Geometry of foliations and flows I: Almost transverse pseudo-Anosov flows and asymptotic behavior of foliations

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Abstract

Let $\mathcal{F}$ be a foliation in a closed 3-manifold with negatively curved fundamental group and suppose that $\mathcal{F}$ is almost transverse to a quasigeodesic pseudo-Anosov flow. We show that the leaves of the foliation in the universal cover extend continuously to the sphere at infinity, hence the limit sets are continuous images of the circle. One important corollary is that if $\mathcal{F}$ is a Reebless, finite depth foliation in a hyperbolic manifold, then it has the continuous extension property. Such finite depth foliations exist whenever the second Betti number is non zero. The result also applies to other classes of foliations, including a large class of foliations where all leaves are dense and infinitely many examples with one sided branching. One key tool is a detailed understanding of asymptotic properties of almost pseudo-Anosov singular 1-dimensional foliations in the leaves of $\mathcal{F}$ lifted to the universal cover.

1 Introduction

A 2-dimensional foliation in a 3-manifold is called Reebless if it does not have a Reeb component: a foliation of the solid torus so that the boundary is a leaf and the interior is foliated by plane leaves spiralling towards the boundary. As such the boundary leaf does not inject in the fundamental group level and is compressible. Novikov [No] showed that Reebless foliations and the underlying manifolds have excellent topological properties. This result was extended by Rosenberg [Ros], Palmeira [Pa] and many others.

The goal of this article is to analyse geometric properties of foliations. Let $\mathcal{F}$ be a Reebless foliation in M^3 with negatively curved fundamental group. Reebless implies that M is irreducible [Ros]. In this article we will not make use of Perelman's fantastic results [Pe1, Pe2, Pe3], which if confirmed imply that the manifold is hyperbolic. Reebless foliations exist for instance whenever M is irreducible, orientable and the second homology of M is not finite [Ga1, Ga3]. They also exist in much more generality by work of Roberts [Ro1, Ro2, Ro3], Thurston [Th5] and many others.

Let M^3 be closed, irreducible with negatively curved fundamental group. The universal cover is canonically compactified with a sphere at infinity (denoted by S^2_∞), with compactification a closed ball [Be-Me]. The covering transformations act by homeomorphisms in the compactified space. Let $\widetilde{\mathcal{F}}$ be the lifted foliation to the universal cover $\widetilde{M}$. The leaves of $\widetilde{\mathcal{F}}$ are topological planes [No] and they are properly embedded. Hence they only limit in the sphere at infinity. For hyperbolic manifolds, the relationship between objects in hyperbolic 3-space (isometric to $\widetilde{M}$) and their limit sets in the sphere at infinity is central to the theory of such manifolds [Th1, Th2, Mar]. The same

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is true if $\pi_1(M)$ is negatively curved [Gr, Gh-Ha]. There is a metric in M so that all leaves of $\mathcal{F}$ are hyperbolic (that is constant curvature -1) [Ca] and so the universal cover of each leaf of $\mathcal{F}$ is isometric to the hyperbolic plane ($\mathbf{H}^2$). The *continuous extension question* asks whether these leaves extend continuously to the sphere at infinity, that is: given the inclusion map from a leaf F of $\widetilde{\mathcal{F}}$ to $\widetilde{M}$ is there a continuous extension to a map $F \cup \partial_\infty F$ to $\widetilde{M} \cup S^2_\infty$? Here $\partial_\infty F$ is the ideal boundary of F which is homeomorphic to a circle. If this is true we say that $\mathcal{F}$ has the *continuous extension property*. In that case the restriction of the map to $\partial_\infty F$ expresses the limit set of F as the continuous image of a circle, showing it is locally connected.

We first review what is known about the continuous extension property. In a seminal work, Cannon and Thurston [Ca-Th] proved this property when when $\mathcal{F}$ is a fibration over the circle. Previously Thurston had showed that the manifold is hyperbolic when the monodromy of the fibration is pseudo-Anosov [Th1, Th3, Th4, Bl-Ca]. Since the fundamental group of a leaf of $\mathcal{F}$ is a normal subgroup of the fundamental group of M, then every limit set of a leaf of $\widetilde{\mathcal{F}}$ is the whole sphere. In this way they produced many examples of group invariant Peano curves.

Another extremely important class of foliations is the following: A foliation is *proper* if the leaves never limit on themselves – this is in the foliation sense and it means that a sufficiently small transversal to a given leaf only meets the leaf in a single point. In particular leaves are not dense. In a proper foliation there are compact leaves which are said to have *depth* 0. The depth of a leaf is inductively defined to be i (for finite i) if $i-1$ is the maximum of the depths of leaves in the (foliation) limit set of the leaf. A foliation has *finite depth* if it is proper and there is a finite upper bound to the depths of all leaves.

Gabai proved that whenever a compact 3-manifold M is irreducible, orientable and the second homology group $H_2(M, \partial M, \mathbf{Z})$ is not finite, then there is a Reebless finite depth, foliation associated to each non torsion homology class [Ga1, Ga3]. The foliation is directly associated to a hierarchy of the manifold and as such is strongly connected with the topological structure of the manifold. These results had several fundamental consequences for the topology of 3-manifolds [Ga1, Ga2, Ga3]. If M is hyperbolic, then one important question is whether these finite depth foliations have the continuous extension property.

Subsequently Gabai and Mosher showed [Mo3] that any Reebless finite depth foliation in a closed, atoroidal 3-manifold admits a pseudo-Anosov flow Φ which is almost transverse to it. Roughly a flow is *pseudo-Anosov* if it has transverse hyperbolic dynamics – even though it may have finitely many singularities. It has stable and unstable two dimensional foliations which in general are singular. The term *almost transverse* means that one may need to blow up one singular orbit (or more) into a finite collection of joined annuli to make the flow transverse to the foliation. See detailed definitions and comments in section 2. Under the atoroidal condition Thurston [Th1, Th3] proved that M is in fact hyperbolic. See more about pseudo-Anosov flows transverse to foliations below.

We proved, jointly with Mosher, that these pseudo-Anosov flows almost transverse to finite depth foliations in hyperbolic 3-manifolds are *quasigeodesic* [Fe-Mo]. This means that flow lines are uniformly efficient in measuring distance in relative homotopy classes, or equivalently, uniformly efficient in measuring distance in the universal cover. This was first proved by Mosher [Mo1, Mo2] for a class of flows transverse to some examples of depth one foliations obtained by handle constructions. Another concept is that of quasi-isometric behavior: a foliation (perhaps singular) is *quasi-isometric* if its leaves are uniformly efficient in measuring distance in the universal cover. There are no non singular 2 dimensional quasi-isometric foliations in closed 3-manifolds with negatively curved fundamental group [Fe2]. As for singular foliations the situation is quite different and there are examples. The stable/unstable singular foliations of the quasigeodesic flows above may be quasi-isometric [Fe8] and may not [Mo3, Fe8]. If both the stable and unstable foliations are quasi-isometric and the flow is actually transverse (as oppposed to being almost transverse) to the finite depth foliation then we proved [Fe8] that $\mathcal{F}$ has the continuous extension property. This result only applies to finite depth foliations − the proof depends on induction in the depth (see theorem D below for a more general result). To apply these results we needed to check the quasi-isometry and transversality conditions. This was very tricky and we could only do that for some depth one foliations. More to the point, it is known that these conditions do not always hold for finite depth foliations.

The continuous extension property has also been proved for another class of foliations: A foliation is *uniform* if any two leaves in the universal cover are a bounded distance apart − the bound depends on the individual leaves. Thurston [Th5] proved that uniform foliations are very common. If in addition $\pi_1(M)$ is negatively curved, then Thurston [Th5] proved that there is a pseudo-Anosov flow transverse to $\mathcal{F}$. From this it is easy to prove that the flow has quasi-isometric stable/unstable foliations. In this case it also easily implies that the foliation $\mathcal{F}$ has the continuous extension property. The arguments are a very clever generalization of the fibering situation.

Notice that in all the results above, there is a pseudo-Anosov flow Φ transverse to $\mathcal{F}$ and so that the stable/unstable foliations of Φ are quasi-isometric singular foliations. Both of these properties were crucial in all proofs.

In this article we prove the continuous extension property for a much larger class of foliations. Our main result is the following:

Main theorem − Let $\mathcal{F}$ be a foliation in M^3 closed, atoroidal. Suppose that $\mathcal{F}$ is almost transverse to a quasigeodesic pseudo-Anosov flow Φ, which has some prong singularity (that is, not a topological Anosov flow). This implies that M has negatively curved fundamental group. Then $\mathcal{F}$ has the continuous extension property. Therefore the limits sets of leaves of $\widetilde{\mathcal{F}}$ are locally connected.

Since M has a singular pseudo-Anosov flow then M is irreducible and the stable/unstable foliations of Φ split to genuine laminations in M. A fundamental result of Gabai and Kazez [Ga-Ka] then implies that M has negatively curved fundamental group. For simplicity of statements we usually use the group negative curvature hypothesis, but in most places that could be substituted by the atoroidal hypothesis. The main theorem implies all the previous results about the continuous extension property.

Notice that it is not necessary to assume that $\mathcal{F}$ is Reebless − we prove that the condition of being almost transverse to a pseudo-Anosov flow implies that $\mathcal{F}$ is Reebless.

As a first consequence we prove the continuous extension property for all Reebless finite depth foliations in hyperbolic 3-manifolds. There are no restrictions on the depth of the foliation, or about transversality of the flow or quasi-isometric behavior of the pseudo-Anosov foliations.

Corollary B − Let $\mathcal{F}$ be a Reebless finite depth foliation in M^3 closed hyperbolic. Then $\mathcal{F}$ has the continuous extension property. In particular the limit sets of the leaves are all locally connected.

This shows that any hyperbolic 3-manifold with non finite second homology has such a foliation with the continuous extension property. Notice that conjecturally any closed, hyperbolic 3-manifold has a finite cover with positive first Betti number, which would imply there would always be a foliation with the continuous extension property in a finite cover. The proof of corollary B is simple given previous results: Mosher and Gabai proved that such $\mathcal{F}$ is almost transverse to a pseudo-Anosov flow Φ [Mo3]. We proved, jointly with Mosher that such flows are quasigeodesic [Fe-Mo]. The main theorem then implies corollary B. By Thurston's geometrization theorem [Th1, Th2, Mor] instead of hyperbolic we could have assumed that M is atoroidal.

The main theorem also applies to other classes of foliations. For example we have the following:

Corollary C − There are infinitely many foliations with all leaves dense which have the continuous extension property. Many of these have one sided branching. These are not uniform foliations.

Foliations with all leaves dense can be obtained for example starting with finite depth foliations and doing small perturbations – keeping it still almost transverse to the same quasigeodesic pseudo-Anosov flow. A construction is carefully explained by Gabai [Ga3], providing infinitely many examples with dense leaves to which corollary C applies. The examples occur whenenver the second Betti number of M is non zero. In fact whenever a foliations $\mathcal{F}$ satisfies the hypothesis of the main theorem, then any $\mathcal{F}'$ sufficiently close to $\mathcal{F}$ will also be transverse to the same flow. By the main theorem again, $\mathcal{F}'$ will have the continuous extension property. This perturbation feature of the main theorem is not shared by any previous result.

A foliation $\mathcal{F}$ is **R**-*covered* if the leaf space of $\widetilde{\mathcal{F}}$ is homeomorphic to the real numbers. Equivalently this leaf space is Hausdorff. A foliation which is not **R**-covered has *branching*, that is there are non separated points in the leaf space. This leaf space is oriented (being a simply connected, perhaps non Hausdorff 1-manifold) and there is a notion of branching in the positive or negative directions. If it branches only in one direction the foliation is said to have *one sided branching*. Foliations with one sided branching, where all leaves are dense and the foliation is transverse to a suspension pseudo-Anosov flow (which is quasigeodesic) were constructed by Meigniez [Me]. This provides infinitely many examples with one sided branching to which corollary C applies.

The main theorem can potentially be widely applicable because of the abundance of pseudo-Anosov flows almost transverse to foliations: Thurston proved this for fibrations [Th4]. It is also true for all **R**-covered foliations [Fe9, Cal1] and Calegari proved it for all foliations with one sided branching [Cal2], all minimal foliations [Cal3] and many other foliations [Cal3]. One main problem is to analyse the geometry of these pseudo-Anosov flows, in particular to decide whether they are quasigeodesic. By the main theorem this would imply the continuous extension property for the corresponding foliations.

The quasigeodesic property of Φ is definitely weaker than the stable and unstable foliations being quasi-isometric, but it still has useful properties. In order to prove the main theorem, one analyses the topological structure of the pseudo-Anosov flow. Let Φ_1 be the original pseudo-Anosov flow almost transverse to $\mathcal{F}$. To make the flow transverse to $\mathcal{F}$ one needs in general to blow up a collection of singular orbits into a collection of flow saturated annuli so that each boundary is a closed orbit of the new flow Φ. The blown up flow is called an *almost pseudo-Anosov flow*. If $\widetilde{\Phi}$ is the lifted flow to the universal cover $\widetilde{M}$ and $\mathcal{O}$ is its orbit space, then $\mathcal{O}$ is homeomorphic to the plane $\mathbf{R}^2$ [Fe-Mo] – this is true for pseudo-Anosov and almost pseudo-Anosov flows. When one blows up some singular orbits into a collection of joined annuli, the stable/unstable singular foliations also blow up. The two new singular foliations Λ^s, Λ^u are everywhere transverse to each other except at the singularities and the blown up annuli. The blown up annuli are part of both singular foliations. Since $\mathcal{F}$ is transverse to the blown up foliations, then the stable/unstable foliations Λ^s, Λ^u induce singular 1-dimensional foliations in leaves of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. The behavior of this is described in the following result, which is of independent interest also:

Theorem D – Let $\mathcal{F}$ be a foliation with hyperbolic leaves in M^3 closed. Let Φ_1 be a pseudo-Anosov flow almost transverse to $\mathcal{F}$ and let Φ be a corresponding almost pseudo-Anosov flow transverse to $\mathcal{F}$. Let Λ^s, Λ^u be the stable/unstable 2-dimensional foliations of Φ and $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ the lifts to $\widetilde{M}$. Given F leaf of $\widetilde{\mathcal{F}}$, let $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ be the induced singular 1-dimensional foliations in F. Then for every ray l in a leaf of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$, it limits in a single point of $\partial_\infty F$. If the stable/unstable foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ of Φ have Hausdorff leaf space, then the leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ are uniform quasigeodesics in F, the bound is independent of the leaf. In general the leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ are not quasigeodesic. Any non Hausdorffness (of say $\widetilde{\Lambda}^s_F$) is associated to a Reeb annulus in a leaf of $\mathcal{F}$ and when projected to M it either projects to or spirals to a Reeb annulus. The set of ideal points of leaves of $\widetilde{\Lambda}^s_F$ is dense in $\partial_\infty F$ and similarly for $\widetilde{\Lambda}^u_F$. Finally if two rays of the same leaf of $\widetilde{\Lambda}^s_F$ limit to the same ideal point in $\partial_\infty F$ then the leaf is not singular and the region in F bounded by the leaf projects in M to a set in a leaf of $\mathcal{F}$ which is either contained in or asymptotic to a Reeb annulus.

