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Discrete Kato-type theorem on inviscid limit of Navier-Stokes flows

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The inviscid limit of wall bounded viscous flows is one of the unanswered central questions in theoretical fluid dynamics. Here we present a somewhat surprising result related to numerical approximation of the problem. More precisely, we show that numerical solutions of the incompressible Navier-Stokes equations converge to the exact solution of the Euler equations at vanishing viscosity and vanishing mesh size provided that small scales of the order of ν/U in the directions tangential to the boundary are not resolved in the scheme. Here ν is the kinematic viscosity of the fluid and U is the typical velocity taken to be the maximum of the shear velocity at the boundary for the inviscid flow. Such a result is somewhat counterintuitive since the convergence is ensured even in the case that small scales predicted by the conventional theory of turbulence and boundary layer are not resolved since under-resolution (which is allowed in our theorem) in advection dominated problem usually leads to oscillation which inhibits convergence in general. The result also indicates possible difficulty in terms of numerical investigation of the vanishing viscosity problem if rigorous fidelity of the numerics is desired since we have to resolve at least small scales of the order of ν/U which is much smaller than any small scales predicted by the conventional theory of turbulence. On the other hand, such a result can be viewed as a discrete version of our result [X. Wang, Indiana Univ. Math. J. **50**, 223 (2001)] which generalized earlier the result of Kato [in *Seminar on PDE*, edited by S. S. Chern (Springer, NY, 1984)] where the relevance of a scale proportional to the kinematic viscosity to the problem of vanishing viscosity is first discovered.
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I. INTRODUCTION

One of the central and most useful systems in fluid dynamics is the Navier-Stokes system for incompressible homogeneous Newtonian fluids which governs the motion of fluids like air and water under normal conditions

$$\frac{\partial u^{\nu}}{\partial t} + (\mathbf{u}^{\nu} \cdot \nabla)\mathbf{u}^{\nu} - \nu\Delta\mathbf{u}^{\nu} + \nabla p^{\nu} = \mathbf{f}, \text{ in } \Omega, \tag{1}$$

$$\text{div } \mathbf{u}^{\nu} = 0, \text{ in } \Omega, \tag{2}$$

$$\mathbf{u}^{\nu} = \mathbf{b}, \text{ on } \Gamma, \tag{3}$$

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$$\mathbf{u}^{\nu} = \mathbf{u}_0, \text{ at } t = 0, \tag{4}$$

where $\mathbf{u}^{\nu} = (u_1^{\nu}, u_2^{\nu}, u_3^{\nu})$ is the velocity field in the Eulerian coordinates, p^{ν} is the kinematic pressure, $\mathbf{f} = (f_1, f_2, f_3)$ is the external body force, and the positive constant ν is the kinematic viscosity. The velocity $\mathbf{b}$ at the boundary satisfies the no-penetration condition

$$\mathbf{b} \cdot \mathbf{n} = 0, \tag{5}$$

where $\mathbf{n}$ is the unit outward normal to the boundary $\Gamma = \partial\Omega$. This includes the case of Taylor-Couette-type flows among others. The boundary condition sometimes is referred to as a characteristic boundary condition since the boundary consists of stream lines all the time.

There is abundant literature on the Navier-Stokes systems. The interested reader may consult the books by Constantin and Foias (1988), Doering and Gibbon (1995), Ladyzhenskaya (1969), Majda and Bertozzi (2001), or Temam (2001) for the mathematical theories of the Navier-Stokes equations.

For realistic fluids like air and water, the kinematic viscosity is very small and hence we may formally set it to zero and arrive at the Euler system for incompressible inviscid (dry) fluids

$$\frac{\partial \mathbf{u}^0}{\partial t} + (\mathbf{u}^0 \cdot \nabla)\mathbf{u}^0 + \nabla p^0 = \mathbf{f}, \text{ in } \Omega, \tag{6}$$

$$\operatorname{div} \mathbf{u^0} = 0, \text{ in } \Omega, \tag{7}$$

$$\mathbf{u^0} \cdot \mathbf{n} = 0, \text{ on } \Gamma, \tag{8}$$

$$\mathbf{u^0} = \mathbf{u_0}, \text{ at } t = 0. \tag{9}$$

More importantly, if the characteristic flow speed U is large or the characteristic length scale L is large, the Reynolds number which is defined as

$$\mathrm{Re} = \frac{LU}{\nu} \tag{10}$$

is large and the nondimensionalized Navier-Stokes system takes the same form except the kinematic viscosity is replaced by the reciprocal of the Reynolds number which is very small. This provides another scenario where inviscid approximation is needed.

Such an approximation has been utilized in many applications. A physically important question is then whether such an approximation can be justified via the zero viscosity limit of the Navier-Stokes equations.

The mathematical investigation of such a problem is extremely difficult due to the singular nature of the problem which involves a boundary layer and the nonlinear nonlocal nature of the systems involved. Extensive efforts have been made to resolve the inviscid limit problem which lead to many partial results [see, for instance, Prandtl (1905), von Karman (1956), Schlichting (1979), etc., from the physical perspective, and Bona and Wu (2002), Weinan and Engquist (1997), Kato (1984), Ladyzhenskaya (1969), Oleinik (1963), Oleinik and Samokhin (1999), Matsui (1994), Sammartino and Caflisch (1996), Temam and Wang (1997, 1998), Wang (2001), and Xin and Zhang (2004) for some of the mathematical results].

