\boldsymbol{\beta^i}\in H^2(0,T;H^m(\Omega))$, $i=0,1$ *and* $\mathbf{f}=\in L^\infty(0,T;H^m)$, $m>8$, *satisfy the compatibility conditions* (2.5) *and* (2.6). *Then, there exist positive constants* C_1, C_2 *independent of* ϵ *such that for any solution* $\mathbf{u}^\epsilon$ *of the system* (2.4) *with initial condition* $\mathbf{u}_0$ *and boundary data* $\boldsymbol{\beta^i}$,

$$\|\mathbf{u}^\epsilon-\tilde{\mathbf{u}}^{\mathbf{app,1}}\|_{L^\infty(0,T;H^1(\Omega))}\leq C_1\sqrt{\epsilon},\tag{6.12}$$

$$\|\mathbf{u}^\epsilon-\tilde{\mathbf{u}}^{\mathbf{app,1}}\|_{L^\infty((0,T)\times\Omega)}\leq C_2\,\epsilon^{\frac{3}{4}},\tag{6.13}$$

where $\mathbf{u}^{\mathbf{app,1}}$ *is given in* (6.6).

Parallel to Corollary 5.3, we also obtain optimal convergence rates.

Corollary 6.2. *Under the hypotheses of Theorem 6.1, we have*

$$C_3\sqrt{\epsilon}\leq\|\mathbf{u}^\epsilon-\mathbf{u^0}-\psi(z)\boldsymbol{\theta}^{\mathbf{0}}-\psi(1-z)\boldsymbol{\theta}^{\boldsymbol{u},\mathbf{0}}\|_{L^\infty(0,T;H^1)}\leq C_4\sqrt{\epsilon},$$

where C_3 *and* C_4 *are constants depending on* $\mathbf{u_0},\boldsymbol{\beta^i},i=0,1$ *and* $\mathbf{f}$, *but not on* ϵ.

In addition, we also improve the results in Corollary 5.4 as follows.

Corollary 6.3. *Under the hypotheses of Theorem 5.2, there exists positive constants* C_i, $i=1,2$, *independent of* ϵ *such that for any* $\delta\in(0,1)$ *satisfying* $\frac{\delta}{\sqrt{\epsilon}}\to\infty$ *as* $\epsilon\to0$,

$$\begin{aligned}\|\mathbf{u}^\epsilon-\mathbf{u^0}\|_{L^\infty(0,T;H^1(\Omega^\delta))}&\leq C_1\sqrt{\epsilon},\\\|\mathbf{u}^\epsilon-\mathbf{u^0}\|_{L^\infty((0,T)\times\Omega^\delta))}&\leq C_2\,\epsilon^{\frac{3}{4}},\end{aligned}$$

where $\Omega^\delta = [0, L] \times [\delta, 1-\delta]$.

We omit the proofs of Theorem 6.1 and Corollaries 6.2 and 6.3, which are very similar to those in Section 4.

APPENDIX A. DECAY ESTIMATES OF THE CORRECTORS

In this appendix, we discuss the solvability of the systems of equations satisfied by the correctors and establish their decay properties. For simplicity, we will only state and prove the estimates needed in Section 5 and 6.

The correctors satisfy Prandtl-type effective equations and some of the techniques we use are similar to those in [25], which deals with the regularity and the decay properties of solutions to the Prandtl-type equations derived from the linearized compressible Navier Stokes equations.

By symmetry in the problem between the lower and upper corrector, we only deal with the correctors in the boundary layer at $z = 0$, $\boldsymbol{\theta}$ and $\boldsymbol{\theta^1}$. Below, we will be rather explicit in the dependence of constants on norms of data in view of the applications to Section 5 and 6.

We begin by studying θ_1^0, which solves the following initial-boundary value problem for the one-dimensional heat equation:

$$\begin{aligned}
&\partial_t\theta_1^0 - \partial_{ZZ}\theta_1^0 = 0,\\
&\theta_1^0|_{Z=0} = \beta_1^0(t) - u_1^0(t,0),\ \theta_1^0|_{Z=\infty} = 0,\\
&\theta_1^0|_{t=0} = 0.
\end{aligned} \tag{A.1}$$

Below, we denote $\langle Z\rangle := \sqrt{Z^2+1}$.