For this result one does not need negatively curved fundamental group or any metric properties of the flow. Theorem D is one of the main technical results used in the proof of the main theorem. We stress that in all the previous results concerning the continuous extension property this was also a crucial property on which the whole analysis hinged. In these other situations, the analysis of leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ was either trivial or substantially easier. In particular in these situations the leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ were always uniform quasigeodesics, which simplified subsequent proofs considerably. Such is not the case here. The proof here works in complete generality. It uses the denseness of contracting directions for foliations as proved by Thurston [Th6, Th7] when he introduced the universal circle for foliations – even though we do not directly use the universal circle here. The basic idea is: if any ray does not limit in a single point then it limits in a non trivial interval of $\partial_\infty F$ and we zoom into this interval and analyse the situation in the limit. This is actually the easiest statement to prove in theorem D. The facts about rays with same ideal point and non Hausdorffness are much trickier, but they will be essential in the analysis of the main theorem. The results of theorem D are also used in other contexts, for example to analyse rigidity of pseudo-Anosov flows almost transverse to a given foliation. This will be explored in a future article [Fe12].

The proof of the main theorem has 2 parts: given a leaf F of $\widetilde{\mathcal{F}}$, one first constructs an extension to the ideal boundary and then show it is continuous. To define the extension, one uses the foliations $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ as they hopefully define a basis neighborhood of an ideal point p of F. The best situation is that the corresponding leaves of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ define basis neighborhoods of unique points in S^2_∞, hence defining the image of p. There are several difficulties here: first the leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ are not quasi-geodesics, so much more care is needed. Another problem is that the foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^s$ in general do not have Hausdorff leaf space. This keeps recurring throughout the proof. A further difficulty is that if intersections with a leaf F of $\widetilde{\mathcal{F}}$ escape, it does not mean that the corresponding stable/unstable leaves in $\widetilde{M}$ escape compact sets. Consequently there are several cases to be analysed.

Another fact that is important for the analysis of the main theorem and theorem D is the following: Let Θ be the projection map from $\widetilde{M}$ to $\mathcal{O}$. A leaf of $\widetilde{\mathcal{F}}$ intersects an orbit of $\widetilde{\Phi}$ at most once defining an injective projection of F to $\Theta(F)$. The projection $\Theta(F)$ is equal to $\mathcal{O}$ if and only if the foliation is **R**-covered. An important problem here is to determine the boundary $\Theta(F)$ as a subset of $\mathcal{O}$. This turns out to be a collection of subsets of stable/unstable leaves in $\mathcal{O}$. This result is different than what happens for pseudo-Anosov flows transverse to foliations and its proof is much more delicate.

The article is organized as follows: In the next section we present basic definitions and results concerning pseudo-Anosov flows and almost pseudo-Anosov flows. In section 3 we analyse the set $\Theta(F)$ for leaves in the universal cover. In sections 4 and 5 we analyse the singular foliations $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ and asymptotic properties of their leaves, proving theorem D. In section 6 we prove the continuous extension property – the main theorem. In the final section we comment on general relationships between foliations and Kleinian groups.

In a subsequent article we analyse other important consequences of quasigeodesic behavior for flows and foliations [Fe12].

2 Preliminaries: Pseudo-Anosov flows and almost pseudo-Anosov flows

Let Φ be a flow on a closed, oriented 3-manifold M. We say that Φ is a *pseudo-Anosov flow* if the following are satisfied:

- For each $x \in M$, the flow line $t \to \Phi(x,t)$ is C^1, it is not a single point, and the tangent vector bundle $D_t\Phi$ is C^0.

- There is a finite number of periodic orbits $\{\gamma_i\}$, called *singular orbits*, such that the flow is "topologically" smooth off of the singular orbits (see below).

- The flowlines are tangent to two singular transverse foliations Λ^s, Λ^u which have smooth leaves off of γ_i and intersect exactly in the flow lines of Φ. These are like Anosov foliations off of the singular orbits. This is the topologically smooth behavior described above. A leaf containing a singularity is homeomorphic to $P \times I/f$ where P is a p-prong in the plane and f is a homeomorphism from $P \times \{1\}$ to $P \times \{0\}$. In a stable leaf, f contracts towards towards the prongs and in an unstable leaf it expands away from the prongs. We restrict to p at least 2, that is, we do not allow 1-prongs.

- In a stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backwards asymptotic.

Basic references for pseudo-Anosov flows are [Mo1, Mo3] and for 3-manifolds [He].

Notation/definition: The singular foliations lifted to $\widetilde{M}$ are denoted by $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$. If $x \in M$ let $W^s(x)$ denote the leaf of Λ^s containing x. Similarly one defines $W^u(x)$ and in the universal cover $\widetilde{W}^s(x), \widetilde{W}^u(x)$. Similarly if α is an orbit of Φ define $W^s(\alpha)$, etc... Let also $\widetilde{\Phi}$ be the lifted flow to $\widetilde{M}$.

We review the results about the topology of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ that we will need. We refer to [Fe6, Fe8] for detailed definitions, explanations and proofs. The orbit space of $\widetilde{\Phi}$ in $\widetilde{M}$ is homeomorphic to the plane $\mathbf{R}^2$ [Fe-Mo] and is denoted by $\mathcal{O} \cong \widetilde{M}/\widetilde{\Phi}$. Let $\Theta : \widetilde{M} \to \mathcal{O} \cong \mathbf{R}^2$ be the projection map. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then $\Theta(L) \subset \mathcal{O}$ is a tree which is either homeomorphic to $\mathbf{R}$ if L is regular, or is a union of p-rays all with the same starting point if L has a singular p-prong orbit. The foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ induce 1-dimensional foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O}$. Its leaves are $\Theta(L)$ as above. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then a *sector* is a component of $\widetilde{M} - L$. Similarly for $\mathcal{O}^s, \mathcal{O}^u$. If B is any subset of $\mathcal{O}$, we denote by $B \times \mathbf{R}$ the set $\Theta^{-1}(B)$. The same notation $B \times \mathbf{R}$ will be used for any subset B of $\widetilde{M}$: it will just be the union of all flow lines through points of B.

Definition 2.1. *Let L be a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. A slice of L is $l \times \mathbf{R}$ where l is a properly embedded copy of the reals in $\Theta(L)$. For instance if L is regular then L is its only slice. If a slice is the boundary of a sector of L then it is called a line leaf of L. If a is a ray in $\Theta(L)$ then $A = a \times \mathbf{R}$ is called a half leaf of L. If ζ is an open segment in $\Theta(L)$ it defines a* flow band *L_1 of L by $L_1 = \zeta \times \mathbf{R}$. Same notation for the foliations $\mathcal{O}^s, \mathcal{O}^u$ of $\mathcal{O}$.*

If $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$ then F and G intersect in at most one orbit. Also suppose that a leaf $F \in \widetilde{\Lambda}^s$ intersects two leaves $G, H \in \widetilde{\Lambda}^u$ and so does $L \in \widetilde{\Lambda}^s$. Then F, L, G, H form a *rectangle* in $\widetilde{M}$ and there is no singularity in the interior of the rectangle [Fe8]. There will be two generalizations of rectangles: 1) perfect fits = rectangle with one corner removed and 2) lozenges = rectangle with two opposite corners removed. We will also denote by rectangles, perfect fits, lozenges and product regions the projection of these regions to $\mathcal{O} \cong \mathbf{R}^2$.

Definition 2.2. *([Fe3, Fe6, Fe8]) Perfect fits - Two leaves $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$, form a perfect fit if $F \cap G = \emptyset$ and there are half leaves F_1 of F and G_1 of G and also flow bands $L_1 \subset L \in \widetilde{\Lambda}^s$ and $H_1 \subset H \in \widetilde{\Lambda}^u$, so that the set*

$$\overline{F}_1 \cup \overline{H}_1 \cup \overline{L}_1 \cup \overline{G}_1$$

separates M and forms an a rectangle R with a corner removed: The joint structure of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in R is that of a rectangle with a corner orbit removed. The removed corner corresponds to the perfect of F and G which do not intersect.

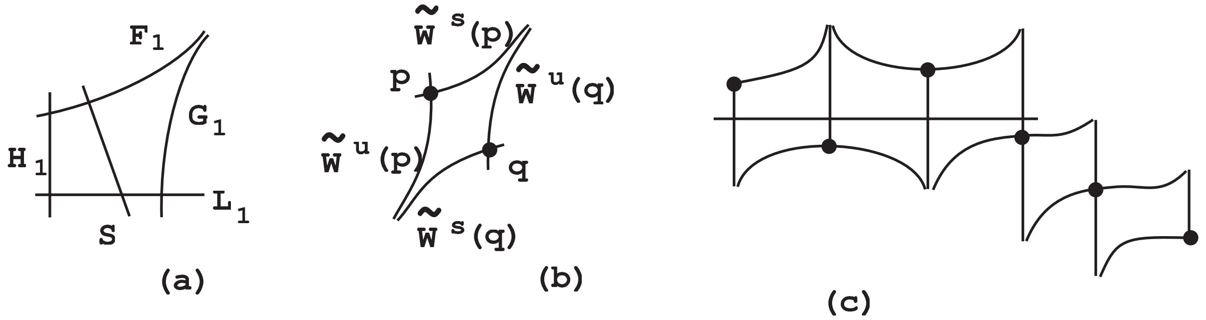

Figure 1: *a. Perfect fits in $\widetilde{M}$, b. A lozenge, c. A chain of lozenges.*

We refer to fig. 1, a for perfect fits. There is a product structure in the interior of R: there are two stable boundary sides and two unstable one. An unstable leaf intersects one stable boundary side (not in the corner) if and only if it intersects the other stable boundary side (not in the corner). We also say that the leaves F, G are *asymptotic*.

Definition 2.3. *([Fe3, Fe6, Fe8]) Lozenges - A lozenge is a region of $\widetilde{M}$ whose closure is homeomorphic to a rectangle with two corners removed. More specifically two points p, q form the corners of a lozenge if there are half leaves A, B of $\widetilde{W}^s(p), \widetilde{W}^u(p)$ defined by p and C, D half leaves of $\widetilde{W}^s(q), \widetilde{W}^u(q)$ so that A and D form a perfect fit and so do B and C. The region bounded by the lozenge is R and it does not have any singularities. The sides are not contained in the lozenge, but are in the boundary of the lozenge. See fig. 1, b.*

There are no singularities in the lozenges, which implies that R is an open region in $\widetilde{M}$. There may be singular orbits on the sides of the lozenge and the corner orbits.

Two lozenges are *adjacent* if they share a corner and there is a stable or unstable leaf intersecting both of them, see fig. 1, c. Therefore they share a side. A *chain of lozenges* is a collection $\{\mathcal{C}_i\}, i \in I$, where I is an interval (finite or not) in $\mathbf{Z}$; so that if $i, i+1 \in I$, then $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ share a corner, see fig. 1, c. Consecutive lozenges may be adjacent or not. The chain is finite if I is finite.

Definition 2.4. *Suppose A is a flow band in a leaf of $\widetilde{\Lambda}^s$. Suppose that for each orbit γ of $\widetilde{\Phi}$ in A there is a half leaf B_γ of $\widetilde{W}^u(\gamma)$ defined by γ so that: for any two orbits γ, β in A then a stable leaf intersects B_β if and only if it intersects B_γ. This defines a stable product region which is the union of the B_γ. Similarly define unstable product regions.*

The main property of product regions is the following: for any $F \in \widetilde{\Lambda}^s$, $G \in \widetilde{\Lambda}^u$ so that (i) $F \cap A \neq \emptyset$ and (ii) $G \cap A \neq \emptyset$, then $F \cap G \neq \emptyset$. There are no singular orbits of $\widetilde{\Phi}$ in A.

We abuse convention and call a leaf L of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ is called *periodic* if there is a non trivial covering translation g of $\widetilde{M}$ with $g(L) = L$. This is equivalent to $\pi(L)$ containing a periodic orbit of Φ. In the same way an orbit γ of $\widetilde{\Phi}$ is *periodic* if $\pi(\gamma)$ is a periodic orbit of Φ.

We say that two orbits γ, α of $\widetilde{\Phi}$ (or the leaves $\widetilde{W}^s(\gamma), \widetilde{W}^s(\alpha)$) are connected by a chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$, if γ is a corner of $\mathcal{C}_1$ and α is a corner of $\mathcal{C}_n$.

If $\mathcal{C}$ is a lozenge with corners β, γ and g is a non trivial covering translation leaving β, γ invariant (and so also the lozenge), then $\pi(\beta), \pi(\gamma)$ are closed orbits of $\widetilde{\Phi}$ which are freely homotopic to the inverse of each other.

Theorem 2.5. *[Fe6, Fe8] Let Φ be a pseudo-Anosov flow in M^3 closed and let $F_0 \neq F_1 \in \widetilde{\Lambda}^s$. Suppose that there is a non trivial covering translation g with $g(F_i) = F_i, i = 0, 1$. Let $\alpha_i, i = 0, 1$ be*

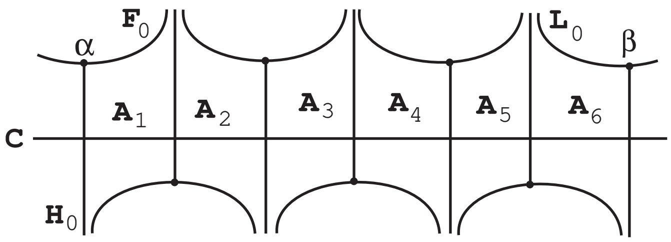

Figure 2: *The correct picture between non separated leaves of* $\widetilde{\Lambda}^s$.

the periodic orbits of $\widetilde{\Phi}$ *in* F_i *so that* $g(\alpha_i) = \alpha_i$. *Then* α_0 *and* α_1 *are connected by a finite chain of lozenges* $\{\mathcal{C}_i\}, 1 \leq i \leq n$ *and* g *leaves invariant each lozenge* $\mathcal{C}_i$ *as well as their corners.*

A chain from α_0 to α_1 is called *minimal* if all lozenges in the chain are distinct. Exactly as proved in [Fe4] for Anosov flows, it follows that there is a unique minimal chain from α_0 to α_1 and also all other chains have to contain all the lozenges in the minimal chain.

The main result concerning non Hausdorff behavior in the leaf spaces of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ is the following:

Theorem 2.6. *[Fe6, Fe8] Let* Φ *be a pseudo-Anosov flow in* M^3. *Suppose that* $F \neq L$ *are not separated in the leaf space of* $\widetilde{\Lambda}^s$. *Then* F *is periodic and so is* L. *Let* F_0, L_0 *be the line leaves of* F, L *which are not separated from each other. Let* V_0 *be the sector of* F *bounded by* F_0 *and containing* L. *Let* α *be the periodic orbit in* F_0 *and* H_0 *be the component of* $(\widetilde{W}^u(\alpha) - \alpha)$ *contained in* V_0. *Let* g *be a non trivial covering translation with* $g(F_0) = F_0$, $g(H_0) = H_0$ *and* g *leaves invariant the components of* $(F_0 - \alpha)$. *Then* $g(L_0) = L_0$. *This produces closed orbits of* Φ *which are freely homotopic in* M. *Theorem 2.5 then implies that* F_0 *and* L_0 *are connected by a finite chain of lozenges* $\{A_i\}, 1 \leq i \leq n$, *consecutive lozenges are adjacent. They all intersect a common stable leaf* C. *There is an even number of lozenges in the chain, see fig. 2. In addition let* $\mathcal{B}_{F,L}$ *be the set of leaves non separated from* F *and* L. *Put an order in* $\mathcal{B}_{F,L}$ *as follows: Put an orientation in the set of orbits of* C *contained in the union of the lozenges and their sides. If* $R_1, R_2 \in \mathcal{B}_{F,L}$ *let* α_1, α_2 *be the respective periodic orbits in* R_1, R_2. *Then* $\widetilde{W}^u(\alpha_i) \cap C \neq \emptyset$ *and let* $a_i = \widetilde{W}^u(\alpha_i) \cap C$. *We define* $R_1 < R_2$ *in* $\mathcal{B}_{F,L}$ *if* a_1 *precedes* a_2 *in the orientation of the set of orbits of* C. *Then* $\mathcal{B}_{F,L}$ *is either order isomorphic to* $\{1, ..., n\}$ *for some* $n \in \mathbf{N}$; *or* $\mathcal{B}_{F,L}$ *is order isomorphic to the integers* $\mathbf{Z}$. *In addition if there are* $Z, S \in \widetilde{\Lambda}^s$ *so that* $\mathcal{B}_{Z,S}$ *is infinite, then there is an incompressible torus in* M *transverse to* Φ. *In particular* M *cannot be atoroidal. Also if there are* F, L *as above, then there are closed orbits* α, β *of* Φ *which are freely homotopic to the inverse of each other. Finally up to covering translations, there are only finitely many non Hausdorff points in the leaf space of* $\widetilde{\Lambda}^s$.