Confronted with such a difficult problem, we naturally resort to numerical methods, especially with today's powerful computer and efficient and accurate numerical schemes. A natural question to ask is the fidelity of the numerical results. More precisely, let

$$\mathbf{u}^k = \mathbf{u}_{h_k}^{\nu_k}$$

be a sequence of numerical solutions of an appropriate numerical scheme with kinematic viscosity ν_k and mesh size h_k satisfying the vanishing viscosity and mesh size assumption

$$\nu_k \to 0,\ h_k \to 0, \text{ as } k \to \infty$$

our questions are as follows:

$$\text{Does } \lim_{k\to\infty} \mathbf{u}_{h_k}^{\nu_k} = \mathbf{u}^0 \text{ imply } \lim_{k\to\infty} \mathbf{u}^{\nu_k} = \mathbf{u}^0? \tag{11}$$

$$\text{Does } \lim_{k\to\infty} \mathbf{u}_{h_k}^{\nu_k} \neq \mathbf{u}^0 \text{ imply } \lim_{k\to\infty} \mathbf{u}^{\nu_k} \neq \mathbf{u}^0? \tag{12}$$

What we will demonstrate below is that numerical solutions to the Navier-Stokes system always converges to the exact solution to the Euler system at vanishing viscosity if small scale of the order of ν/U tangential to the boundary is not resolved in the scheme. Therefore, numerically observed convergence at vanishing viscosity may have nothing to do with the convergence of the continuous solutions [solutions of the Navier-Stokes system (1)] at vanishing viscosity to the solution of the Euler system (6). This indicates the difficulty in studying such an inviscid limit problem. Such a result is eluded to in Wang (2001) and is somewhat surprising since convergence is ensured even in the case that small scales predicted by the conventional theory of turbulence and boundary layer theories are not resolved in the scheme (recall that under-resolution in an advection dominated problem usually leads to oscillation which inhibits convergence in general).

The rest of the manuscript is organized as follows. In the next section we introduce the notion of appropriate truncation of the Navier-Stokes system and formulate our main result. We then compare the small scale in our theorem with other small scales predicted by conventional theory of turbulence and boundary layer theory. A sketch of the proof of the main result is presented in the third section. Some numerical results will be presented in Sec. IV, and we offer our concluding remarks in the last section.

II. MAIN RESULT AND REMARKS

It is apparent that the convergence of numerical solutions of the Navier-Stokes system to that of the Euler system should not be expected for arbitrary truncation, but for suitable approximations of the Navier-Stokes system. Thus we need to introduce the notion of appropriate truncation. Also the problem involves several limits: time step; spatial scale; and viscosity. The essential ingredients of an appropriate truncation are the consistency (as required by all convergent numerical schemes) and a bound on the truncated time averaged energy dissipation rate that is independent of the kinematic viscosity [as is consistent with the Kolmogorov theory, see, for instance, Doering and Gibbon (1995), Foias *et al.* (2001), and Frisch (1995)].

In order to focus on the main issue and for the sake of exposition, we consider flow in a two-dimensional (2D) channel. Moreover, we consider discretization in the direction tangential to the boundary only (no time discretization or spatial discretization in the direction normal to the wall). This allows us to concentrate on the phenomena related to tangential (to the wall) spatial discretization only as it is the focus of our main result. The result stated here remains valid for 3D general domain with discretization in the directions tangential to the wall in a boundary layer done using local curvilinear coordinates, and the additional assumption that the Euler system possesses a smooth enough solution on that fixed time interval under consideration.

For the channel geometry with periodicity in the horizontal direction, it is natural to use Fourier spectral truncation in the horizontal direction and thus a natural (suitable) truncation would be the following Galerkin truncation:

$$\frac{\partial \mathbf{u}^k}{\partial t} + P_k((\mathbf{u}^k \cdot \nabla)\mathbf{u}^k) - \nu_k \Delta \mathbf{u}^k + \nabla p^k = P_k \mathbf{f}, \tag{13}$$

$$\operatorname{div} \mathbf{u}^k = 0, \tag{14}$$

$$\mathbf{u}^k|_{z=0,h} = P_k \mathbf{b}, \tag{15}$$

$$\mathbf{u}^k|_{t=0} = P_k \mathbf{u}_0, \tag{16}$$

where P_k is the projection onto the first K_k modes in x, i.e.,

$$P_k \mathbf{u} = \sum_{|j| \le K_k} e^{2\pi i j x/L} \hat{\mathbf{u}}^j, \quad \left(\mathbf{u} = \sum_j e^{2\pi i j x/L} \hat{\mathbf{u}}^j\right). \tag{17}$$

The consistency of such a truncation is obvious. An appropriate bound on the energy dissipation rate will be derived later in the next section.

We also introduce the following quantity as a typical velocity:

$$U = \sup_k \max_{[0,T]\times\Gamma} \{|P_k(b_1 - u_1^0)|\}. \tag{18}$$

Our main result is as follows.

Theorem 1: *Suppose that we have a smooth solution $\mathbf{u}^0$ of the Euler system (6) on the time interval $[0,T]$. [This is guaranteed in the 2D case with smooth enough data satisfying certain compatibility condition, see Temam (1975).] Let $\mathbf{u}^k$ be the solution of the truncated Navier-Stokes system (13) with kinematic viscosity ν_k. Assume that the following conditions are satisfied:*

$$K_k \to \infty \text{ (consistency)}, \tag{19}$$

$$\nu_k \to 0 \text{ (vanishing viscosity)}, \tag{20}$$

$$K_k \frac{\nu_k}{LU} \to 0 \text{ (under-resolved condition)}. \tag{21}$$

Then

$$\mathbf{u}^k \to \mathbf{u}^0. \tag{22}$$

More precisely, there exists a generic constant κ independent of k such that

$$\|\mathbf{u}^k - \mathbf{u}^0\|_{L^\infty(O,T;L^2)} \le \kappa\left((K_k \nu_k)^{\frac{1}{5}} + \|\mathbf{u}^0 - P_k \mathbf{u}^0\|_{L^2(O,T;H^1)} + \|\mathbf{u}^0 - P_k \mathbf{u}^0\|_{L^\infty(O,T;L^2)}\right). \tag{23}$$

The under-resolved condition (21) can be written in terms of the smallest scale, denoted l_s, resolved by the numerical method in the direction tangential to the boundary. Indeed, since $K_k l_s = L$, the under-resolved condition is equivalent to

$$\frac{\nu_k/U}{l_s} \to 0. \tag{24}$$

This means that scales of the order ν/U are not resolved in the scheme.