Lemma A.1. *Assume that* $\beta_1^0 \in L^\infty(0,T)$ *and* $u_1^0 \in L^\infty(0,T;H^1(0,1))$. *Then, for any* $l \in \mathbb{Z}_+$ *there exists a constant* $C_l > 0$ *depending on* T *such that*

$$\|\langle Z\rangle^l\theta_1^0\|_{(L^\infty(0,T)\times\Omega_\infty)} \leq C_l(\|\beta_1^0\|_{L^\infty(0,T)}, \|u_1^0(t,z)\|_{L^\infty(0,T;H^1(0,1))}), \tag{A.2}$$

Proof. We let $G := \|\beta_1^0(\cdot) - u_1^0(0,\cdot)\|_{L^\infty(0,T)}$. Then, by the maximum principle for the heat equation from (A.1), we have

$$\|\theta_1^0\|_{L^\infty((0,T)\times\Omega_\infty)} \leq G$$

Next, we define $\kappa(t,Z) := Ge^{t-Z}$. The function κ satisfies the following initial-boundary value problem for the heat equation on $[0,+\infty)$.

$$\begin{aligned}
&\kappa_t - \partial_{ZZ}\kappa = 0,\\
&\kappa|_{t=0} = Ge^{-Z} \geq 0,\ \kappa|_{Z=0} = Ge^t > G,\ \kappa|_{Z=\infty} = 0.
\end{aligned} \tag{A.3}$$

By the comparison principle for the heat equation it follows that

$$\|\theta_1^0\|_{L^\infty((0,T)\times\Omega_\infty)} \leq G\,e^T. \tag{A.4}$$

Then, for any $l \geq 1$,

$$\|\langle Z\rangle^l \theta_1^0\|_{L^\infty((0,T)\times\Omega_\infty)} \leq G\|Z^l e^{T-Z}\|_{L^\infty([0,+\infty))} \leq e^T\|\beta_1^0(\cdot)-u_1^0(\cdot,0)\|_{L^\infty(0,T)},$$

which gives the desired result. Above, we have used the property that $\langle Z^l\rangle\, e^{-Z}$ is uniformly bounded in Z for any l. ☐

We use Lemma A.1 to prove the decay properties of θ_2^0. First, we recall that θ_2^0 satisfies

$$\begin{aligned}
&\partial_t\theta_2^0 + \theta_1^0\partial_x\theta_2^0 + u_1^0(t,0)\partial_x\theta_2^0 - \partial_{ZZ}\theta_2^0 = -\theta_1^0\partial_x u_2^0(t,x,0)\\
&\theta_2^0|_{Z=0} = \beta_2^0(t,x) - u_2^0(t,x,0),\ \theta_2^0|_{Z=\infty} = 0,\\
&\theta_2^0|_{t=0} = 0.
\end{aligned} \tag{A.5}$$

We begin with deriving bounds on $\|\langle Z\rangle^l\partial_{xxx}\theta_2^0\|_{L^\infty(0,T;L^2(\Omega_\infty))}$, $\|\partial_{xxx}\theta_2^0\|_{L^\infty((0,T)\times\Omega_\infty)}$, and $\|\partial_{xx}\partial_Z\theta_2^0\|_{L^2(0,T;L^2(\Omega_\infty))}$, which are employed in Section 5.

Lemma A.2. *Assume that* $\beta_2^0 \in H^2(0,T;H^5(0,L))$, $f_2 \in L^\infty(0,T;H^{3+s}(0,L))$, $u_1^0 \in L^\infty(0,T;H^1(0,L)) \subset L^\infty([0,T)\times[0,L])$, *and* $u_2^0 \in L^\infty(0,T;H^5(\Omega))$, $s>0$. *Then, for each* $l \in \mathbb{Z}_+$, *there exists* $C_1 > 0$ *depending on* $\|f_2\|_{L^2(0,T;H^{3+s})}$, $\|u_1^0\|_{L^\infty((0,T)\times(0,1))}$, $\|\beta_2^0\|_{H^1(0,T;H^5(0,L))}$, $\|u_2^0\|_{L^\infty((0,T);H^5(\Omega))}$, T, *and* l, *such that for* $i=0,1,2,3$

$$\|\langle Z\rangle^l\partial_x^i\theta_2^0\|_{L^\infty(0,T;L^2(\Omega_\infty))} + \|\langle Z\rangle^l\partial_x^i\partial_Z\theta_2^0\|_{L^2(0,T;L^2(\Omega_\infty))} \leq C_1, \tag{A.6}$$