Notice that $\mathcal{B}_{F,L}$ is a discrete set in this order. For detailed explanations and proofs, see [Fe6, Fe8].

Theorem 2.7. *([Fe8]) Let* Φ *be a pseudo-Anosov flow. If there is a stable or unstable product region, then* Φ *is topologically conjugate to a suspension Anosov flow. In particular* Φ *is non singular.*

Proposition 2.8. *Let* φ *be a (topological) Anosov flow so that every leaf of its stable foliation* $\widetilde{\Lambda}^s$ *intersects every leaf of its stable foliations* $\widetilde{\Lambda}^u$. *Then* φ *is topologically conjugate to a suspension Anosov flow. In particular* M *fibers over the circle with fiber a torus and Anosov monodromy.*

Proof. This result is proved by Barbot [Ba1] when φ is a smooth Anosov flow. That means it is C^1 and it has also strong stable/unstable foliations and contraction on the level of tangent vectors along the flow. Here we only have the weak foliations and orbits being asymptotic in their leaves. With proper understanding all the steps carry through to the general situation.

Lift to a finite cover where Λ^s, Λ^u are transversely orientable. A cross section in the cover projects to a cross section in the manifold (after cut and paste following Fried [Fr]) and so we can prove the result in the cover.

First, the flow φ is expanding: there is $\epsilon > 0$ so that no distinct orbits are always less than ϵ away from each other. Inaba and Matsumoto then proved that this flow is a topological pseudo-Anosov flow [In-Ma]. The main thing is the existence of a Markov partition for the flow. This implies that if F is a leaf of $\widetilde{\Lambda}^s$ which is left invariant by g, then there is a closed orbit of φ in $\pi(F)$ and all orbits are asymptotic to this closed orbit. Similarly for $\widetilde{\Lambda}^u$.

What this means is the following: consider the action of $\pi_1(M)$ in the leaf space of $\widetilde{\Lambda}^s$ which is the reals. Hence we have a group action in $\mathbf{R}$. Let g in $\pi_1(M)$ which fixes a point. There is L in $\widetilde{\Lambda}^s$ with $g(L) = L$. So there is orbit γ of $\widetilde{\varphi}$ with $g(\gamma) = \gamma$. Let U be the unstable leaf of $\widetilde{\varphi}$ with γ contained in U. Then $g(U) = U$. If g is associated to the positive direction of γ then g acts as a contraction in the set of orbits of U with γ as the only fixed point. Since every leaf of $\widetilde{\Lambda}^u$ intersects every leaf of $\widetilde{\Lambda}^s$ then the set of orbits in U is equivalent to the set of leaves of $\widetilde{\Lambda}^s$. This implies the important fact:

<u>Conclusion</u> - If g is in $\pi_1(M)$ has a fixed point in the leaf space of $\widetilde{\Lambda}^s$ then it is of hyperbolic type and has a single fixed point.

Using this topological characterization Barbot [Ba1] showed that $G = \pi_1(M)$ is metabelian, in fact he showed that the commutator subgroup $[G, G]$ is abelian. In particular $\pi_1(M)$ is solvable. This used only an action by homeomorphisms in $\mathbf{R}$ satisfying the conclusion above. Barbot [Ba1] also proved that the leaves of Λ^s, Λ^u are dense in M.

Plante [Pl1], showed that if $\mathcal{F}$ a minimal foliation in $\pi_1(M)$ solvable then $\mathcal{F}$ is transversely affine: there is a collection of charts $f_i : U_i \to \mathbf{R}^2 \times \mathbf{R}$, so that the transition functions are affine in the second coordinate. Using this Plante [Pl1, Pl2] constructs a homomorphism

$$C: \ \pi_1(M) \ \rightarrow \ \mathbf{R}$$

which measures the logarithm of how much distortion there is along an element of $\pi_1(M)$. This is a cohomology class. Every closed orbit γ of φ has a transversal fence which is expanding - this implies that $C(\gamma)$ is positive. Plante then refers to a criterion of Fried [Fr] to conclude that φ has a cross section and therefore it is easily seen that φ is topologically conjugate to a suspension Anosov flow. This finishes the proof of the proposition. $\square$

We now describe almost pseudo-Anosov flows.

Definition 2.9. *Given a pseudo-Anosov flow Φ_1 in a closed 3-manifold, then Φ is an almost pseudo-Anosov flow associated to Φ_1 if Φ is obtained from Φ_1 by blowing some singular orbits of Φ_1 into a collection of flow annuli. Specifically if γ is such a singular orbit of Φ_1, then it blows up into a connected collection of annuli $\{A_i, 1 \leq i \leq n\}$, each of which is flow invariant. The collection is embedded and the annuli have disjoint interiors. In each annulus the boundary components are closed orbits of Φ isotopic to γ as oriented orbits. In the interior of each annulus all orbits are forward asymptotic to one boundary component and backwards asymptotic to the other one. There is a blow down map $\xi : M \to M$, homotopic to the identity and isotopic to the identity in the complement of the A_i and sending each connected collection of A_i into a periodic orbit of Φ_1. The map ξ sends orbits of Φ to orbits of Φ_1 preserving orientation.*

The reason for considering almost pseudo-Anosov flows is as follows. All of the constructions of pseudo-Anosov flows transverse to foliations are in fact constructions of a pair of laminations – stable

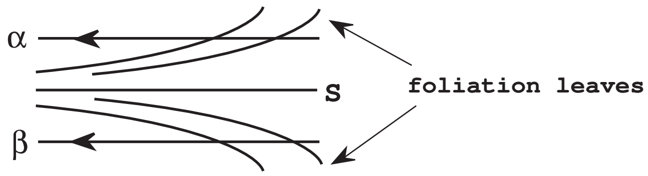

Figure 3: *Obstruction to transversality.*

and unstable − which are transverse to each other and to the foliation [Th4, Mo3, Fe9, Cal1, Cal2]. The intersection of the laminations is oriented producing a flow in this intersection. One then collapses the complementary regions to the laminations to produce transverse singular foliations and a pseudo-Anosov flow.

The transversality problem occurs in this last step, the blow down of complementary regions. In certain situations, for example for finite depth foliations, one cannot guarantee total transversality after the blow down. We briefly explain a possible problem. Mosher's construction [Mo3] of flows (almost) transverse to foliations is done inductively on the depth of the leaves (starting with the top depth leaves), associated to a sutured manifold hierarchy and the ensuing foliations construction of Gabai. At each step there is a foliation which is partially tangent/transverse to the boundary and also two laminations (stable/unstable) which are transverse to each other and to the foliation. There is a flow in the intersection of the laminations and a flow direction in "periodic" leaves, since all orbits in say a stable leaf are forward asymptotic. The next step topologically involves glueing two subsurfaces in the boundary in the construction of the foliation and laminations/flow.

One of the problems that can easily happen is the following. Suppose the glueing is done along surface S and after the glueing there are closed orbits α, β of the flow, which are oriented isotopic to the same simple closed curve of S and are in opposite sides of S, see fig. 3.

In the resulting pseudo-Anosov flow, α, β will be (oriented) freely homotopic to each other. By theorem 2.5 when lifted to $\widetilde{M}$ they are connected by a finite chain of lozenges. This forces the existence of another closed orbit α_1 which is freely homotopic to the inverse of α (in the opposite corner of a lozenge in $\widetilde{M}$). The problem is there is no guarantee such an orbit α_1 will be produced in the inductive process. In order to fix that, then in the collapsing step Mosher collapses α and β into a single orbit. This allows for the collapsed flow to be pseudo-Anosov. Unfortunately the transversality is lost locally near this region of S. There may be more collapsing forced by the inductive process. In order to recover the transversality, in this particular case one blows up the collapsed orbit into an embedded annulus, with boundaries α, β and puts a flow going from one orbit to the other, crossing S in the correct direction. Since other collapsings may be forced we may have a collection of annuli which are joined together and collapse to a single periodic orbit.

We still denote by Λ^s, Λ^u the stable/unstable laminations of an almost pseudo-Anosov flows. They are transverse to each other except at the blown up annuli. The same notation is used for $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$, etc..

The objects perfect fits, lozenges, product regions, etc.. all make sense in the setting of almost pseudo-Anosov flows: they are just the blow ups of the same objects for the corresponding pseudo-Anosov flows. Since the interior of these objects does not have singularities, the blow up operation does not affect these interiors. There may be singular orbits in the boundary which get blown into a collection of annuli. All the results in this section still hold for almost pseudo-Anosov flows, with the blow up operation. For example if F, L in $\widetilde{\Lambda}^s$ are not separated from each other, then they are connected by an even number of lozenges all intersecting a common stable leaf. Since parts of the boundary of these may have been blown into annuli, there is not a product structure in the closure of the union of the lozenges, but there is still a product structure of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in the interior.

3 Projections of leaves of $\widetilde{\mathcal{F}}$ to the orbit space

Let Φ be an almost pseudo-Anosov flow which is transverse to a foliation $\mathcal{F}$. This implies that $\mathcal{F}$ is Reebless – we provide a proof of this at the end of this section. An orbit of $\widetilde{\Phi}$ intersects a leaf of $\widetilde{\mathcal{F}}$ at most once – because the leaves of $\widetilde{\mathcal{F}}$ are properly embedded and $\widetilde{\Phi}$ is transverse to $\widetilde{\mathcal{F}}$. Hence the projection $\Theta : F \to \Theta(F)$ is injective. We want to determine the set of orbits a leaf of $\widetilde{\mathcal{F}}$ intersects – in particular we want to determine the boundary $\partial\Theta(F)$. As it turns out, $\partial\Theta(F)$ is composed of a disjoint union of slice leaves in $\mathcal{O}^s, \mathcal{O}^u$.

Since Φ is transverse to $\mathcal{F}$, there is $\epsilon > 0$ so that if a leaf F of $\widetilde{\mathcal{F}}$ intersects an orbit of $\widetilde{\Phi}$ at p then it intersects every orbit of $\widetilde{\Phi}$ which passes ϵ near p and the intersection is also very near p. To understand $\partial\Theta(F)$ one main ingredient is that when considering pseudo-Anosov flows, then flow lines in the same stable leaf are forward asymptotic. So if F intersects a given orbit in a very future time then it also intersects a lot of other orbits in the same stable since in future time they converge. In the limit this produces a stable boundary leaf of $\Theta(F)$. The blow up operation disturbs this: it is not true that orbits in the same stable leaf of an almost pseudo-Anosov flow are forward asymptotic: when they pass arbitrarily near a blow up annulus the orbits are distorted and their distance can increase enormously. This is the key difficulty in this section. Hence we first analyse the blow up operation more carefully.

<u>Notation</u> – Given Φ an almost pseudo-Anosov flow, let Φ_1 be a corresponding pseudo-Anosov flow associated to Φ. The term $\widetilde{W}^s(x)$ will denote the stable leaf of $\widetilde{\Phi}$ or $\widetilde{\Phi}_1$, where the context will make clear which one it is.

Recall that $\pi : \widetilde{M} \to M$ denotes the universal covering map.

We will start with Φ_1 and understand the blow up procedure. The blown up annuli come from singular orbits. The *lift annuli* are the lifts of blown up annuli to $\widetilde{M}$. Their projections to $\mathcal{O}$ are called *blown segments*. If L is a blown up leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ the components of L minus the lift annuli are called the *prongs*. A *quarter* associated to an orbit γ of $\widetilde{\Phi}_1$ is the closure of a connected component of $\widetilde{M} - (\widetilde{W}^u(\gamma) \cup \widetilde{W}^s(\gamma))$. Its boundary is a union of γ and half leaves in the stable and unstable leaves of γ. We will be interested in a neighborhood V of γ in this quarter which projects to M near the closed orbit $\pi(\gamma)$. We will understand the blow up in the projection of a quarter. Glueing up different quarter gives the overall picture of the blow up operation. In the projected quarter $\pi(V)$ in M there is a cross setion to the flow Φ_1. The orbits across the cross section are determined by which stable and unstable leaf they are in. The return map on the stable direction is a contraction and an expansion in the unstable direction. Any contraction is topologically conjugate to say $x \to x/2$ and an expansion is conjugate to $x \to 2x$. Hence the local return map is topologically conjugate to

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

a linear map. The whole discussion here is one of topological conjugacy. The flow is conjugate to $(x, y, 0) \to (2^{-t}x, 2^t y, t)$. Think of the blow up annulus as the set of unit tangent vectors to γ associated to the quarter region. The flow in the annulus is given by the action of DV_t on the tangent vectors. It has 2 closed orbits (the boundary ones corresponding to the stable and unstable leaves). The other orbits are asymptotic to the stable closed orbit in negative time and to the unstable closed orbit in positive time. This makes it into a continuous flow in this blown up part. See detailed explanation in [Fr] or [Ha-Th] (Fried, Handel-Thurston). For future reference recall this fact that in a blow up annulus the boundary components are orbits of the flow and in the interior the flow lines go from one boundary to the other without a Reeb annulus picture (there is a cross section to the flow in the annulus). Do this for each quarter region that is blown up. One can then glue up the 2 sides of the appropriate annuli because they are all of the same topological picture (using the standard model above). This describes the blown up operation in a quarter. There is clearly a blow down map which sends orbits of the blown up flow Φ to orbits of Φ_1 and collapses connected unions of annuli into a single p-prong singular orbit.

We quantify these: let ϵ very small so that any two orbits of Φ_1 which are always less than ϵ apart in forward time, then they are in the same stable leaf. Let $\mathcal{Z}$ the union of the singular orbits of Φ_1 which are blown up. Let $\epsilon' << \epsilon$ and let U be the ϵ' tubular neighborhood of $\mathcal{Z}$. Let U' (resp. U) be the $\epsilon'/2$ (resp. ϵ') tubular neighborhood of $\mathcal{Z}$. Choose the blow up map to be the identity in the complement of U', that is the blown up annuli are also contained in U'. The blow down map is then an isometry of the Riemannian metric outside U'. Choose the blow down to move points very little in U'. Isotope $\mathcal{F}$ so that it is transverse to the flow Φ. We are now ready to analyse $\partial\Theta(F)$.

Proposition 3.1. *Let F in $\widetilde{\mathcal{F}}$. Then $\Theta(F)$ is an open subset of $\mathcal{O}$. Any boundary component of $\Theta(F)$ is a slice of a leaf of $\mathcal{O}^s$ or $\mathcal{O}^u$. If it is a slice of $\mathcal{O}^s$, then as $\Theta(F)$ approaches l, the corresponding points of F escape in the positive direction. Similarly for unstable boundary slices.*

Proof. First notice that since F is transverse to $\widetilde{\Phi}$ then $\Theta(F)$ is an open set. Hence $\partial\Theta(F)$ is disjoint from $\Theta(F)$. The important thing is to notice that the metric is the same outside the small neighborhood U' of the blown up annuli. If two points are in the same stable leaf, then their orbits under the blow down flow Φ_1 are asymptotic in forward time. The same is true for Φ, for big enough time if the point is outside U. This is because the points of the corresponding orbits of Φ_1 will be <u>both</u> outside U' $-$ this is the reason for the construction of two neighborhoods U', U. The following setup will be used in <u>all</u> cases.