The appearance of this small scale is a little bit surprising since it is smaller than any of the known scales predicted by conventional theory of turbulence and boundary layer theory. Here we recall a few well-known small scales (Foias *et al.*, 2001; Doering and Gibbon, 1995; Prandtl, 1905; Frisch, 1995; among others):

- Prandtl boundary layer thickness:

$$\sqrt{\nu T}; \tag{25}$$

- Kolmogorov dissipation length (3D):

$$\left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \sim \nu^{3/4}, \tag{26}$$

where ε is the energy dissipation rate per unit volume and is presumably independent of the kinematic viscosity. [The energy dissipation rate per unit volume scales as U^3/h. In the case of boundary driven flow, the typical velocity is specified by the boundary value and thus is independent of the viscosity (see, for instance, Doering and Gibbon, 1995; Foias *et al.*, 2001; Wang, 1997; among others). In the case of body force driven flow, it is easy to derive that $\varepsilon(\sim U^3/h) \leq \kappa|f|_{L^2}Uh^{-3/2}$, where f is the external body force, the typical velocity U is defined as the root-mean-square (space and time averaged) velocity. It is also know (see Doering and Foias, 2002, among others) that $\mathrm{Re}(=hU/\nu) \geq \kappa Gr^{1/2}(=|f|_{L^2}^{1/2}h^{3/4}\nu)$. Therefore, $U \sim |f|_{L^2}^{1/2}h^{-1/4}$. Hence the ε is independent of viscosity for fixed body force and for turbulent flow which saturates the Kolmogorov scaling. Similar arguments can be made for the entropy dissipation rate. The Taylor microscale can be estimated in the same fashion.]

- Kraichnan dissipation length (2D):

$$\left(\frac{\nu^3}{\eta}\right)^{1/6} \sim \nu^{1/2}, \tag{27}$$

where η is the enstrophy dissipation rate per unit volume which is presumably independent of the kinematic viscosity.

- Taylor microlength:

$$\left(\frac{\nu U^3}{\varepsilon}\right)^{1/2} \sim \nu^{1/2}. \tag{28}$$

Notice that these small length scales are all much bigger than ν/U. Even the thickness of a viscous sublayer $[(\nu/U)\log \mathrm{Re}]$ predicted by some boundary layer theory is bigger than ν/U at large Reynolds number (and the thickness of viscous layer is a small scale in the direction normal to wall only). Thus, if one follows the conventional wisdom, one would just resolve the small scales predicted by conventional theory and thus the numerical results would indicate convergence of numerical solutions to that of the Euler system (see Sec. IV below).

Of course, ν/U appear as the natural small scale in certain circumstances such as the boundary layer thickness in the presence of suction at the boundary. The appearance of the thickness ν/U is directly related to the suction which makes the boundary layer thinner and stable (see Temam and Wang, 2000,2002). Even in that case, the scale of ν/U appears *only* in the direction *normal* to the boundary in the boundary layer.

The relevance of small scales of the order of ν/U to the inviscid limit problem was first discovered by Kato (1984) and was improved to the case of small scale of the order of ν/U in the directions tangential to the boundary in an appropriate boundary layer by Temam and Wang (1998) and Wang (2001). The main result here is essentially a discrete version of the main result stated in Wang (2001) and thus a discrete Kato-type result.

Notice that the main result implies convergence even if small scales predicted by the conventional theory of turbulence and boundary layer theory listed above are not resolved in the scheme. For instance, we may choose

$$K_k = \left(\frac{LU}{\nu_k}\right)^{\alpha}, \quad \alpha \in \left(0, \frac{1}{2}\right). \tag{29}$$

This is what we mean by under-resolution. The convergence of numerical solutions under the under-resolved condition is puzzling since the small scale resolved here can be much bigger than any of the small scales predicted by the conventional wisdom as we discussed in the previous paragraphs. It is well-known that we usually expect oscillation (Gibbs-type phenomena) in convection dominated systems if small scales are not well-resolved (see, for instance, Gottlieb and Orszag, 1977; Fletcher, 1988; Morton, 1996; Cheng and Temam, 2002; Cheng, Temam, and Wang 2000; among others). Oscillation usually inhibits convergence which is contradictory to our main result.

III. SKETCH OF THE PROOF

Throughout this section, κ will denote a generic constant independent of the kinematic viscosity ν or truncation wave number K_k.

Our proof is along the line of Kato (1984) and Temam and Wang (1998) with some modification. The basic idea is to construct a so-called *background* flow [Hopf- type technique (Hopf, 1955)] with a free parameter α which interpolates between the viscous sublayer (Kato-type result) and laminar boundary layer (Prandtl theory).

For simplicity we consider channel flow (flat boundary) and two-dimensional case only. The case with curved boundary can be treated in the same way as in our previous work (Temam and Wang, 1997; Wang, 1997) using curvilinear coordinates. The three-dimensional case is very similar to our work on energy dissipation rate (Wang, 2000).

Our approach is close to the idea of Vishik and Lyusternik (1957) (see also Lions, 1973) in the sense that we seek a corrector which approximates the difference between the viscous and inviscid solution. Hence it is slightly different from Kato's (Kato, 1984) approach.