Furthermore, for more regular data such that $\|u_2^0\|_{L^\infty((0,T);H^5(\Omega))}$ *and* $\|f_2(t,x,z)\|_{L^\infty(0,T;H^{4+s}(\Omega))}$, $s>0$, *there exists a constant* $C_2>0$ *such that for* $i=0,1,2,3$,

$$\|\langle Z\rangle^l\partial_x^i\theta_2^0\|_{L^\infty((0,T)\times\Omega_\infty)} \leq C, \tag{A.7}$$

where C_2 *depends on* $\|u_1^0(t,z)\|_{L^\infty(0,T;H^1(0,1))}$, $\|\beta_2^0\|_{H^{1+s}(0,T;H^5(0,L))}$, $\|u_2^0(t,x,z)\|_{L^\infty(0,T;H^{5+s}(\Omega))}$, $\|f_2(t,x,z)\|_{L^\infty(0,T;H^{4+s}(\Omega))}$, T, *and* l.

Proof. Since $\partial_x^i\theta_2^0, i=1,2,3$ satisfies the same equation as θ_2^0, with boundary data given by derivatives of $\boldsymbol{\beta_i}$, $i=1,2$, we will establish only the bounds for θ_2^0. Similar arguments apply for the derivatives provided the data has sufficient regularity.

We will use again comparison estimates for parabolic equations. We henceforth define $w(t,Z) = \theta_2^0(t,x,Z) - (\beta_2^0(t,x) - u_2^0|_{z=0}(t,x))e^{-Z}$. Next, we observe that we have imposed enough regularity on the data and $\mathbf{u}^0$ such that u_2^0 is a classical solution of (2.7) and we can extrapolate the validity of this equation at $z=0$ to obtain an equation satisfied by $u_0^2|_{z=0}$ in t and x.

Consequently, we derive the system satisfied by w as follows:

$$\begin{aligned}&\partial_t w+\theta_1^0\partial_x w+u_1^0(t,0)\partial_x w-\partial_{ZZ}w=-\theta_1^0\partial_x u_2^0(t,\cdot,0)+H,\\&w|_{Z=0}=0,\ w|_{Z=\infty}=0,\ w|_{t=0}=0,\end{aligned}\tag{A.8}$$

where

$$\begin{aligned}H(t,x,Z):=-[\partial_t\beta_2^0(t,x)-f_2(t,x,0)+\theta_1^0(t,Z)\partial_x\beta_2^0(t,x)\\-\theta_1^0(t,Z)\partial_x u_2^0(t,x,0)+u_1^0(t,0)\partial_x\beta_2^0(t,x)+\beta_2^0(t,x)-u_2^0(t,x,0)]e^{-Z}\end{aligned}$$

acts as a forcing term.

Multiplying the first equation in (A.8) by $\langle Z\rangle^{2l}w$ and integrating by parts over Ω_∞ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\langle Z\rangle^l w\|_{L^2(\Omega_\infty)}^2+\|\langle Z\rangle^l\partial_Z w\|_{L^2(\Omega_\infty)}^2\\&\le\|\langle Z\rangle^l w\|_{L^2(\Omega_\infty)}^2+\|u_2^0(t,x,0)\|_{H^1(0,L)}\|\langle Z\rangle^l\theta_1^0\|_{L^2(\Omega_\infty)}\|\langle Z\rangle^l w\|_{L^2(\Omega_\infty)}\\&\quad+[\|\partial_t\beta_2^0\|_{L^2(0,L)}+\|f_2(t,x,0)\|_{L^2(0,L)}+|u_1^0(t,0)|\|\beta_2^0\|_{H^1(0,L)}\\&\quad+(\|\beta_2^0\|_{H^1(0,L)}+\|u_2^0(t,x,0)\|_{H^1(0,L)})\\&\qquad(\|\langle Z\rangle^l\theta_1^0\|_{L^2(0,\infty)}+\|\langle Z\rangle^l e^{-Z}\|_{L^2(0,\infty)})]\|\langle Z\rangle^l w\|_{L^2(\Omega_\infty)},\end{aligned}$$