<u>Setup</u> $-$ Let v in $\partial\Theta(F)$ and v_i in $\Theta(F)$ with v_i converging to v. Let p_i in F with $\Theta(p_i) = v_i$ and let w in $\widetilde{M}$ with $\Theta(w) = v$. Let D be any small disk in $\widetilde{M}$ transverse to $\widetilde{\Phi}$ with w in the interior of D. For i big enough v_i is in $\Theta(D)$ so there are t_i real numbers with $p_i = \widetilde{\Phi}_{t_i}(w_i)$ and w_i are in D. As v is not in $\Theta(F)$, then $|t_i|$ grows without bound. Without loss of generality assume up to subsequence that $t_i \to \infty$. We will prove that there is a slice leaf L of $\widetilde{W}^s(w)$ so that $\Theta(L) \subset \Theta(F)$ and F goes up as it "approaches" L. The stable/unstable leaves here are those of the almost pseudo-Anosov flow and they may have blown up annuli.

<u>Case 1</u> $-$ Suppose that w is not in a blown up leaf.

First we show that we can assume no w_i is in $\widetilde{W}^s(w)$. Otherwise up to subsequence assume all w_i are in $\widetilde{W}^s(w)$. The orbits through w_i and w start out very close and aside from the time they stay in $\pi^{-1}(U)$ they are always very close. Let B be the component of the intersection of F with the flow band from $\widetilde{\Phi}_{\mathbf{R}}(w_i)$ to $\widetilde{\Phi}_{\mathbf{R}}(w)$ in the stable leaf $\widetilde{W}^s(w)$, which contains p_i. Then B does not intersect $\widetilde{\Phi}_{\mathbf{R}}(w)$ so it has to either escape up or down. If it escapes down it will have to intersect a small segment from w_i to w and hence so does F. For i big enough w_i is arbitrarily near w, so transversality of $\mathcal{F}$ and Φ then implies that F will intersect $\widetilde{\Phi}_{\mathbf{R}}(w)$ near w, contradiction see fig. 4, a.

We now consider the case that B escapes up. If the forward orbit through w is not always in $\pi^{-1}(U)$ then at those times outside of $\pi^{-1}(U)$ it will be arbitrarily close to $\widetilde{\Phi}_{\mathbf{R}}(w_i)$ and transversality implies again that F intersects $\widetilde{\Phi}_{\mathbf{R}}(w)$. If the forward orbit of w always stays in $\pi^{-1}(U)$ the same happens after the blow down so the blow down orbit is in the stable leaf of the singular orbit which is being blown up. This does not happen in case 1.

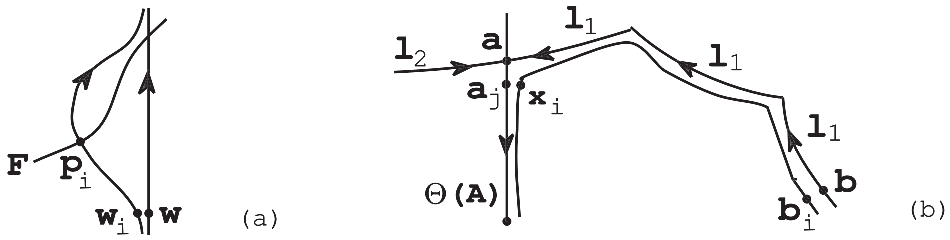

Figure 4: *a. A strangling neck is being forced, b. A slice in a leaf of* $\mathcal{O}^s$ *or* $\mathcal{O}^u$. $x_i = \Theta(z_i)$.

We can now assume that all v_i are in a sector of $\mathcal{O}^s(v)$ with l the boundary of this sector and $L = l \times \mathbf{R}$, the line leaf of $\widetilde{W}^s(w)$ which is the boundary of this sector.

Let now q in l. We will show that q is in $\partial\Theta(F)$ so $l \subset \partial\Theta(F)$. There is a segment $[q, v]$ contained in l. Choose x in L with $\Theta(x) = q$. Let α be a segment in $\widetilde{W}^s(w)$ transverse to the flow lines and going from x to w. Let x_i converging to x and x_i in $\widetilde{W}^s(w_i)$. We can do that since all w_i are in the same sector of $\widetilde{W}^s(w)$. Choose segments α_i from x_i to w_i in $\widetilde{W}^s(w_i)$ and transverse to the flowlines of $\widetilde{\Phi}$ in $\widetilde{W}^s(w_i)$.

<u>Claim</u> – For every orbit γ of $\widetilde{\Phi}$ intersecting α_i in y then γ intersects F in $\widetilde{\Phi}_s(y)$ where s converges to ∞ as $i \to \infty$.

Suppose there is $a_0 > 0$ so that for some i_0 then

$$\widetilde{\Phi}_{[a_0,t_i]}(w_i) \ \subset\ \pi^{-1}(U) \quad \text{for all} \ \ i \geq i_0$$

Then $\widetilde{\Phi}_{[a_0,\infty)}(w)$ is contained in the closure of $\pi^{-1}(U)$. As seen before this implies that w is in a blown up stable leaf, which is not the hypothesis of case 1. Therefore up to subsequence, there are arbitrary big times s_i between 0 and t_i so that $\widetilde{\Phi}_{s_i}(w_i)$ is not in $\pi^{-1}(U)$. Hence $\widetilde{\Phi}_{\mathbf{R}}(x_i)$ is very close to $\widetilde{\Phi}_{s_i}(w_i)$ and since F cannot escape up or down then F intersects $\widetilde{\Phi}_{\mathbf{R}}(x_i)$. Hence the segment $[\Theta(x_i), v_i]$ of $\mathcal{O}^s(v)$ is contained in $\Theta(F)$ and so $[x, v]$ is contained in the closure of $\Theta(F)$. Also the time s so that $\widetilde{\Phi}_s(y)$ hits F goes to ∞, hence $[x, v]$ cannot intersect $\Theta(F)$ – else there would be bounded times where it intersects F, by transversality. We conclude that $[x, v] \subset \partial\Theta(F)$, hence $l \subset \partial\Theta(F)$ as desired. If there is a sequence z_i in F escaping down with $\Theta(z_i)$ converging to a point in l, then by connectedness there is one intersecting a compact middle region – this would force an intersection of F with $l \times \mathbf{R}$ which is impossible.

This finishes the proof of case 1. In this case we proved there is a <u>line leaf</u> l of $\Theta(L)$ with $l \subset \partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches l.

<u>Case 2</u> – w is in a blown up leaf, but F does not intersect a lift annulus in $\widetilde{W}^s(w)$.

Refer to the setup above. As before we first show we can assume w_i are not in $\widetilde{W}^s(w)$. Otherwise, up to subsequence assume all w_i are in $\widetilde{W}^s(w)$. Since F does not intersect lift annuli in $\widetilde{W}^s(w)$, then w_i are all in prongs of $\widetilde{W}^s(w)$. Up to subsequence we can assume they are all in the same prong C of $\widetilde{W}^s(w)$ which has boundary an orbit γ of $\widetilde{\Phi}$. It follows that w is in γ. All the orbits in C are forward asymptotic to γ, even in the blown up situation. The strangling necks analysis of case 1 shows that F will be forced to intersect $\widetilde{\Phi}_{\mathbf{R}}(w)$. This cannot occur.

Hence assume all v_i are in a sector of $\mathcal{O}^s(v)$ bounded by a line leaf l. Let L be $l \times \mathbf{R}$. Let q be a point in l and choose x, α, x_i and α_i as in the proof of case 1. Choose a small disc D which is transverse to $\widetilde{\Phi}$ and has α in its interior. For i big enough then D intersects lift annuli only in $\widetilde{W}^s(x)$.

This is because the union of the blown annuli forms a compact set in M, so either α intersects a lift annulus, in which case there is no other lift annulus nearby or D is entirely disjoint from lift annuli. From now on the arguments of case 1 apply perfectly. This shows that $\Theta(L)$ is contained in $\overline{\Theta(F)}$, it is disjoint from $\Theta(F)$ and so it is $\partial\Theta(F)$ and F escapes up as it approaches L. This finishes the proof of case 2.

Now we need to understand what happens when F intersects a lift annulus in general. We separate that in a special case. We need the following facts before addressing this case. A lift annulus W through b is contained in $\widetilde{W}^s(b)$ and $\widetilde{W}^u(b)$ so there is not stable/unstable flow directions in W. However there are still such directions in ∂W, because one attracts nearby orbits of $\widetilde{\Phi}$ in W and the other one repels nearby orbits in W. In this generalized sense the first one is stable and the second one is unstable. In this sense if a is in an endpoint of a blown segment, then all local components of $\mathcal{O}^s(a) - a, \mathcal{O}^u(a) - a$ near a are either generalized stable or unstable. With this understanding there is an even number of such components and they alternate between generalized stable and unstable. Some local components of $\mathcal{O}^s(a) - a$ are also local components of $\mathcal{O}^u(a) - a$ if they are blown segments. One key thing to remember is that generalized stable and unstable alternate.

Case 3 $-$ Suppose that F intersects some lift annulus A contained in $\widetilde{W}^s(u_1)$.

Then F does not intersect both boundary orbits of A. Otherwise we could collapse $\pi(A)$ to a single orbit, still keeping Φ transverse to $\mathcal{F}$. Hence either $F \cap A$ is contained in the interior of A or it intersects only one boundary leaf.

Assume without loss of generality that F escapes up in one direction. This defines an orbit γ of $\widetilde{\Phi}$ with $a = \Theta(\gamma)$ in $\partial\Theta(F)$. The orbit γ has to be in the boundary of the lift annulus A. This is because an interior orbit is asymptotic to both boundary orbits and hence would intersect F. We now look at the picture in $\mathcal{O}$. Consider the stable leaf $\mathcal{O}^s(a)$. Notice that $\Theta(F)$ intersects $\Theta(A)$. From the point of view of γ, orbits in A move away from γ in future time, that is A is an unstable direction from γ. This means that $\Theta(A)$ is generalized unstable as seen from a. It follows that there are two generalized stable sides of $\mathcal{O}^s(a)$ one on each side of $\Theta(A)$ which are the closest to $\Theta(A)$. Choose one side, start at a and follow along $\mathcal{O}^s(a)$ either through blown segments and eventually into a prong in $\mathcal{O}^s(a)$ so as to produce a piece of a line leaf of $\mathcal{O}^s(a)$ in that direction. This path is regular on the side associated to $\Theta(A)$ and defines a half leaf l_1 of $\mathcal{O}^s(a)$. Similarly define l_2 in the other direction, see fig. 4, b. Let l be the union of l_1 and l_2. Then l is a slice leaf of $\mathcal{O}^s(a)$ but is not a line leaf since $\Theta(A)$ is in $\mathcal{O}^s(a)$ and is not in l.

Claim $-$ l is contained in $\partial\Theta(F)$ and F escapes positively as $\Theta(F)$ approaches l.

Let b in l_1 with b not in blown segment, that is, b in a prong. Choose b_i in $\mathcal{O}^u(b)$, with $b_i \to b$ and in that component of $\mathcal{O} - l$. Let D be an embedded disc in $\widetilde{M}$ which is transverse to $\widetilde{\Phi}$ and projects to $\mathcal{O}$ to a neighborhood of the arc ξ in l_1 from a to b. Let y_i in D with $\Theta(y_i) = b_i$, $y_i \to y$ with $\Theta(y) = b$. Assume that y is not in $\pi^{-1}(U)$. Choose b so that it is not in the unstable leaf of one singular orbit, hence $\widetilde{W}^u(y)$ does not contain lift annuli. In addition choose y_i so that $\widetilde{W}^s(y_i)$ does not contain lift annuli either.

Choose points u_j in $F \cap A$ so that $\Theta(u_j) = a_j$ converges to a. For each j the set $\Theta(F)$ contains a small neighborhood V_j of $\Theta(u_j)$ with V_j converging to a when j converges to infinity. The leaves $\mathcal{O}^s(b_i)$ are getting closer and closer to l_1 and $\Theta(A)$. For j fixed there is i big enough so that $\mathcal{O}^s(b_i)$ intersects V_j. Let

$$z_i \ \in \ F \cap \widetilde{W}^s(y_i) \ \text{ with } \ \Theta(z_i) \in V_j$$

here i depends on j. Let $z_i = \widetilde{\Phi}_{t_i}(r_i)$ with r_i in D. By choosing j and i converging to infinity we get that $\Theta(z_i)$ converges to a and we can ensure that the arc of $D \cap \widetilde{W}^s(y_i)$ between r_i and y_i is converging to an arc η of $\widetilde{W}^s(a) \cap D$ with $\Theta(\eta) = \xi$. We can also choose V_j small enough so that t_i converges to infinity.

The orbits $\widetilde{\Phi}_{\mathbf{R}}(y_i), \widetilde{\Phi}_{\mathbf{R}}(r_i)$ are very close in the forward direction as long as they are outside $\pi^{-1}(U)$. Since $\widetilde{W}^s(y_i)$ does not contain lift annuli then for times s converging to infinity $\widetilde{\Phi}_s(y_i)$ is not in $\pi^{-1}(U)$. Consider the flow band C in $\widetilde{W}^s(y_i)$ between $\widetilde{\Phi}_{\mathbf{R}}(r_i)$ and $\widetilde{\Phi}_{\mathbf{R}}(y_i)$. The leaf F intersects $\widetilde{\Phi}_{\mathbf{R}}(r_i)$ in $\widetilde{\Phi}_{t_i}(r_i)$ with t_i converging to infinity. Then an analysis exactly as in case 1 considering strangling necks and the arcs B in that proof, shows that $F \cap \widetilde{W}^s(y_i)$ cannot escape up down before intersecting $\widetilde{\Phi}_{\mathbf{R}}(y_i)$.

Suppose that F escapes down before intersecting $\widetilde{\Phi}_{\mathbf{R}}(y_i)$. We show that this is impossible. Since $F \cap \widetilde{W}^s(y_i)$ has points z_i in the forward direction from D and points in the backwards direction from D it follows that $F \cap \widetilde{W}^s(y_i)$ must intersect D in at least a point q_i. Up to subsequence we may assume that q_i converges to q in $\widetilde{W}^s(y)$. This will be an iterative process. Let $u_1 = q$. It is crucial to notice that in the flow band of $\widetilde{W}^s(y)$ between $\widetilde{\Phi}_{\mathbf{R}}(y)$ and γ the flow lines tend to go closer to γ, that is, either they project to closed orbits freely homotopic to $\pi(\gamma)$ or they are asymptotic to one of these orbits moving closer to γ. We now consider the component of $F \cap \widetilde{W}^s(y)$ containing u_0 and follow it towards γ. This component does not intersect γ and by the above it can only escape down in $\widetilde{W}^s(y)$. As it escapes down it produces points c_i in $\widetilde{W}^s(r_i)$ and as before produces points c_i' in D, which up to subsequence converge to c in $D \cap F$. By construction c is not u_1 and its orbit is closer to γ. Let $u_2 = c$. We can iterate this process. Notice the u_i cannot accumulate in D, or else all the corresponding points of F are in a compact set of $\widetilde{M}$. On the other hand the process does not terminate. This produces a contradiction.

The contradiction shows that in fact the arc $\Theta(C)$ is in $\Theta(F)$ which implies that $\xi = \Theta(\eta)$ is contained in $\overline{\Theta(F)}$. As the time to hit F from D grows with i, this shows that $\Theta(F)$ does not intersect ξ and hence ξ is contained in $\partial\Theta(F)$. As b is arbitrary this shows that $l \subset \partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches l. This finishes the analysis of case 3.

<u>Case 4</u> $-$ w is in a blown up stable leaf and F intersects some lift annulus A in $\widetilde{W}^s(w)$.