Since we are interested in the asymptotic behavior of the solution $\mathbf{u}^k$ to the Galerkin truncated Navier-Stokes system (13), we naturally compare $\mathbf{u}^k$ to the spectral truncation of the solution to the Euler equation, namely, $P_k\mathbf{u}^0$. Notice that $P_k\mathbf{u}^0$ satisfies the system

$$\frac{\partial}{\partial t}P_k\mathbf{u}^0 + P_k((P_k\mathbf{u}^0\cdot\nabla)P_k\mathbf{u}^0) + \nabla P_k p^0 = P_k\mathbf{f} + \mathbf{g}_k, \tag{30}$$

$$\text{div } P_k\mathbf{u}^0 = 0, \tag{31}$$

$$P_k\mathbf{u}^0\cdot\mathbf{n}|_z = 0, \quad h = 0, \tag{32}$$

$$P_k\mathbf{u}^0|_{t=0} = P_k\mathbf{u}_0, \tag{33}$$

where

$$\mathbf{g}_k = -P_k(((I-P_k)\mathbf{u}^0\cdot\nabla)\mathbf{u}^0) - P_k((P_k\mathbf{u}^0\cdot\nabla)(I-P_k)\mathbf{u}^0). \tag{34}$$

It is easy to see that $\mathbf{g}_k$ is small for large k due to the consistency assumption and the smoothness assumption on the inviscid solution $\mathbf{u}^0$. Indeed

$$\|\mathbf{g}_k\|_{L^2} \le \kappa(\|\nabla\mathbf{u}^0\|_{L^\infty}\|(I-P_k)\mathbf{u}^0\|_{L^2} + \|P_k\mathbf{u}^0\|_{L^\infty}\|\nabla(I-P_k)\mathbf{u}^0\|_{L^2} \le \kappa\|\nabla\mathbf{u}^0\|_{L^\infty}\|(I-P_k)\mathbf{u}^0\|_{H^1}. \tag{35}$$

We now follow the strategy of the continuous case and compare $\mathbf{u}^k$ to $P_k\mathbf{u}^0$ with the aid of a corrector (background flow). For this purpose we need to first establish the upper bound on the energy dissipation rate independent of the kinematic viscosity for the truncated Navier-Stokes system (13) just as in the continuous case.

Let ϕ be a fixed (smooth) incompressible flow that matches $\mathbf{b}$ on the boundary of the domain. The existence of such flows is classical (see, for instance, Temam, 2001 and Wang, 2001).

Consider

$$\mathbf{v}^k = \mathbf{u}^k - P_k\varphi.$$

We then deduce that $\mathbf{v}^k$ satisfies the following system:

$$\frac{\partial \mathbf{v}^k}{\partial t} + P_k((\mathbf{v}^k\cdot\nabla)\mathbf{v}^k) + P_k((\mathbf{v}^k\cdot\nabla)P_k\phi) + P_k((P_k\phi\cdot\nabla)\mathbf{v}^k) - \nu_k\Delta\mathbf{v}^k + \nabla p^k$$

$$= P_k\mathbf{f} - \frac{\partial}{\partial t}P_k\phi - P_k((P_k\phi\cdot\nabla)P_k\phi) + \nu_k\Delta P_k\phi,$$

$$\operatorname{div}\,\mathbf{v}^k = 0,$$

$$\mathbf{v}^k|_{z=0,h} = 0,$$

$$\mathbf{v}^k|_{t=0} = P_k(\mathbf{u}_0 - \phi(0)).$$

Multiplying both sides by $\mathbf{v}^k$, integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{v}^k|_{L^2}^2 + \nu_k|\nabla\mathbf{v}^k|_{L^2}^2 \le |\nabla P_k\phi|_{L^\infty}|\mathbf{v}^k|_{L^2}^2 + \kappa\left(|\mathbf{f}|_{L^2} + \left|\frac{\partial\phi}{\partial t}\right|_{L^2} + |\phi|_{H^2}|\nabla\phi|_{L^2} + \nu_k|\Delta\phi|_{L^2}\right)|\mathbf{v}^k|_{L^2},$$

which implies

$$\|\mathbf{v}^k\|_{L^\infty(0,T;L^2)\le\kappa},$$

which further implies

$$\nu_k\int_0^T|\nabla\mathbf{v}^k|_{L^2}^2\,\mathrm{d}t \le \kappa,$$

where κ is a constant independent of k (or ν_k). Since $\mathbf{u}^k$ and $\mathbf{v}^k$ differ by $P_k\phi$, we also have

$$\nu_k\int_0^T|\nabla\mathbf{u}^k|_{L^2}^2\,\mathrm{d}t \le \kappa. \tag{36}$$

Next we move on to the issue of convergence of $\mathbf{u}^k$ to $\mathbf{u}^0$ under the under-resolved condition. We first introduce a corrector (background flow) just as in the continuous case. The key idea, in addition to the ones that we had for the continuous case, is a reverse Poincaré inequality which implies the smallness of energy dissipation rate due to the tangential derivative of the flow.

Define the stream function

$$\psi^k(x,z,t) = P_k(b_1(x,0,t) - u_1^0(x,0,t))\int_0^z\rho\left(\frac{\alpha Us}{\nu_k}\right)\mathrm{d}s, \tag{37}$$

where the cutoff function ρ satisfies the following properties:

$$\rho\in C^\infty[0,\infty),$$

$$\rho(0)=1,$$

$$\rho'(0)=0,$$

$$\operatorname{supp}\,\rho\subset[0,\ 1),$$

$$\int_0^1 \rho = 0,$$

$$|\rho|_{L^\infty} \leq 1,$$

$$|\rho'|_{L^\infty} \leq 2,$$

and the typical velocity U is defined as in Eq. (18).

The corresponding velocity field is

$$\theta^k(x,z,t) = \text{curl } \psi^k(x,z,t) = \left(\frac{\partial \psi^k}{\partial z}, - \frac{\partial \psi^k}{\partial x} \right). \tag{38}$$

The typical velocity defined is a natural generalization of the continuous one $(\max_{[0,T]\times\Gamma}|b_1 - u_1^0|)$ to this truncated case. This new typical velocity dominates the continuous version since we have, for smooth enough $b_1 - u_1^0$,

$$\lim_{k\to\infty} P_k(b_1 - u_1^0) = b_1 - u_1^0.$$

Next, we consider the adjusted differences

$$\mathbf{w}^k = \mathbf{u}^k - P_k\mathbf{u}^0 - \theta^k. \tag{39}$$

Our goal is to prove $\mathbf{w}^k \to 0$ which implies our final result since $\theta^k \to 0$ in $L^\infty(0,T;L^2)$ and $P_k\mathbf{u}^0 \to \mathbf{u}^0$ in $L^\infty(0,T;L^2)$ as k approaches infinity.