where we have used the Sobolev inequality $H^1([0,L])\subset L^\infty([0,L])$ and Lemma A.1. By Grönwall's inequality, we then obtain that

$$\begin{aligned}&\|\langle Z\rangle^l w\|_{L^\infty(0,T;L^2(\Omega_\infty))}+\|\langle Z\rangle^l\partial_Z w\|_{L^2(0,T;L^2(\Omega_\infty))}\\&\le C(\|\beta_2^0\|_{H^1(0,T;H^1(0,L))},\|u_2^0\|_{L^2(0,T;H^2(0,L))},\\&\qquad\|f_2\|_{L^2(0,T;L^2(\Omega))},\|u_1^0\|_{L^2(0,T;H^1(0,L))},T,l),\end{aligned}$$

from which (A.6) easily follows.

Similarly, by multiplying both sides of the first equation in (A.8) by $p|w|^{p-2}w, p>2$ and integrating by parts, we have

$$\begin{aligned}&\frac{d}{dt}\|w\|_{L^p(\Omega_\infty)}^p+\frac{4(p-1)}{p}\|\partial_Z|w|^{\frac{p}{2}}\|_{L^2(\Omega_\infty)}^2\\&\le p\|u_2^0(t,\cdot,0)\|_{L^\infty(0,L)}\|\theta_1^0\|_{L^p(0,\infty)}\|w\|_{L^p(\Omega_\infty)}^{p-1}+p\|H\|_{L^p(\Omega_\infty)}\|w\|_{L^p(\Omega_\infty)}^{p-1}.\end{aligned}$$

We then conclude, by dividing for $\|w\|_{L^p(\Omega_\infty)}^{p-1}$ (which is finite by the regularity of the system (A.8)), that

$$\frac{d}{dt}\|w\|_{L^p(\Omega_\infty)}\le\|u_2^0(t,\cdot,0)\|_{L^\infty(0,L)}\|\theta_1^0\|_{L^p(0,\infty)}+\|H\|_{L^p(\Omega)}\tag{A.9}$$

Taking the limit $p \to \infty$ and integrating in time yields similarly

$$\begin{aligned}&\|w\|_{L^\infty((0,T)\times\Omega_\infty)}\\ &\quad\le C(\|\theta_1^0\|_{L^\infty(0,\infty)}, \|u_2^0\|_{L^\infty(0,T;H^{2+s}(\Omega))}, \|\beta_2^0\|_{H^2((0,T)\times(0,L))},\\ &\qquad\quad \|u_1^0\|_{L^\infty((0,T;H^1(0,L))}, \|f_2\|_{L^\infty((0,T;H^1(\Omega))}, l).\end{aligned}$$

Then the desired conclusion (A.7) follows. □

Completely analogous results hold for $\theta_1^{u,0}$ and $\theta_2^{u,0}$, which we state below for completeness.

Lemma A.3. *Under the regularity assumptions on* $\mathbf{u}^0$ *and* f_2 *of Lemmas A.1 and A.2 and similar ones on* β_2^1*, it holds for* $i = 0, 1, 2, 3$*,*

$$\begin{aligned}&\|\langle Z^u\rangle^l \theta_1^{u,0}\|_{(L^\infty(0,T)\times\Omega_\infty)} \le C,\\ &\|\langle Z^u\rangle^l \partial_x^i \theta_2^{u,0}\|_{L^\infty(0,T;L^2(\Omega_\infty))} + \|\langle Z^u\rangle^l \partial_x^i \partial_Z \theta_2^{u,0}\|_{L^2(0,T;L^2(\Omega_\infty))} \le C,\\ &\|\langle Z\rangle^l \partial_x^i \theta_2^{u,0}\|_{L^\infty((0,T)\times\Omega_\infty)} \le C,\end{aligned} \tag{A.10}$$

where and C *depends on* $\|f_2\|_{L^\infty(0,T;H^{4+s}(\Omega))}$, $\|u_1^0(t,z)\|_{L^\infty(0,T;H^1(0,1))}$, $\|\beta_2^1\|_{H^2(0,T;H^5(0,L))}$, $\|u_2^0(t,x,z)\|_{L^\infty((0,T);H^{5+s}(\Omega))}$, $s > 0$*,* T*, and* l*.*

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