The difference from case 3 is that in case 3 we obtained a slice boundary l of $\Theta(F)$ $-$ but in our situation we do not yet know if it contains $\Theta(w)$ and whether it is a stable or unstable. Here we prove it is a stable slice and it contains $\Theta(w)$.

Recall the setup: $v = \Theta(w)$ is in $\partial\Theta(F)$ and there are v_i in $\Theta(F)$ with v_i converging to v and with p_i in $(v_i \times \mathbf{R}) \cap F$. Also $p_i = \widetilde{\Phi}_{t_i}(w_i)$ with w_i converging to w in $\widetilde{M}$ and t_i converging to infinity. Let ξ be the blown segment $\Theta(A)$.

The analysis of case 3 shows that $\Theta(F)$ contains the interior of $\Theta(A)$. Suppose first that v is in ξ. Then v is in the boundary of ξ and by case 3 again F escapes up or down when $\Theta(F)$ approaches a slice which contains v. If it escapes up, then the slice is a stable slice and we obtain the desired result in this case. We now show that F does not escape down. Let l be the unstable slice in $\partial\Theta(F)$ associated to this. Then l cuts in half a small disk neighborhood of v in $\mathcal{O}$. The set $\Theta(F)$ intersects only one component of the complement, the one which intersects ξ. As F escapes down when $\Theta(F)$ approaches l, then for all points in $\Theta(F)$ near enough v the corresponding point in F is flow backwards from D. This contradicts the fact that t_i is converging to infinity. Therefore F cannot escape down as it approaches l.

We can now assume that v is not in ξ. By changing ξ if necessary assume that ξ is the blown segment in $\mathcal{O}^s(v)$ intersected by $\Theta(F)$ which is closest to v. Let z be the endpoint of ξ separating the rest of ξ from v in $\mathcal{O}^s(v)$.

We first show that z is not in $\Theta(F)$. Suppose that is not the case and let b the intersection point of $z \times \mathbf{R}$ and F. Since ξ is the last blown segment of $\mathcal{O}^s(v)$ between ξ and v intersected by $\Theta(F)$ and $\Theta(F)$ contains an open neighborhood of z, it follows that v is in a prong B of $\mathcal{O}^s(v)$ with endpoint z. Let τ be the component of $F \cap \widetilde{W}^s(b)$ containing b. Since F does not intersect $v \times \mathbf{R}$ then it escapes. As the region between $b \times \mathbf{R}$ and $z \times \mathbf{R}$ is a prong, then F cannot escape up. As seen in the arguments for case 3, F cannot escape down either. This shows that z cannot be in $\Theta(F)$.

It follows that F escapes either up or down as $\Theta(F)$ approaches z. Suppose first that it escapes up. Then we are in the situation of case 3 and we produce a stable slice l in $\partial\Theta(F)$ with F going up as $\Theta(F)$ approaches l. If v is not in l then l separates v from $\Theta(F)$. This contradicts v_i in $\Theta(F)$ with v_i converging to v. Hence v is in l with F escaping up as $\Theta(F)$ approaches l. This is exactly what we want finishing the analysis in this case.

The last situation is F escaping down in A as $\Theta(F)$ approaches z. By case 3 there is a slice leaf l in $\mathcal{O}^u(z)$ with l contained in $\partial\Theta(F)$ and F escaping down as $\Theta(F)$ approaches l. We want to show that this case cannot happen. Notice that the blown segments of $\mathcal{O}^s(z)$ are exactly the same as the blown segments of $\mathcal{O}^u(z)$. The sets $\mathcal{O}^s(z), \mathcal{O}^u(z)$ differ exactly in the prongs and as they go around the collection of blown segments. The collection of all prongs in $\mathcal{O}^s(z), \mathcal{O}^u(z)$ also alternates between stable and unstable as it goes around the union of the blown segments.

Suppose first that v is in l. This contradicts F escaping down and $t_i \to \infty$. Finally suppose that v is not in l. We claim that in this case l separates v from $\Theta(F)$. Let α be the path in $\mathcal{O}^s(v)$ from z to v. If α only intersects l in z, then the separation property follows because l_1 and l_2 contain the local components of $\mathcal{O}^s(z) \cup \mathcal{O}^u(z) - z$ which are closest to $\Theta(A)$. This was part of the construction of l in case 3. Here the ξ is generalized stable at z and l_1, l_2 are generalized unstable at p. The path from z to v in $\mathcal{O}^s(v)$ cannot start in ξ or l_1 or l_2, hence l separates $\Theta(F)$ from v.

If on the other hand $\alpha \cap l = \delta$ is not a single point, then it is a union of blown segments. Let u be the other endpoint of δ. By regularity of l_1 and l_2 on the $\Theta(F)$ side it follows that each blown up segment in δ has flow direction away from z. Hence δ is generalized stable at u. Therefore the closest component of $\mathcal{O}^s(u) \cup \mathcal{O}^u(u) - u$ on the $\Theta(F)$ side is generalized unstable and that is contained in l. In this case it also follows that l separates v from $\Theta(F)$. As seen before this is a contradiction.

This finishes the proof of proposition 3.1 ☐

This has an important consequence that will be used extensively in this article.

Proposition 3.2. *Let F in $\widetilde{\mathcal{F}}$ and L in $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. Then the intersection $F \cap L$ is connected.*

Proof. By transversality of $\mathcal{F}$ and Φ, the intersection $C = \Theta(F) \cap \Theta(L)$ is open in $\Theta(L)$. Supppose there are 2 disjoint components A, B of C. Then there is v in ∂A with v separating A from B. There are v_i in A with v_i converging to v. By the previous proposition F escapes up or down in $A \times \mathbf{R}$ as $\Theta(F)$ approaches v. Assume wlog that F escapes up. Then there is a slice leaf l of $\mathcal{O}^s(v)$ with $l \subset \partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches l. Since l and $\Theta(F)$ are disjoint then B is disjoint from l. In addition v separates B from A in $\Theta(L)$. It follows from the construction of the slice l as being the closest to A, that l separates A from B. Hence $\Theta(F)$ cannot intersect B, contrary to assumption. This finishes the proof. ☐

As promised, we now prove that $\mathcal{F}$ being almost transverse to a pseudo-Anosov flow implies that $\mathcal{F}$ is Reebless.

Proposition 3.3. *Let $\mathcal{F}$ be a foliation almost transverse to a pseudo-Anosov flow Φ_1 and transverse to a corresponding almost pseudo-Anosov flow Φ. Then $\mathcal{F}$ is Reebless.*

Proof. Suppose that $\mathcal{F}$ is not Reebless and consider a Reeb component which is a solid torus V bounded by a torus T. Assume that the flow Φ is incoming along T.

Recall that there are some singular orbits of Φ_1 which blow up into a collection of flow annuli of Φ. Suppose that V intersects one of these annuli A. Then since Φ is incoming along T, the torus T cannot intersect the closed orbits in ∂A. Hence it intersects the interior of A, say in a point p and the forward orbit of p will limit in a closed orbit which is contained in the interior of V.

If on the other hand V does not intersect these blown annuli then the blow down operation does not affect the flow in V. That means we can assume that Φ_1 is equal to Φ in V. Since orbits of Φ_1 are trapped inside V once they enter V, the shadow lemma for pseudo-Anosov flows [Han, Man, Mo1], shows that there is also a periodic orbit of Φ_1 (and hence also of Φ) in $V - T$. Notice that the shadow lemma is for pseudo-Anosov flows and not for almost pseudo-Anosov flows and that is why we split the analysis into 2 cases.

In any case there is a closed orbit γ of Φ contained in the interior of V. Consider the generalized stable/unstable local leaves at γ. Since Φ is incoming along T, the generalized unstable leaves have to be contained in V. We eventually obtain that a whole half leaf of $W^s(\gamma)$ is contained in V. A lift $\widetilde{V}$ to $\widetilde{M}$ is homeomorphic to $D^2 \times \mathbf{R}$, because closed orbits of Φ are not null homotopic. The procedure above produces a half leaf of $\widetilde{W}^u(\widetilde{\gamma})$ contained in $\widetilde{V}$. This contradicts the fact that $\widetilde{W}^u(\widetilde{\gamma})$ is properly embedded [Ga-Oe]. This shows that $\mathcal{F}$ is Reebless. □

4 Asymptotic properties in leaves of the foliation

Let Φ be an almost pseudo-Anosov flow transverse to a foliation $\mathcal{F}$ with hyperbolic leaves. Let Λ^s, Λ^u be the singular foliations of Φ. Given leaf F of $\widetilde{\mathcal{F}}$ let $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ be the induced one dimensional singular foliations in F. In this section we study asymptotic properties of rays in $\widetilde{\Lambda}^s_F$. First we mention a result of Thurston [Th5, Th7] concerning contracting directions, which for convenience we state for 3-manifolds:

Theorem 4.1. *(Thurston) Let $\mathcal{F}$ be a codimension one foliation with hyperbolic leaves in M^3 closed. Then for every x in any leaf F of $\widetilde{\mathcal{F}}$ and every $\epsilon > 0$ there is a dense set of geodesic rays of F starting at x such that: for any such ray r there is a transversal β to $\widetilde{\mathcal{F}}$ starting at x so that any leaf L intersecting β and any y in r, then the distance between y and L is less than ϵ. If there is not a holonomy invariant transverse measure whose support contains $\pi(F)$ then one can show that the directions are actually contracting, that is: if y escapes in r then the distance between y and L converges to 0. Finally if $\pi(F)$ is not closed one can choose the β above to have x in the interior.*

There is a carefully written published version of this result in [Ca-Du]. The directions above where distance to nearby L goes to 0 are called contracting directions. The other ones where distance is bounded by ϵ are called ϵ non expanding directions. We first prove a preliminary result:

Theorem 4.2. *Let Φ be a pseudo-Anosov flow almost transverse to a foliation $\mathcal{F}$ in M^3 closed with $\mathcal{F}$ having hyperbolic leaves. Suppose there is a leaf L of $\widetilde{\mathcal{F}}$ and l a ray in a leaf of $\widetilde{\Lambda}^s_L$ so that l does not limit in a single point in $\partial_\infty L$. Then $\mathcal{F}$ is an $\mathbf{R}$-covered foliation. Similarly for rays of $\widetilde{\Lambda}^u_L$*

Proof. We assume at the start that $\mathcal{F}$ is not $\mathbf{R}$-covered. Let ϵ positive so that if p in $\widetilde{M}$ is less than ϵ from a leaf F of $\widetilde{\mathcal{F}}$, then the flow line through p intersects F less than 2ϵ away from p. Let l be a ray in $\widetilde{\Lambda}^s_L$. Because $\mathcal{F}$ and Φ are transverse, L is properly embedded in $\widetilde{M}$ and leaves of $\widetilde{\Lambda}^s$ are properly embedded, it follows that l is a properly embedded ray in L. Therefore it can only limit in $\partial_\infty L$. Suppose that l limits on 2 points a_0, b_0 in $\partial_\infty L$. Fix p a basepoint in L.

Since l limits in a_0, b_0, there are compact arcs l_i of l with endpoints which converge to a_0, b_0 respectively in $L \cup \partial_\infty L$ and so that the distance from l_i to p in L converges to infinity. Also we can assume that the l_i converges to a segment v in $\partial_\infty L$, where v connects a_0, b_0. This is in the Hausdorff topology of closed sets in $L \cup \partial_\infty L$, which is a closed disk.

The key idea is to bring this situation to a compact part of $\widetilde{M}$. Choose a sequence p_i a bounded distance from points in l_{k_i} so that that p_i converges to a point a in the interior of v. The bound depends on the sequence. Up to subquence assume that there are convering translations g_i in $\pi_1(M)$ so that $g_i(p_i)$ converges to a point p_0 in $\widetilde{M}$.

We claim that the set of points obtained as above projects to a sublamination of $\mathcal{F}$. Clearly if $g_i(p_i)$ converges to p_0 and q is in the same leaf L_0 of $\widetilde{\mathcal{F}}$ as p, then the distance from p_0 to q is finite and there are q_i in L with $d_L(q_i, p_i)$ bounded and $g_i(q_i)$ converging to q. Also q_i converges to a in $\partial_\infty L$. In addition if a sequence of such limits c_j converges to c_0 then a diagonal process shows that c_0 is also obtained as a single limit. This proves the claim. We extract a minimal sublamination $\mathcal{L}$.

A leaf F of $\widetilde{\mathcal{F}}$ is isometric to the hyperbolic plane. A *wedge* W in F with corner b and ideal set an interval $B \subset \partial_\infty F$ is the union of the rays in F from b with ideal point in B. The angle of the wedge is the angle that the boundary rays of W make at b. For any such sequence p_i as above, then the visual angle at p_i subintended by the arc v in $\partial_\infty L$ grows to 2π. Therefore the angle of wedge with corner p_i and ideal set $\partial_\infty L - v$ converges to 0. This is called the *bad wedge*.

Assume up to subsequence that $g_i(p_i)$ is converging to p_0 in a leaf L_0 of $\widetilde{\mathcal{F}}$ and that the directions of the bad wedges with corners $g_i(p_i)$ in $g_i(L)$ are converging to the direction r_0 of L_0. Let c be the ideal point of r_0 in $\partial_\infty L_0$.

Suppose first that $\pi(L_0)$ is not compact $-$ we shall see briefly that this is in fact always the case. Thurston's theorem shows that the set of two sided contracting directions (or ϵ non expanding directions) in L_0 is dense in $\partial_\infty L_0$. We will use these to transport a lot of the structure of $\widetilde{\Lambda}^s_{L_0}$ to nearby leaves. Choose s_0, s_1 to be rays in L_0 defining contracting directions (or ϵ non expanding directions) very near r_0 so that together they form a small wedge W in L_0 with corner p_0. There is an interval of leaves near L_0 so that any such leaf V is less than ϵ away from s_0, s_1. Then a flow line of $\widetilde{\Phi}$ through any point in s_0 or s_1 intersects V less than 2ϵ away. So s_0 flows to a curve in V, where we can assume it has geodesic curvature very close to 0, if ϵ is sufficiently small. It is therefore a quasigeodesic with a well defined ideal point. The same happens for s_1 and the flow images u_0, u_1 of s_0, s_1 in V define a generalized wedge W' in V. The ideal points e_0, e_1 of u_0, u_1 are close and bound an interval I which is almost all of $\partial_\infty g_i(L)$.

By construction $g_i(l)$ is a ray which limits in an interval of $\partial_\infty g_i(L)$ which contains I in its interior if i is big enough. There are then subarcs τ_j of $g_i(l)$ with endpoints a_j, b_j in u_0, u_1 respectively so that a_j converges to e_0 and b_j converges to e_1 and τ_j converges to I, see fig. 5. Here i is fixed and j varies. Since a_j, b_j are in u_0, u_1 then they flow (by $\widetilde{\Phi}$) to points in L_0. The images in L_0 are in the same leaf of $\widetilde{\Lambda}^s$. By proposition 3.2 these images are in the same leaf of $\widetilde{\Lambda}^s_{L_0}$. Hence the whole segment τ_j flows into L_0.

The point p_0 flows into p' in $g_i(L)$ under the flow. The arc τ_j together with subarcs or u_0, u_1 from a_j, b_j to p' bound a disc D_j in $g_i(L)$. The arguments above show that the boundary of D_j flows into L_0 producing a curve in L_0 bounding a disc B_j. The segments of $\widetilde{\Phi}$ connecting points in ∂D_j to points in ∂B_j produce an annulus C_j. Then $D_j \cup C_j \cup B_j$ is an embedded sphere in $\widetilde{M}$ and hence bounds an embedded ball. Since orbits of $\widetilde{\Phi}$ are properly embedded in $\widetilde{M}$, it follows that all orbits of $\widetilde{\Phi}$ intersecting D_j will also intersect B_j. Hence there is product flow in this ball. Since this is true for all j then the union of the D_j flows into L_0. The union of the D_j is the closure of $g_i(L) - W'$. The image is contained in the closure of $L_0 - W$ in L_0 $-$ call the closure J.