It is easy to verify that $\mathbf{w}^k$ satisfies

$$\begin{aligned} \frac{\partial \mathbf{w}^k}{\partial t} + P_k((\mathbf{u}^k\cdot\nabla)\mathbf{w}^k) - \nu_k\Delta\mathbf{w}^k + \nabla q^k = &- \frac{\partial\theta^k}{\partial t} + \nu_k\Delta\mathbf{u}^0 + \nu_k\Delta\theta^k - P_k((\theta^k\cdot\nabla)\theta^k) - P_k((\mathbf{w}^k\cdot\nabla)\theta^k) \\ &- P_k((\mathbf{u}^0\cdot\nabla)\theta^k) - P_k((\mathbf{w}^k\cdot\nabla)P_k\mathbf{u}^0) - P_k((\theta^k\cdot\nabla)P_k\mathbf{u}^0) + \mathbf{g}_k, \end{aligned} \tag{40}$$

$$\text{div } \mathbf{w}^k = 0, \tag{41}$$

$$\mathbf{w}^k|_{z=0,h} = 0, \tag{42}$$

$$\mathbf{w}^k|_{t=0} = 0. \tag{43}$$

Thanks to the explicit construction of our θ^k, we have

$$\left|\frac{\partial\theta^k}{\partial t}\right|_{L^2}^2 \leq U_t^2\frac{L\nu_k}{\alpha U} + U_{tx}^2\frac{L\nu_k^3}{\alpha^3U^3},$$

$$|\nabla\theta^k|_{L^2}^2 \leq 2U_x^2\frac{L\nu_k}{\alpha U} + U^2\frac{L\alpha U}{\nu_k} + U_{xx}^2\frac{L\nu_k^3}{\alpha^3U^3},$$

$$|P_k(\theta^k\cdot\nabla)\theta^k|_{L^2}^2 \leq 5U^2U_x^2\frac{L\nu_k}{\alpha U} + U^2U_{xx}^2\frac{L\nu_k^3}{\alpha^3U^3} + U_x^4\frac{L\nu_k^3}{\alpha^3U^3},$$

$$|P_k(P_k\mathbf{u}^0\cdot\nabla)\theta^k|_{L^2}^2 \le 2\left(|P_k\mathbf{u}_1^0|_{L^\infty}^2\left|\frac{\partial\theta^k}{\partial x}\right|_{L^2}^2+\left|\frac{P_k\mathbf{u}_2^0}{z(h-z)}\right|_{L^\infty}^2\left|z(h-z)\frac{\partial\theta^k}{\partial z}\right|_{L^2}^2\right)\le\kappa\left(\frac{L\nu_k}{\alpha U}+\frac{L\nu_k^3}{\alpha^3U^3}\right),$$

$$|P_k(\mathbf{w}^k\cdot\nabla)P_k\mathbf{u}^0|_{L^2}\le|\nabla P_k\mathbf{u}^0|_{L^\infty}|\mathbf{w}^k|_{L^2},$$

$$|P_k(\theta^k\cdot\nabla)P_k\mathbf{u}^0 \leqslant |_{L^2}^2|\nabla P_k\mathbf{u}^0|_{L^\infty}^2\left(U^2\frac{L\nu_k}{\alpha U}+U_x^2\frac{L\nu_k^3}{\alpha^3U^3}\right)\le\kappa|\mathbf{u}^0|_{H^3}^2\left(U^2\frac{L\nu_k}{\alpha U}+U_x^2\frac{L\nu_k^3}{\alpha^3U^3}\right),$$

where we have used the impermeable wall boundary condition (32), and

$$U_t=\sup_k\max_{[0,T]\times\Gamma}\left\{\left|P_k\left(\frac{\partial b_1}{\partial t}-\frac{\partial u_1^0}{\partial t}\right)\right|\right\},$$

$$U_x=\sup_k\max_{[0,T]\times\Gamma}\left\{\left|P_k\left(\frac{\partial b_1}{\partial x}-\frac{\partial u_1^0}{\partial x}\right)\right|\right\},$$

$$U_{tx}=\sup_k\max_{[0,T]\times\Gamma}\left\{\left|P_k\left(\frac{\partial^2 b_1}{\partial x\,\partial t}-\frac{\partial^2 u_1^0}{\partial x\,\partial t}\right)\right|\right\},$$

$$U_{xx}=\sup_k\max_{[0,T]\times\Gamma}\left\{\left|P_k\left(\frac{\partial^2 b_1}{\partial x^2}-\frac{\partial^2 u_1^0}{\partial x^2}\right)\right|\right\}.$$

We then deduce, via the standard energy method,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{w}^k|_{L^2}^2+\nu_k|\nabla\mathbf{w}^k|_{L^2}^2\le\nu_k\sqrt{2U_x^2\frac{L\nu_k}{\alpha U}+U^2\frac{L\alpha U}{\nu_k}+U_{xx}^2\frac{L\nu_k^3}{\alpha^3U^3}}|\nabla\mathbf{w}^k|_{L^2}+\nu_k^2|\Delta\mathbf{u}^0|_{L^2}^2+\kappa|\mathbf{w}^k|_{L^2}^2+\kappa|\mathbf{u}^0$$
$$-P_k\mathbf{u}^0|_{H^1}^2+\kappa\left(\frac{\nu_k}{\alpha}+\frac{\nu_k^3}{\alpha^3}\right)+\int_\Omega(\mathbf{w}^k\cdot\nabla)\mathbf{w}^k\cdot\theta^k. \tag{44}$$