We claim that the image is in fact J. All the τ_j are in the same leaf of $\widetilde{\Lambda}^s$ and hence all their flow images in L_0 also are. Since rays of $\widetilde{\Lambda}^s_{L_0}$ are properly embedded in L_0 then when j converges to infinity the images of τ_j in L_0 escape compact sets. This shows the claim. Therefore the flow

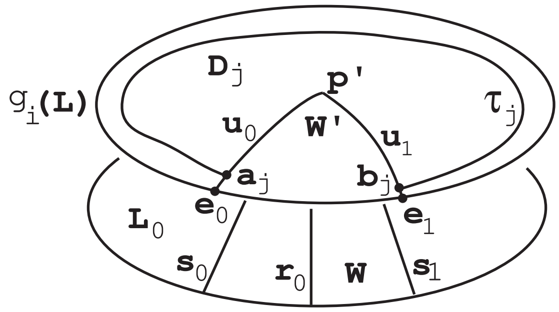

Figure 5: *Transporting the structure between leaves $g_i(L)$ and L_0.*

produces a homeomorphism between the closure of $L_0 - W$ and the closure of $g_i(L) - W'$. Clearly the same is true for any leaf in the interval associated to the contracting (non expanding) directions s_0, s_1. In particular we have the following conclusions:

Conclusion – In any limit leaf L_0 with a limit direction r_0 of bad wedges the following happens: Let c be the ideal point of r_0 and A a closed interval of $\partial_\infty L_0 - \{c\}$. Then there is a leaf l of $\widetilde{\Lambda}^s_{L_0}$ with compact subsegments l_i so that the endpoints of l_i converge to the endpoints a, b of A and l_i converges to A. In particular l_i escapes compact sets. There are also subsegments v_i with both endpoints converging to a and so that v_i converges to sets in $\partial_\infty L_0$ which contain A. Finally for sufficiently near leaves there is a wedge in L_0 which forms a product flow region with these nearby leaves.

To get the second assertion above just follow l beyond the endpoint of l_i near b until it returns near a again. As a preliminary step to obtain theorem 4.2 we prove the following:

Lemma 4.3. *Either $\mathcal{F}$ is $\mathbf{R}$-covered or for any limit $g_i(p_i)$ converging to p_0, the distinguished direction of the bad wedge associated to $g_i(p_i)$ converges to a single direction at p_0. In the second case this direction varies continuously with the leaves in $\widetilde{\mathcal{L}}$.*

Proof. Suppose there are subsequences q_i, p_i converging to points in (interior) v with $g_i(p_i), h_i(q_i) \to p_0 \in L_0 \in \widetilde{\mathcal{F}}$, but the directions of the wedges converge to r_0, r_1 distinct geodesic rays in L_0. We will first show that there is an interval of leaves of $\widetilde{\mathcal{F}}$ so that the flow $\widetilde{\Phi}$ is a product flow in this region.

Using the limit direction r_0 we produce a wedge W in L_0 so that the closure of $L_0 - W$ is part of a product flow region with nearby leaves of $\widetilde{\mathcal{F}}$. Using the other limit direction r_1 we produce a flow product region associated to another wedge region W_* disjoint from $W - p_0$. Together they produce a global product structure of the flow in a neighborhood of L_0.

This shows that there is a neighborhood N of L_0 in the leaf space of $\widetilde{\mathcal{F}}$ so that the flow is a product flow in N. In particular there is no non Hausdorffness of $\widetilde{\mathcal{F}}$ in this neighborhood. This is a very strong property as we shall see below. It implies a global product structure of the flow.

Notice that the structure of $\widetilde{\Lambda}^s_{g_i(L)}$ in $g_i(L) - W'$ flows over to L_0. In particular there are many rays of $\widetilde{\Lambda}^s_{L_0}$ which do not have a single limit in $\partial_\infty L_0$. This implies that $\pi(L_0)$ is not compact. This is because Levitt [Le] proved that given any singular foliation with prong singularities in a closed hyperbolic surface R, then the rays of the lift to $\widetilde{R}$ all have unique limit points in the ideal boundary. This shows that the minimal lamination $\mathcal{L}$ is not a compact leaf and hence it has no compact leaves.

Consider the neighborhood N as above. Consider the translates $g(N)$ where g runs through all elements of the fundamental group. Let P be the component of the union containing N. It is easy to see that the set P is precisely invariant: if g is in $\pi_1(M)$ and $g(P)$ intersects P then $g(P)$ is equal to P. In addition $\mathcal{F}$ restricted to P has leaf space homeomorphic to $\mathbf{R}$ because of the product flow property. We are assuming that N is open.

Suppose first that P is not all of $\widetilde{M}$, hence ∂P is a non empty collection of leaves of $\widetilde{\mathcal{F}}$. Let C be the projection of P to M. Then C is open, saturated by leaves of $\mathcal{F}$. Notice that $g(P)$ does not intersect ∂P for any g in $\pi_1(M)$ for otherwise $g(P)$ intersects P and so $g(P) = P$. It follows that $\pi(\partial P)$ is disjoint from C hence C is a proper open, foliated subset of M.

Dippolito [Di] developed a theory of such open, saturated subsets. Let $\overline{C}$ be the metric completion of C. There is an induced foliation in $\overline{C}$, which we will also denote by $\mathcal{F}$. Then

$$\overline{C} \ = \ V \ \cup \ \bigcup_1^n V_i$$

where V is compact and may be all of $\overline{C}$. Each nonempty V_i is an I-bundle over a non compact surface with boundary, so that $\mathcal{F}$ is a foliation transverse to the I-fibers. Each component of the intersection $\partial V_i \cap V$ is an annulus (or Moebius band) with induced foliation transverse to the I fibers. In our situation with Φ transverse to the flow, if V is not $\overline{C}$, we can choose V big enough so that the flow is transverse to $\mathcal{F}$ in each V_i and induces an I-fibration there.

Consider a component R of ∂C with lift $\widetilde{R}$ a subset of ∂P. Suppose first that R is closed. We show this is impossible, basically using holonomy. Parametrize the leaves of $\widetilde{\mathcal{F}}$ in P as $F_t, 0 < t < 1$ with t increasing with flow direction. A leaf in the boundary of P which is the limit of leaves in P which are limiting from the positive side above has to be the limit of F_t as t goes to 0: Suppose that S is in the boundary of P and there are x_i in F_{t_i} with t_i converging to $t_0 > 0$ and x_i convergint to x in S. Then S and F_{t_0} are not separated from each other. For i big enough the flow line through x_i will intersect S and therefore this flow line will not intersect F_{t_0}. This contradicts the fact that F_{t_o} and F_{t_i} have a flow product structure.

Suppose then that $\widetilde{R}$ is a limit of F_t where t converges to 0. Suppose first that R is compact. Suppose there are t_i converging to 0 so that F_{t_i} are in $\widetilde{\mathcal{L}}$. Then since $\mathcal{L}$ is a closed subset of M it follows that $\widetilde{R}$ is in $\widetilde{\mathcal{L}}$ and so R is in $\mathcal{L}$. But R is closed, contradicting the fact that $\mathcal{L}$ has no closed leaves. There is then $a > 0$ which is the smallest a so that F_a is in $\widetilde{\mathcal{L}}$ – notice that $\widetilde{\mathcal{L}}$ has leaves in P. For any g in $\pi_1(R)$ then $g(N) \cap N$ is not empty hence $g(N) = N$. It follows that $g(F_a) = F_b$ for some b. If b is not a then by taking g^{-1} if necessary we may assume that $b < a$. But as F_b is in $\widetilde{\mathcal{L}}$, this contradicts the definition of a. Hence $g(F_a) = F_a$ for any g in $\pi_1(R)$. This implies that $\pi(F_a)$ is a closed surface, again contradiction.

We conclude that R is not compact, hence it eventually enters some V_i (the point here is that V is not $\overline{C}$). The flow restricted to any component of $\partial V_i \cap \overline{C}$ goes from one component to the other in the annulus. This implies that all $\pi(F_t)$ intersect this annulus. There is then a leaf B of $\mathcal{L}$ which enters V_i. Going deeper and deeper in this non compact I-bundle will produce a limit point which is not in C. This shows the very important fact that $\mathcal{L}$ is not contained in C and therefore

$$\mathcal{E} \ = \ \mathcal{L} \cap (M - C) \ \neq \ \emptyset$$

In addition $\mathcal{E}$ is not equal to $\mathcal{L}$ since $\mathcal{L}$ has leaves in C and $(M - C)$ is closed. Hence $\mathcal{E}$ is a non trivial, proper sublamination of $\mathcal{L}$. This contradicts the fact that $\mathcal{L}$ is a minimal lamination.

This shows that the assumption $P \neq \widetilde{M}$ is impossible. Hence $P = \widetilde{M}$, which implies the flow $\widetilde{\Phi}$ produces a global product picture of $\widetilde{\mathcal{F}}$ and in particular $\mathcal{F}$ is $\mathbf{R}$-covered.

This shows that if $\mathcal{F}$ is not $\mathbf{R}$-covered, then the limits of the bad wedges are unique directions in the limit leaves. It also shows that they vary continuously from leaf to leaf, for otherwise one obtains bad wedges in very near leaves which have definitely separated directions. The same proof above then applies. This finishes the proof of lemma 4.3 $\square$

<u>Continuation of the proof of theorem 4.2</u>

We continue the proof of the theorem, assuming that $\mathcal{F}$ is not $\mathbf{R}$-covered. By the previous lemma we know that limit directions of bad wedges are unique and they vary continuously in leaves of $\widetilde{\mathcal{L}}$. These unique directions are distinguished in their respective leaves.

We first show that any complementary region of $\mathcal{L}$ (if any) is an I-bundle with a product flow.

Lift to a double cover if necessary to assume that M is orientable. Assume this is the original foliation $\mathcal{F}$, flow Φ, etc.. Let Z be a leaf of $\widetilde{\mathcal{L}}$. Since Z has a distinguished ideal point, then the fundamental group of $\pi(Z)$ can be at most $\mathbf{Z}$. Since there is a transverse flow and M is orientable this implies that $\pi(Z)$ is either a plane or an annulus.

Let U be a complementary region of $\mathcal{L}$ with boundary leaves R_1, R_2, R_3, etc.. As explained before the completion of U has a compact thick part and the non compact arms which are in thin, I-bundle regions. Suppose first that R_1 is a plane. There is a big disk D so that $R_1 - D$ is contained in the thin arms and flows across U to another boundary components of U. By connectedness it flows into a single boundary component R_2 of U. Then ∂D flows into a curve γ in R_2 which is null homotopic in M. The flow segments in M produce an annulus C in the completion of U. Since $\mathcal{F}$ is Reebless then γ bounds a disk D' in R_2 and so R_2 is a plane. The union $D \cup C \cup D'$ is an embedded sphere in M which bounds a ball B. Since orbits of $\widetilde{\Phi}$ are properly embedded in $\widetilde{M}$, it follows that the flow has to a product flow in B as well. This shows that flow is a product in the completion of U.

Suppose now that each R_i is an annulus. Let F be a lift of R_1 to $\widetilde{M}$ with F in the boundary of a component $\widetilde{U}$ of $\pi^{-1}(U)$. In R_1 there are two disjoint open annuli A_1, A_2 contained in the thin arms so that $B = R_1 - (A_1 \cup A_2)$ is a closed annulus in the core. Then A_1, A_2 flow into two annuli leaves R_2, R_3 in the boundary of U. Lifting to $F = \widetilde{R}_1$ we see leaves of $\widetilde{\Lambda}^s_F$ limiting in an interval of $\partial_\infty F$ with very small complement (near the distinguished ideal point of F). This implies they will have points in the lifts $\widetilde{A}_1, \widetilde{A}_2$ of A_1, A_2 to F. This shows that $\widetilde{A}_1, \widetilde{A}_2$ are in the same leaf of $\widetilde{\mathcal{F}}$. This implies that $R_2 = R_3$. In the same way a half of the infinite strip $\widetilde{B}$ flows into $\widetilde{R}_2$. Since B is compact, then all of B flows into R_2. This implies that the region U is an I-bundle. It is also easy to show that the flow is a product in this I-bundle.

This implies that we can collapse this complementary region along flow lines to completely eliminate it. This is because even in the universal cover we are eliminating product regions of the flow and the asymptotic behavior is stil preserved in the remaining regions. This can be done to all complementary regions and therefore we can assume there are no complementary regions, that is $\mathcal{L} = \mathcal{F}$ or that $\mathcal{F}$ is minimal.

Suppose now that $\mathcal{F}$ is not $\mathbf{R}$-covered. Let F_1, F_2 be leaves of $\widetilde{\mathcal{F}}$ which are not separated from each other. Consider leaves F of $\widetilde{\mathcal{F}}$ which are very close to points in both F_1 and F_2. As stated in the conclusion in the beginning of the proof of this theorem, there is a wedge of F which flows into F_1 and similarly for F_2. Hence there are half planes E_1, E_2 of F which flow into F_1, F_2. As F_1, F_2 are not separated this implies that E_1, E_2 are disjoint. Fix a point w in F and a big enough radius r so that the disk D of radius r around w intersects both E_1, E_2. Again as seen in the conclusion above there is an arc l in a leaf of $\widetilde{\Lambda}^s_F$ so that both endpoints of l are outside D and in E_1 and so that l is entirely outside D and as seen from p the visual measure of l is almost 2π. This implies that l intersects E_2. Since the endpoints of l are in E_1, which flows to F_1, then proposition 3.2 implies that the whole arc l flows into F_1. The points of l in E_2 will also flow to F_2. This is a contradiction.

This contradiction shows that $\mathcal{F}$ has to be $\mathbf{R}$-covered and finishes the proof of theorem 4.2 $\square$

Theorem 4.4. *Let $\mathcal{F}$ be an $\mathbf{R}$-covered foliation and Φ be a pseudo-Anosov flow almost transverse to $\mathcal{F}$. Then Φ is in fact transverse to $\mathcal{F}$. In addition for any leaf F of $\widetilde{\mathcal{F}}$ and for any ray l in $\widetilde{\Lambda}^s_F$ it converges to a unique ideal point in $\partial_\infty F$.*

Proof. If Φ is not transverse to $\mathcal{F}$, let Φ^* be an almost pseudo-Anosov flow which is transverse to $\mathcal{F}$ and is a blow up of Φ. There is flow annulus A of Φ^* with closed orbits γ_1, γ_2 in the boundary, so that A blows down to a single orbit of Φ.

The foliation induced by $\mathcal{F}$ in A has leaves which spiral to at least one boundary component – which they do not intersect. Lifting this picture to the universal cover one obtains an orbit of $\widetilde{\Phi}^*$ which does not intersect every leaf of $\widetilde{\mathcal{F}}$. This means that the flow $\widetilde{\Phi}^*$ is not *regulating* for $\widetilde{\mathcal{F}}$ [Th6, Th7]. We also say that Φ^* does not regulate $\mathcal{F}$. In [Fe11] we analysed a similar situation and proved the following: if Ψ is a pseudo-Anosov flow transverse to an $\mathbf{R}$-covered foliation and Ψ is not regulating, then Ψ is an $\mathbf{R}$-covered Anosov flow. The same arguments work with an almost pseudo-Anosov flow transverse to an $\mathbf{R}$-covered foliation. This shows that Φ^* is an $\mathbf{R}$-covered Anosov flow and has no (topological) singularities. In particular Φ^* is equal to Φ, that is the original flow is already transverse to $\mathcal{F}$. This proves the first assertion of the theorem.