Notice the last (nonlinear) term can be rewritten as

$$\int_\Omega(\mathbf{w}^k\cdot\nabla)\mathbf{w}^k\cdot\theta^k=\int_W w_1^k\frac{\partial w_1^k}{\partial x}\theta_1^k+\int_\Omega w_3^k\frac{\partial w_1^k}{\partial z}\theta_1^k+\int_\Omega w_1^k\frac{\partial w_3^k}{\partial x}\theta_3^k+\int_\Omega w_3^k\frac{\partial w_3^k}{\partial z}\theta_3^k, \tag{45}$$

and hence we have the following estimates on the nonlinear term, thanks to the explicit construction of the corrector (see Wang, 2001):

$$2\int_\Omega w_1^k\frac{\partial w_1^k}{\partial x}\theta_1^k=\int_\Omega\frac{\partial}{\partial x}(w_1^k)^2\theta_1^k=-\int_\Omega(w_1^k)^2\frac{\partial\theta_1^k}{\partial x}\le U_x|w_1^k|_{L^2}^2,$$

$$2\int_\Omega w_3^k\frac{\partial w_1^k}{\partial z}\theta_1^k\le 2U|w_3^k|_{L^2(\Gamma_\delta)}\left|\frac{\partial w_1^k}{\partial z}\right|_{L^2(\Gamma_\delta)}\le 2U\delta\left|\frac{\partial w_3^k}{\partial z}\right|_{L^2(\Gamma_\delta)}\left|\frac{\partial w_1^k}{\partial z}\right|_{L^2(\Gamma_\delta)}$$
$$=\frac{2\nu}{\alpha}\left|\frac{\partial w_1^k}{\partial x}\right|_{L^2(\Gamma_\delta)}\left|\frac{\partial w_1^k}{\partial z}\right|_{L^2(\Gamma_\delta)}\le\frac{\nu}{4}\left|\frac{\partial w_1^k}{\partial z}\right|_{L^2(\Gamma_\delta)}^2+\frac{4\nu}{\alpha^2}\left|\frac{\partial w_1^k}{\partial x}\right|_{L^2(\Gamma_\delta)}^2,$$

$$2\int_\Omega w_1^k \frac{\partial w_3^k}{\partial x}\theta_3^k \le \kappa\frac{\nu}{\alpha}|w_1^k|_{L^2(\Gamma_\delta)}\left|\frac{\partial w_3^k}{\partial x}\right|_{L^2(\Gamma_\delta)} \le \frac{\nu}{4}\left|\frac{\partial w_3^k}{\partial x}\right|^2_{L^2(\Gamma_\delta)} + \kappa\frac{\nu}{\alpha^2}|w_1^k|^2_{L^2(\Gamma_\delta)}.$$

Similarly

$$2\int_\Omega w_3^k \frac{\partial w_3^k}{\partial z}\theta_3^k \le \frac{\nu}{4}\left|\frac{\partial w_3^k}{\partial z}\right|^2_{L^2(\Gamma_\delta)} + \kappa\frac{\nu}{\alpha^2}|w_3^k|^2_{L^2(\Gamma_\delta)}.$$

Thus we have

$$2\int_\Omega (\mathbf{w}^k\cdot\nabla)\mathbf{w}^k\cdot\theta^k \le \frac{\nu_k}{4}|\nabla\mathbf{w}^k|^2_{L^2} + \frac{4\nu_k}{\alpha^2}\left|\frac{\partial w_1^k}{\partial x}\right|_{L^2(\Gamma_\delta)} + \kappa\frac{\nu_k}{\alpha^2}|\mathbf{w}^k|^2_{L^2} + U_x|\mathbf{w}^k|^2_{L^2}, \tag{46}$$

where

$$\delta = \frac{\nu_k}{\alpha U} \tag{47}$$

is the thickness of the boundary layer.

We now make the following assumption on the free parameter α (and thus δ):

$$\alpha = \alpha_k \to 0, \text{ as } k\to\infty, \text{ and } \frac{\nu_k}{\alpha_k^2} \le 1. \tag{48}$$

The first part of the condition is equivalent to saying that the chosen boundary layer must be thicker than ν_k/U since $\delta_k = \frac{\nu_k}{\alpha_k U}$, and the second part of the assumption is equivalent to saying that the thickness of the chosen boundary layer is at most that of the laminar boundary layer $\sqrt{\nu T}$ since

$$\frac{\delta_k^2}{\nu_k} = \frac{\nu_k}{\alpha_k^2 U^2}.$$

The condition also implies that

$$\frac{\nu_k}{\alpha_k} = U\delta_k \to 0, \quad \text{as } k\to\infty.$$

It is then easy to see, that under the assumption on the parameter (48), together with the vanishing viscosity condition (20), and the key estimate on trilinear term (46), the energy inequality on $\mathbf{w}^k$ becomes

$$\frac{d}{dt}|\mathbf{w}^k|^2_{L^2} + \nu_k|\nabla\mathbf{w}^k|^2_{L^2} \le \kappa\left(|\mathbf{w}^k|^2_{L^2} + |\mathbf{u}^0 - P_k\mathbf{u}^0|^2_{H^1} + \frac{\nu_k}{\alpha}\right) + \alpha L U^3 + \frac{8\nu_k}{\alpha^2}\left|\frac{\partial w_1^k}{\partial x}\right|^2_{L^2(\Gamma_\delta)},$$

which implies, after utilizing the Gronwall inequality,

$$\|\mathbf{w}^k\|_{L^\infty(0,T;L^2)} \le \kappa\left(\sqrt{\frac{\nu_k}{\alpha}} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^2(0,T;H^1)} + \left(\alpha L U^3 + \frac{8\nu_k}{\alpha^2}\frac{1}{T}\int_0^T\int_{\Gamma_{\delta_k}}\left|\frac{\partial \mathbf{w}_1^k}{\partial x}\right|^2\right)^{\frac{1}{2}}\right).$$

Therefore

$$\begin{aligned}\|\mathbf{u}^k - \mathbf{u}\|^0_{L^\infty(0,T;L^2)} &\le \|\mathbf{w}^k\|_{L^\infty(0,T;L^2)} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^\infty(0,T;L^2)} + \|\theta^k\|_{L^\infty(0,T;L^2)}\\ &\le \kappa\left(\sqrt{\frac{\nu_k}{\alpha}} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^2(0,T;H^1)} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^\infty(0,T;L^2)}\right)\end{aligned}$$

$$+\kappa\left(\alpha L U^3+\frac{8\nu_k}{\alpha^2}\frac{1}{T}\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial w_1^k}{\partial x}\right|^2\right)^{\frac{1}{2}}. \tag{49}$$

Here α is a free parameter that we may adjust provided the constraints specified in Eq. (48) are met.