Assume by way of contradiction that there is L' in $\widetilde{\Lambda}^s$ and l in $\widetilde{\Lambda}^s_{L'}$ which does not converge to a single point in $\partial_\infty L'$. As in the proof of theorem 4.2 we construct a minimal sublamination $\mathcal{L}$ of $\mathcal{F}$ such that: for every L in $\widetilde{\mathcal{L}}$ there is an ideal point u in $\partial_\infty L$ so that for every closed segment J in $\partial_\infty L - \{u\}$ there is a ray l of $\widetilde{\Lambda}^s_L$ which has subsegments limiting to J.. As shown in the proof of theorem 4.2, $\mathcal{L}$ cannot be a compact leaf.

Suppose first that every leaf of $\mathcal{F}$ is a plane. Then Rosenberg [Ros] proved that M is the 3-dimensional torus T^3. This manifold is a Seifert fibered space. In this case Brittenham [Br1] proved that an essential lamination is isotopic to one which is either vertical (a union of Seifert fibers) or horizontal (transverse to the fibers). So after isotopy assume $\mathcal{L}$ has one of these types. If $\mathcal{L}$ has a vertical leaf B, then geometrically it is a product of the reals with the circle. Hence it is an Euclidean leaf and in the universal cover it has polynomial growth of area. If $\mathcal{L}$ has a horizontal leaf B, then because the fibration is a product, there is a projection to a T^2 fiber, which distorts distances by a bounded amount. Again the same growth properties hold. But the leaves of $\mathcal{F}$ are hyperbolic, which is a contradiction. We conclude that M cannot be T^3.

Let then F in $\widetilde{\mathcal{L}}$ with $\pi(F)$ not simply connected. Let g in $\pi_1(M)$ non trivial with $g(F) = F$ and ξ be the axis of g in F. At least one ideal point of ξ, call it u, is not the direction of a fixed limit of bad wedges. Then as explained before there is a ray l of $\widetilde{\Lambda}^s_F$ and segments l_i of l, bounded by a_i, b_i both points in ξ, so that l_i escapes compact sets and converges to a non trivial segment in $\partial_\infty F$. We may assume that $l_i \cap \xi = \{a_i, b_i\}$ and also that all l_i are in the same side of ξ. Let e_0 be the translation length of g in F.

If the distance from a_i to b_i along ξ is bigger than e_0 then this produces a contradiction as follows: There is an integer n so that $g^n(a_i)$ is in the open segment (a_i, b_i) of ξ and and $g^n(b_i)$ is outside of the closed segment $[a_i, b_i]$. Since the arc l_i only intersects ξ in a_i, b_i, then l_i, together with $[a_i, b_i]$ bounds a closed disk in F and $g^n(a_i)$ is in (a_i, b_i). But $g^n(b_i)$ is outside and $g^n(l_i)$ is also on this side of ξ, so this produces a transverse self intersection of $\widetilde{\Lambda}^s_F$. If $g^n(l_i)$ is contained in the leaf v which contains l_i, then $g^n(v) = v$ and this produces infinitely many singularities in v, which is impossible. Hence $g^n(l_i)$ is not in v and the transverse intersection is impossible. The same arguments deal with the case that l_i intersects ξ in other points besides a_i, b_i.

We conclude that the distance in ξ from a_i to b_i is bounded. Up to subsequence we may assume there are integers n_i so that $g^{n_i}(a_i)$ converges to a_0 and $g^{n_i}(b_i)$ converges to b_0, both limits in ξ of course. Since the lengths of $g^{n_i}(l_i)$ are converging to infinity, it follows that a_0, b_0 are not in the same leaf of $\widetilde{\Lambda}^s_F$. By proposition 3.2 it follows that a_0, b_0 are not in the same leaf of $\widetilde{\Lambda}^s$. But for each i, the pair of points $g^{n_i}(a_i), g^{n_i}(b_i)$ is in the same leaf of $\widetilde{\Lambda}^s$. This implies that the leaf space of $\widetilde{\Lambda}^s$ is not Hausdorff.

First of all this implies that Φ is regulating for $\mathcal{F}$, for otherwise the aforementioned result from

[Fe11] shows that Φ is an $\mathbf{R}$-covered Anosov flow $-$ in particular $\widetilde{\Lambda}^s$ has Hausdorff leaf space. Also by theorem 2.6 the fact that $\widetilde{\Lambda}^s$ has non Hausdorff leaf space implies that there are closed orbits α, β of Φ so that α is freely homotopic to the inverse of β. Let h be a covering translation associated to α and $\widetilde{\alpha}$, $\widetilde{\beta}$ lifts of α, β to $\widetilde{M}$ which are left invariant by h. Without loss of generality assume that h acts in $\widetilde{\alpha}$ sending points forwards. As $\alpha \cong \beta^{-1}$ this implies that h acts on $\widetilde{\beta}$ taking points backwards. But since both of them intersects all leaves of $\widetilde{\mathcal{F}}$ (by the regulating property) then as seen from $\widetilde{\alpha}$ the translation h acts increasingly in the leaf space of $\widetilde{\mathcal{F}}$, with opposite behavior when considering $\widetilde{\beta}$. This is a contradiction, which shows that this cannot happen. This finishes the proof of theorem 4.4. $\square$

Remark - Group invariance and compactness of M are both essential here. For example start with a nicely behaved singular foliation of $\mathbf{H}^2$, so that all rays converge. It could be a foliation by geodesics or for instance the lift of the stable singular foliation associated to a suspension. Fix a base point p. Now rotate the leaves at a distance d of p by an angle d. In this situation all rays limit in all points of $\partial_\infty L$, in fact they spiral indefinitely into it. Another operation is to fix a ray through p and then distort the rest more and more one way and the other way. Here we have the leaves getting closer and closer to segments in $\partial_\infty F$ which are complementary to the ideal point associated to the ray.

5 Properties of leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ and their ideal points

In this section Φ is an almost pseudo-Anosov flow transverse to a foliation $\mathcal{F}$. As in the previous section there is no restriction on M here. In the previous section we proved that for any ray r of a leaf of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$, then it has a unique ideal point in $\partial_\infty F$. The notation for this ideal point will be r_∞. We now analyse further properties of leaves of $\widetilde{\Lambda}^s_F$ and their ideal points. Analogous results hold for $\widetilde{\Lambda}^u_F$.

First we want to show that if E is a fixed leaf of $\widetilde{\Lambda}^s$ (or $\widetilde{\Lambda}^u$) then the ideal points in $\partial_\infty F$ of rays of $E \cap F$ vary continuously with F. In order to do that we first put a topology on the union of ideal boundaries of an interval of leaves. Let p in F leaf of $\widetilde{\mathcal{F}}$ and τ a transversal to $\widetilde{\mathcal{F}}$ with p in the interior. For any L in $\widetilde{\mathcal{F}}$ intersecting τ, the ideal boundary is in 1-1 correspondence with the unit tangent bundle to L at $\tau \cap L$: ideal points correspond to rays in L starting at $L \cap \tau$. This is a homeomorphism. This puts a topology in

$$\mathcal{A} \ = \ \cup \, \{\partial_\infty L \ \mid \ L \cap \tau \neq \emptyset\}$$

making it into an annulus homeomorphic to $\cup \, \{T^1_q \widetilde{\mathcal{F}}, q \in \tau\}$ as a subspace of the unit tangent bundle of M. This topology in $\mathcal{A}$ is independent of the choice of transversal τ. The following definition/result is proved in [Fe9] or [Cal1].

Definition 5.1. *(markers) Given a foliation $\mathcal{F}$ by hyperbolic leaves of M^3 closed, then there is $\epsilon > 0$ so that: Let v be a geodesic ray in a leaf F so that it is associated to a contracting (or ϵ non expanding direction of F. For any leaf L sufficiently near F, then all the points of v flow into L and define a curve denoted by v_L. Then v_L has a unique ideal point denoted by a_L. The union m of the a_L is called a marker and is a subset of $\mathcal{A} = \cup \, \{\partial_\infty L\}$. Then m is an embedded curve in $\mathcal{A}$ in the topology defined above.*

In addition the markers are dense in $\mathcal{A}$ in the following sense: Let z in $\partial_\infty F$ and a_i, b_i in $\partial_\infty F$ which are in markers associated to contracting (non expanding) directions on a fixed side of F. Suppose that the sequence of open intervals (a_i, b_i) in $\partial_\infty F$ contains z and converges to z as i converges to infinity. Let α_i, β_i be the markers in that side of $\partial_\infty F$ containing a_i, b_i respectively.

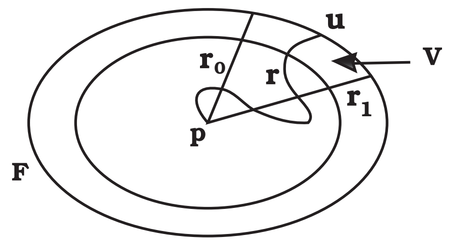

Figure 6: *Leaf in wedge defined by markers.*

Let L_i in $\widetilde{\mathcal{F}}$ be a sequnence of leaves converging to F and on that side of F so that $\partial_\infty L_i$ intersects both α_i and β_i. In the annulus $\mathcal{A}$ of circles at infinity, consider the rectangle R_i bounded by (a_i, b_i) in $\partial_\infty F$, the parts of α_i, β_i between $\partial_\infty F$ and $\partial_\infty L_i$ and the small segment in $\partial_\infty L_i$ bounded by $\partial_\infty L_i \cap \alpha_i$ and $\partial_\infty L_i \cap \beta_i$. Then the sets R_i converge to z as i converges to infinity. This is proved in [Fe9].

From now on the ϵ is chosen small enough to also satisfy the conclusions of the definition above and also that any set in $\widetilde{M}$ of diameter less than 10ϵ is in a product box of $\widetilde{\mathcal{F}}$ and $\widetilde{\Phi}$. Given a curve ζ in a leaf F with starting point p and limiting on a unique point q in $\partial_\infty F$, let ζ^* denote the geodesic ray of F with same starting and ideal points.

Lemma 5.2. *Let E be a leaf of $\widetilde{\Lambda}^s$ and p the starting point of the ray r of $E \cap F$. Assume that r does not have any singularity. For any L near F, then $E \cap L$ has a ray r_L which is near r. The ideal points of r_L in $\partial_\infty L$ vary continuously with L in the topology of $\mathcal{A}$ defined above.*

Proof. We do the proof for say the positive side of F. We consider r without singularity or else we would have to check the 2 exterior rays in $\widetilde{\Lambda}^s_F$ emanating from p. We can always get a subray of r which has no singularities.

Let $u = r_\infty$. Choose contracting (or ϵ non expanding) directions in both sides of u, with ideal points very close to u. Let them be defined by geodesic rays r_0, r_1 starting at p. There is τ a small flow segment starting at p and in that side of F so that for any L intersecting τ, then L is asymptotic to F along the r_0, r_1 rays, or at least always $\leq \epsilon$ from F. Hence r_0, r_1 flow along $\widetilde{\Phi}$ to L. Let s_0, s_1 be the flow images in L. The ϵ is also chosen small enough so that s_0, s_1 have geodesic curvature very small (this ϵ depends only on M and $\mathcal{F}$). In particular the curves s_0, s_1 are a small bounded distance (depending only on ϵ) from the corresponding geodesic arcs s_0^*, s_1^*. Let the ideal points of s_0, s_1 in $\partial_\infty L$ be denoted by v_0, v_1 and let J_L be the small closed interval in $\partial_\infty L$ bounded by v_0, v_1. Then v_0, v_1 are in the markers associated to r_0, r_1 respectively and so they vary continuously with L.

Consider $\xi = E \cap L$ and the rays l of ξ starting at $\tau \cap L$ and containing some points which flow back to points in r. It may be that ξ has singularities – even if r does not – but there are only finitely many such rays. We want to prove that the ideal point of any such is in J_L. As the rectangles R_i defined above converge to u in $\mathcal{A}$ this will prove the continuity property of the lemma.

Choose $d > 0$ so that outside of a disk D of radius d in F, then r is in the small wedge W of F defined by r_0, r_1, see fig. 6. Choose τ small enough so that if L intersects τ, then the entire disk D is ϵ near L. Let V be the closure in F of $W - D$. The boundary ∂V consists of subrays of r_0, r_1 and an arc in ∂D. Therefore all points in ∂V are less than ϵ from L and flow to L under $\widetilde{\Phi}$ with image a curve γ. This curve contains subrays of s_0, s_1 and it is properly embedded in L. Points of F near ∂V also flow to L so there is a unique component U of $L - \gamma$ which has some points flowing back to points in V. We want to show that the ray l is eventually contained in U.

Let r_{init} be the subarc of r between p and the last point c_0 of r in D. As p and c_0 flow into L, then proposition 3.2 shows that the entire arc r_{init} flows into L and let δ be its image in L. As r is singularity free, then so is δ and hence δ is contained in any ray l of $E \cap L$ in that direction. After c_0 the curve r enters V and so l must enter U after δ. If after that the ray l exits U then it must cross $\partial U = \gamma$ in some point, call it c_1. But c_1 flows back to F and one can apply proposition 3.2 again in the backwards direction to show that c_1 has to flow to a point in r. This contradicts the choice of c_0.

This shows that l is eventually entirely contained in U and therefore l_∞ is a point in J_L. This shows the continuity property as desired and finishes the proof of the lemma. $\square$

Now we have a property which will be crucial to a lot of our analysis.

Proposition 5.3. *Suppose that $\mathcal{F}$ is not topologically conjugate to the stable foliation of a suspension Anosov flow. Then the set of ideal points of rays of $\widetilde{\Lambda}^s_F$ is dense in $\partial_\infty F$.*

Proof. Suppose that there is F in $\widetilde{\mathcal{F}}$ so that the set of ideal points in $\widetilde{\Lambda}^s_F$ is not dense in $\partial_\infty F$. Let J be an open interval in $\partial_\infty F$ free of such ideal points. Choose p_i in F, p_i converging to a point in J. The visual angle of J as seen from p_i converges to 2π, so the complementary wedge W_i with corner p_i has angle which converges to zero. Up to subsequence assume that $g_i(p_i)$ converges to p_0 in a leaf L of $\widetilde{\mathcal{F}}$ and the small wedges $g_i(W_i)$ converge to a geodesic ray s in L with ideal point z.

Claim – In L all the rays of $\widetilde{\Lambda}^s_L$ converge to z.

Suppose there is x different from z which is an ideal point of a ray r in $\widetilde{\Lambda}^s_L$. Then r is contained in $\widetilde{W}^s(c_0)$ for some c_0 in $\widetilde{M}$ and for $g_i(F)$ sufficiently near L then $\widetilde{W}^s(c_0)$ intersects $g_i(F_0)$. Any ray of $\widetilde{W}^s(c_0) \cap g_i(F)$ which is near r will have ideal point near x in the topology of corresponding annulus $\mathcal{A}$ of ideal circles near $\partial_\infty L$. This is a consequence of the previous lemma. But $g_i(W_i)$ converges to r in this topology of $\mathcal{A}$, so the sets $g_i(\partial_\infty F - J)$ converge to z in $\mathcal{A}$. There are no ideal points of leaves of $\widetilde{\Lambda}^s_{g_i(F)}$ in $g_i(J)$. This contradicts the fact that the ideal points above are very near x and proves the claim.