Next, we estimate the integral on the right-hand side of Eq. (49) as follows. Notice that

$$\begin{aligned}
\nu_k\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial}{\partial x}u_1^k\right|^2 &\le 2\nu_k\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial}{\partial x}(u_1^k-P_k\varphi_1)\right|^2+2\nu_k\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial}{\partial x}P_k\varphi_1\right|^2\\
&\le 2\nu_k\delta^2\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial^2}{\partial x\,\partial z}(u_1^k-P_k\varphi_1)\right|^2+\kappa\nu_k\delta\\
&\le \kappa\nu_k\delta^2K_k^2\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial}{\partial z}(u_1^k-P_k\varphi_1)\right|^2+\kappa\nu_k\delta\\
&\le \kappa\nu_k\delta^2K_k^2\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial u_1^k}{\partial z}\right|^2+\kappa\nu_k\delta^2K_k^2\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial}{\partial z}P_k\varphi_1\right|^2+\kappa\nu_k\delta\\
&\le \kappa(\delta^2\mathrm{K}_\kappa^2+\nu_k\delta)\le\kappa\left(\frac{\nu_\kappa^2\mathrm{K}_\kappa^2}{\alpha^2}+\frac{\nu_k^2}{\alpha}\right),
\end{aligned}$$

where we have applied the direct and inverse Poincaré inequality, and utilized the bound on energy dissipation rate (36). This further implies

$$\begin{aligned}
\nu_k\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial w_1^k}{\partial x}\right|^2 &\le 2\nu_k\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial u_1^k}{\partial x}\right|^2+2\nu_k\int_0^T\int_{\Gamma_\delta}\left|\frac{\partial(P_k\mathbf{u}^0-\theta^k)}{\partial x}\right|^2\\
&\le \kappa\left(\delta^2K_k^2+\nu_k\delta\right)\le\kappa\left(\frac{\nu_k^2K_k^2}{\alpha^2}+\frac{\nu_k^2}{\alpha}\right).
\end{aligned}$$

We may then rewrite the estimates on $\mathbf{u}^k-\mathbf{u}^0$ as

$$\begin{aligned}
\|\mathbf{u}^k-\mathbf{u}^0\|_{L^\infty(0,T;L^2)} &\le \kappa(\|\mathbf{u}^0-P_k\mathbf{u}^0\|_{L^2(0,T;H^1)}+\|\mathbf{u}^0-P_k\mathbf{u}^0\|_{L^\infty(0,T;L^2)})+\kappa\left(\frac{\nu_k}{\alpha}+\alpha+\frac{\nu_k^2K_k^2}{\alpha^4}+\frac{\nu_k^2}{\alpha^3}\right)^{\frac{1}{2}}\\
&\le \kappa(\|\mathbf{u}^0-P_k\mathbf{u}^0\|_{L^2(0,T;H^1)}+\|\mathbf{u}^0-P_k\mathbf{u}^0\|_{L^\infty(0,T;L^2)})+\kappa\left(\alpha+\frac{\nu_k^2K_k^2}{\alpha^4}\right)^{\frac{1}{2}},
\end{aligned} \tag{50}$$

since ν_k/α is dominated by α as $\frac{\nu_k/\alpha}{\alpha}=\frac{\nu_k}{\alpha^2}\le 1$, while $\frac{\nu_k^2}{\alpha^3}$ is dominated by $\frac{\nu_k}{\alpha}$ as $\frac{\nu_k^2/\alpha^3}{\nu_k/\alpha}=\frac{\nu_k}{\alpha^2}\le 1$.

The last piece of work is to choose an appropriate α which minimizes the expression $\alpha+\frac{\nu_k^2K_k^2}{\alpha^4}$. This is roughly accomplished if we set

$$\alpha=\alpha_k=\left(\frac{\nu_kK_k}{LU}\right)^{\frac{2}{5}}. \tag{51}$$

Obviously α_k approaches zero as k approaches infinity thanks to the under-resolved condition (21). Moreover,

$$\frac{\nu_k}{\alpha_k^2}=\nu_k^{\frac{1}{5}}K_k^{-\frac{4}{5}}(LU)^{\frac{4}{5}}\to 0,\ \ \text{as } k\to\infty,$$

thanks to the consistency condition (19) and the vanishing viscosity condition (20). Thus the α determined by Eq. (51) satisfies the constraint (48) and hence is allowed.

TABLE I. Scaled L^2 norm of the difference between the viscous and inviscid flows.

Viscosity	$\|\nabla^{\perp}\psi - \nabla^{\perp}\psi^0\|_{L^2}\nu^{-1/4}$	K (horizontal wave No.)
10^{-1}	3.40139810188282	32
10^{-2}	3.33091578998929	32
10^{-3}	3.80526863781140	64
10^{-4}	5.09509568114280	64
10^{-5}	5.32291259447423	128
10^{-6}	7.89684211938674	256

In the last step, we plug Eq. (51) into Eq. (50) and we deduce

$$\|\mathbf{u}^k - \mathbf{u}^0\|_{L^\infty(0,T;L^2)} \le \kappa(\|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^2(0,T;H^1)} + \|\mathbf{u}^0 - P_k\mathbf{u}^0\|_{L^\infty(0,T;L^2)} + (\nu_k K_k)^{\frac{1}{5}}), \tag{52}$$

which is exactly what we desired. This ends the proof.

IV. NUMERICAL RESULTS

Here we report numerical results performed on a two-dimensional channel flow with zero flux. In this case the Navier-Stokes system can be formulated in the stream-function only (or stream-function vorticity formulation)

$$\frac{\partial}{\partial t}\Delta\psi + \nabla^{\perp}\psi\cdot\nabla\Delta\psi - \nu\Delta^2\psi = F,$$

$$\psi|_{y=\pm1} = \frac{\partial}{\partial y}\psi|_{y=\pm1} = 0.$$

We assume the channel is $-1 \le y \le 1$ with periodicity in x with period 2π. The initial data is set to zero and the external forcing in the stream-function formulation is set to

$$F(x,y,t) = (3 - y^2)\sin x + 4ty^2 \sin x \cos x.$$

Therefore the Euler equation possess an exact solution of the form

$$\psi^0(x,y,t) = t(1 - y^2)\sin x.$$

A standard spectral method is used to solve the Navier-Stokes equation (see Peyret, 2002). More specifically, we use Fourier series in the x direction and Chebyshev polynomial in the y direction for spatial discretization. A second-order Adams-Bashforth time scheme which is implicit in the viscous term and explicit in the nonlinear advection term is applied.

Table I lists the scaled L^2 norm of the difference between the viscous and inviscid flows, i.e., $\|\nabla^{\perp}\psi - \nabla^{\perp}\psi^0\|_{L^2}\nu^{-1/4}$ at time $t=1$ and the horizontal wave number that is sufficient for numerical convergence for various values of the viscosity. The factor $\nu^{-1/4}$ is used since the convergence would be of the order of $\nu^{1/4}$ if this is a completely laminar case where Prandtl's theory is valid (see, for instance, Temam and Wang 1996, 1998; Sammartino and Calflisch, 1996; Weinan, 2000 among others).

Our result indicates convergence of the solutions of the Navier-Stokes system to that of the Euler system at vanishing viscosity. Of course, the smallest horizontal scales resolved is much bigger than ν/U (in fact at least of the order of $\sqrt{\nu}$). In fact, our main result guarantees the convergence by simply looking at the small scales resolved in the horizontal direction.

Numerical results obtained by Johnston, Liu, and E (2006) on two-dimensional flow past a cylinder indicate the same phenomena: numerical convergence of the viscous solutions to the inviscid solution with a rate of $\nu^{1/4}$ and the smallest scale resolved (needed for numerical convergence) is of the order of $\nu^{1/2}$ (see the figure obtained by Johnston, Liu, and E at http://www.math.temple.edu/~hej/ZVDATA/veldiff.html). Again, our main result guarantees the convergence by looking at the small scales resolved in the scheme.

V. CONCLUDING REMARKS

We have shown that if small scales of the order ν/U are not resolved in the direction tangential to the boundary in numerical scheme for Navier-Stokes equation (NSE) (1), the numerical solutions will always converge to the solution of the Euler system (6) at vanishing viscosity and mesh size for any suitable (reasonable) numerical scheme. This implies that numerical solutions to the Navier-Stokes system will converge to that of the Euler system in the vanishing viscosity limit in the under-resolved case (under-resolved in the sense that small scales predicted by conventional wisdom such as boundary layer theory and turbulence theory are not resolved). This is surprising since we usually expect oscillation in an advection dominated problem in the under-resolved case. The oscillation in turn should inhibit convergence in general.

Numerical results obtained by Johnston, Liu, and E (2006) as well as ours confirm this fact. Of course the numerics can be interpreted in two different ways:

1. No small scales of the order ν/U or smaller are detected in the numerical experiment, and thus numerics provide further evidence that the inviscid limit of viscous flows is the inviscid Euler flow.
2. Small scales of the order ν/U are not resolved in the numerics and thus the numerical solutions must converge to the solution of the inviscid Euler system (6) regardless of whether the solutions of the Navier-Stokes system (1) converge to the solution of the Euler system at vanishing viscosity. In another word, the numerical results may have nothing to do with the continuous problem.

This indicates that in order to guarantee that the convergence of the numerical solutions implies the convergence of the continuous solutions, i.e., providing an affirmative answer to Eq. (11), small scales of the order of ν/U in the direction tangential to the boundary must be resolved in the numerical scheme. This gives us a flavor on the difficulty of the problem of numerical investigation of the vanishing viscosity problem.

A natural question to ask then is what is the smallest scale that has to be resolved in the numerics in order to ensure that convergence of numerical solutions imply convergence of continuous solutions, i.e., we have an affirmative answer to Eq. (11). It is natural to speculate that it is suffice to resolve small scales of the order of ν/U. Unfortunately we cannot establish that this is the smallest scale in a rigorous fashion. The best available rigorous result indicates that the smallest scale is at most exponentially small in ν (see, for instance, Foias *et al.*, 2001; Doering and Gibbon 1995). What we can prove is that if we resolve an exponentially small scale $\left[L\exp\left(-c_0\frac{\nu_k}{LU}\right)\right]$, then $\mathbf{u}^k\to\mathbf{u}^0$ does imply $\mathbf{u}^{\nu_k}\to\mathbf{u}^0$. Of course such a small scale is physically irrelevant. The appearance of such a small scale is due to the very presence of boundary layer and is typical in rigorous analysis of wall bounded flows (see, for instance, Temam 1997; Foias *et al.*, 2001). It still remains a great challenge to establish that the effective smallest scale is an algebraic function of the Reynolds number.

We also remark that a similar result involving small scales in the direction normal to the boundary in the boundary layer can be derived as well.

Theorem 2. *If the smallest scales resolved in the direction normal to the boundary in a thick enough boundary layer is at least of the order of ν/U, then we always observe numerical convergence of the solutions to the suitably truncated Navier-Stokes system to that of the Euler system at vanishing viscosity and mesh size.*

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