The proof of the proposition is similar to that of theorem 4.2. As in that theorem consider the set of possible limits $g_i(p_i)$ as above. This projects to a lamination in M and let $\mathcal{L}$ be a minimal sublamination. The claim shows that each leaf of $\widetilde{\mathcal{L}}$ has a distinguished ideal point towards which all rays of $\widetilde{\Lambda}^s_L$ converge. The arguments in the claim also prove that if τ is a transversal to $\widetilde{\mathcal{F}}$, then the ideal points of leaves of $\widetilde{\mathcal{L}}$ intersecting τ vary continuously in the corresponding ideal annulus. Because of the distinguished ideal point property, then each leaf of $\mathcal{L}$ has fundamental group at most $\mathbf{Z}$. If needed lift to a double cover so that all leaves of $\mathcal{F}$ are orientable. Hence a leaf of $\mathcal{L}$ is either a plane or an annulus.

Consider a complementary component U of $\mathcal{L}$ and a boundary leaf A of U. If A is a plane then as in the proof of theorem 4.2, the region U is an I-bundle over A and the flow Φ is a product in U. This region can be collapsed away.

Suppose now that A is an annulus. Assume that flow lines through A flow into U. Again we want to show that U is a product region. As in the proof of theorem 4.2 let A_1, A_2 be two noncompact, disjoint annuli in A with $A - (A_1 \cup A_2)$ a compact annulus and A_1, A_2 contained in the thin, I-bundle region. Then A_1, A_2 flow entirely into leaves B and C in ∂U. Suppose first that B, C are different. Lift to the universal cover to produce lifts $\widetilde{U}, \widetilde{A}, \widetilde{A}_1, \widetilde{A}_2, \widetilde{B}, \widetilde{C}$. Then $\widetilde{A}_1, \widetilde{A}_2$ are disjoint half planes of $\widetilde{A}$ which flow positively respectively into $\widetilde{B}$ and $\widetilde{C}$. Let g be the generator of the isotropy group of $\widetilde{A}$, which has fixed points in z, x where z is the distinguished ideal point in $\widetilde{A}$. The argument will show there is a leaf in $\widetilde{\Lambda}^s_{\widetilde{A}}$ which also has ideal point in x, contradiction.

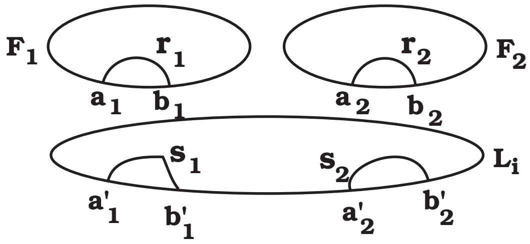

Figure 7: *Pushing ideal points near.*

From a point in $\widetilde{A}_1$ draw a geodesic segment of $\widetilde{A}$ to a point in $\widetilde{A}_2$. Let p be the first point of this segment which does not flow positively into $\widetilde{B}$. Then $\Theta(p)$ is in the boundary of $\Theta(\widetilde{B})$. Also points in the segment near p flow to $\widetilde{B}$ in positive time, hence there is a slice leaf l of $\mathcal{O}^s(\Theta(p))$ which is in the boundary of $\Theta(\widetilde{B})$. Notice that every point in l is a limit of points in $\Theta(B)$ on that side. The set $(l \times \mathbf{R})$ intersects $\widetilde{A}$ in at least p: if l is contained in $\Theta(\widetilde{A})$ then it generates a properly embedded copy of the reals in a leaf s of $\Lambda^s_{\widetilde{A}}$ otherwise the part that is contained in $\Theta(\widetilde{A})$ also does. Every point of s is a limit of points that flow positively into $\widetilde{B}$. Therefore no point in s can flow positively in $\widetilde{C}$ or else we would have points flowing both in $\widetilde{B}$ and $\widetilde{C}$.

This shows that the leaf s of $\Lambda^s_{\widetilde{A}}$ is a bounded distance from the axis r of g. Iterate s by powers of g acting with z as an expanding fixed point. The iterates $g^n(s)$ with $n > 0$ are all distinct. Either they are all nested or they are disjoint. If they are not nested since they all have to be in a bounded distance neighborhood of the axis of g and have both endpoints in z, then eventually they will have two points which are far along the leaf, but close in $\widetilde{A}$. By Euler characteristic reasons, this would force a center or one prong singularity, which is impossible. Hence they are nested, increasing and they limit to a leaf of $\Lambda^s_{\widetilde{A}}$ which has ideal limit points in z and x. This is a contradiction. This shows that $B = C$. In fact the same arguments show that all of the points in A flow into B, since that happens for the complement of a compact annulus in A and then the arguments above apply here. Hence U is a product region. Therefore we can collapse $\mathcal{F}$ to a minimal foliation.

As in theorem 4.2 we can then show that $\mathcal{F}$ is $\mathbf{R}$-covered. Suppose this is not the case and let F_1, F_2 be non separated leaves. Let L_i in $\widetilde{\mathcal{F}}$ leaves converging to both F_1, F_2. Let u_1, u_2 be the distinguished ideal points in $\partial_\infty F_1$, $\partial_\infty F_2$ respectively. Let a_1, b_1 be points in $\partial_\infty F_1$ very near u_1 and on opposite sides of u_1 and which are in markers associated to contracting or ϵ non expanding directions in F_1 associated to the L_i side. Let r_1 be the geodesic in F_1 with ideal points a_1, b_1. Similarly for F_2 producing a_2, b_2, r_2. For i big enough L_i is at most ϵ far from all points in r_1, r_2. Therefore r_1 flows (by $\widetilde{\Phi}$) into a curve s_1 in L_i and r_2 flows into s_2. This implies that s_1, s_2 are disjoint in L_i. Also s_1 has ideal points a'_1, b'_1 which are in markers containing a_1, b_1 respectively (this is using a transversal to $\widetilde{\mathcal{F}}$ through a point in F_1). Similarly s_2 has ideal points a'_2, b'_2 in markers containing a_2, b_2 (using transversal to $\widetilde{\mathcal{F}}$ through a point in F_2). As s_1, s_2 are disjoint then a'_1, b'_1 do not link a'_2, b'_2 in $\partial_\infty L_i$, see fig. 7.

The ideal point a'_1 cannot be in a marker to $\partial_\infty F_1$ and to $\partial_\infty F_2$ at the same time since they are non separated leaves. Hence the points a'_1, b'_1, a'_2, b'_2 are all distinct. Let J_1 be the interval of $\partial_\infty L_i$ bounded by a'_1, b'_1 and not containing the other points and similarly define J_2. For simplicity we are ommitting the dependence of J_1, J_2 on L_i (or on i). Now consider E a leaf of $\widetilde{\Lambda}^s$ intersecting F_1. Then $E \cap F_1$ has a ray with ideal point u_1, which is in the interval (a_1, b_1) of $\partial_\infty F_1$. The proof of lemma 5.2 shows that if L_i is close enough to F_1 then the ideal points of the corresponding rays of $(E \cap L_i)$ have to be in J_1. In the same way using F_2 one shows that the distinguished ideal point has to be in J_2. Since J_1, J_2 are disjoint, this is a contradiction. This shows that $\mathcal{F}$ is $\mathbf{R}$-covered.

Since $\mathcal{F}$ is $\mathbf{R}$-covered then theorem 4.4 implies that Φ can be chosen to be a pseudo-Anosov flow.

Also as $\mathcal{F}$ is $\mathbf{R}$-covered we can choose a transversal τ intersecting all the leaves of $\widetilde{\mathcal{F}}$. This shows that the union of all the circles at infinity has a natural topology making it into a cylinder $\mathcal{A}$. This situation of $\mathbf{R}$-covered foliations is carefully analysed in [Fe9]. The fundamental group of M acts in $\mathcal{A}$ by homeomorphisms. The union of the distinguished ideal points of leaves of the distinct leaves of $\widetilde{\mathcal{F}}$ is a continuous curve ζ in $\mathcal{A}$ which is group invariant.

Suppose first that $\mathcal{F}$ admits a holonomy invariant transverse measure. Since $\mathcal{F}$ is minimal then the transverse measure has full support. Under these conditions Imanishi [Im] proved that M fibers over the circle with fiber a closed surface. In addition $\mathcal{F}$ is approximated arbitrarily near by a a fibration. The pseudo-Anosov flow is also transverse to these nearby fibrations and so the same situation occurs for the fibrations: there is a global invariant curve in the cylinder at infinity. Since now there are compact leaves, this is impossible.

We conclude that there is no holonomy invariant transverse measure. Therefore Thurston's theorem shows the existence of contracting directions and not just ϵ non expanding directions. So the markers are associated to contracting directions. If ζ intersects a marker m, that corresponds to a direction in a leaf of $\widetilde{\mathcal{F}}$ which is contracting. Under the flow $\widetilde{\Phi}$ this gets reflected in the contracted leaves nearby, that is the marker is contained in ζ. Since $\mathcal{F}$ is minimal and ζ is $\pi_1(M)$ invariant, this shows that the entire curve ζ is a marker associated to contracting directions. The results from [Fe9] apply here, in particular lemma 3.17 through proposition 3.21 of [Fe9]: they show that no other direction in $\widetilde{\mathcal{F}}$ (outside of ζ) is a contracting direction. By Thurston's theorem again, there would be a holonomy invariant transverse measure, contradiction.

Therefore ζ has no contracting directions. The same analysis of [Fe9] now shows that for any leaf F in $\widetilde{\mathcal{F}}$ and every direction other than the distinguished direction, then it is a contracting direction. In fact it is a contracting direction with any other leaf of the foliation.

This is a very interesting situation. Let a_F be the distinguished ideal point of F leaf of $\widetilde{\mathcal{F}}$. Consider a one dimensional foliation in $\widetilde{M}$ whose leaves are geodesics in leaves F of $\widetilde{\mathcal{F}}$ and which have one ideal point a_F. Let $\widetilde{\xi}$ be the flow which is unit speed tangent to this foliation and moves towards the ideal point a_F.

This is a flow in $\widetilde{M}$. Clearly in each leaf of $\widetilde{\mathcal{F}}$, it is a smooth flow. If q_i in L_i of $\widetilde{\mathcal{F}}$ converge to q in L, then the geodesics of L_i with ideal point a_{L_i} converge to the geodesic through q in L with ideal point a_L. This is because the ideal points a_F vary continuously with F and q_i converges to q $-$ this is the local trivialization of the union of the circles at infinity using the tangent bundles to a transversal. Hence $\widetilde{\xi}$ varies continuously.

Since ζ is group invariant, this induces a flow in M, which is tangent to the foliation $\mathcal{F}$. Clearly it is smooth along the leaves of $\mathcal{F}$ and usually just continuous in the transverse direction.

This flow is a topological Anosov flow: the stable foliation is just the original foliation $\mathcal{F}$. The unstable foliation: Let p in leaf L of $\widetilde{\mathcal{F}}$, let γ be the flow line of $\widetilde{\xi}$ through p. Then γ has positive ideal point a_L and negative ideal point v. As explained above v is in a marker m which is associated to a contracting direction and so that m intersects all ideal circles. For each F in $\widetilde{\mathcal{F}}$, let m_F be the intersection of m and $\partial_\infty F$. Let γ_F be the geodesic in F with ideal points a_F and m_F. Let E_p be the union of these γ_F. Then all orbits of $\widetilde{\xi}$ in E_p are backwards asymptotic by construction. By construction the E_p are either disjoint or equal as p varies in $\widetilde{M}$ and they form a group invariant foliation in $\widetilde{M}$. This is the unstable foliation. Hence ξ is a topologically Anosov flow. Notice that in the universal cover every stable leaf intersects every unstable leaf and vice versa.

By proposition 2.8 it follows that ξ is topologically conjugate to a suspension Anosov flow. The foliation $\mathcal{F}$ is then topologically conjugate to the stable foliation of this flow. This finishes the proof

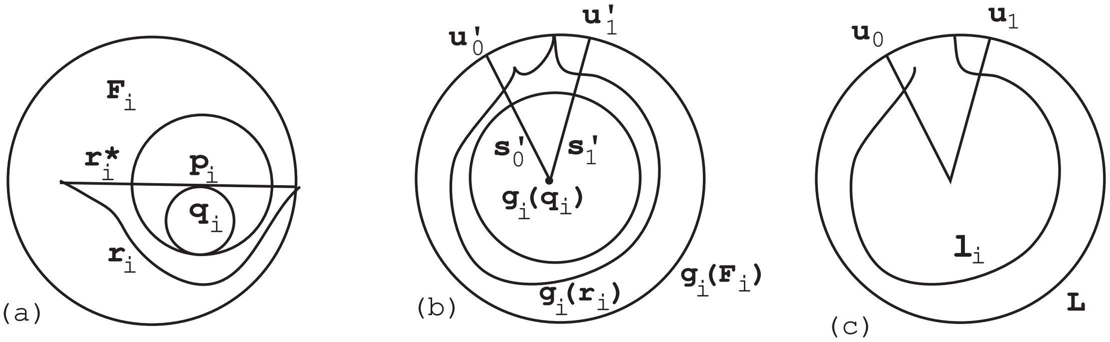

Figure 8: *a. Limits of points, b. Going around disks in F_i, c The picture in L.*

of this proposition. ☐

Remark – The hypothesis is necessary. Suppose that $\mathcal{F}$ is the stable foliation of a suspension Anosov flow, ξ so that it is transversely orientable. Perturb the flow slightly so that flow lines are still tangent to the original unstable foliation of ξ. The new flow, call it Φ is transverse to $\mathcal{F}$, it has the same unstable foliation as ξ but different stable foliation. The flow Φ is not regulating for $\mathcal{F}$. The intersections of leaves of $\widetilde{\Lambda}^s$ with leaves F of $\widetilde{\mathcal{F}}$ are all horocycles with the same ideal point which is the positive ideal point of flow lines in F. So the ideal points of rays of leaves of $\widetilde{\Lambda}^s_F$ are not dense in $\partial_\infty F$. Notice these leaves are not quasigeodesics in F either. This example is studied in detail in section 7 of [Fe11].

Now we want to study metric properties of slices of leaves of $\widetilde{\Lambda}^s_F$. The best metric property such leaves could have is that they are *quasigeodesic*: this means that length along the curve is at most a bounded multiplicative distortion of length in the leaf F of $\widetilde{\mathcal{F}}$ [Th1, Gr, Gh-Ha, CDP]. If the bound is k then we say the curve is a k-quasigeodesic. Since F is hyperbolic this would imply that such leaves (the non singular ones) are a bounded distance from true geodesics. Very unfortunate for us, this is not true in general. But there are still some good properties.

Let $\mathcal{H}^s$ be the leaf space of $\widetilde{\Lambda}^s$ and $\mathcal{H}^u$ be the leaf space of $\widetilde{\Lambda}^u$. Clearly since $\mathcal{H}^s$ may be non Hausdorff, it could be that some $\widetilde{\Lambda}^s_F$ does not have Hausdorff leaf space. This easily would imply that the slices of $\widetilde{\Lambda}^s_F$ are not uniformly quasigeodesic [Fe2]. This in fact occurs, see Mosher [Mo1, Mo3]. Still it could be that given a ray in $\widetilde{\Lambda}^s_F$, it is a quasigeodesic – with the quasigeodesic constant depending on the particular ray. We are not able to prove this and we cannot conjecture what happens in generality. But we are able to prove a weaker property, which will be enough for our purposes. If r is a ray in a leaf of $\widetilde{\Lambda}^s_F$, recall that r^* is the unique geodesic ray in F with same starting point as r and same ideal point. We would like to prove that r, r^* are a bounded distance apart, but we do not know if that is true. But we can prove the following important property:

Lemma 5.4. *There is $\delta_0 > 0$ so that for any F in $\widetilde{\mathcal{F}}$ and any ray r in a leaf of $\widetilde{\Lambda}^s_F$, then given any segment of length δ_0 in r^*, there is a point in this segment which is less than δ_0 from r in F. That implies that r^* is in the neighborhood of radius $2\delta_0$ of r in F.*

Proof. This means that $r^* \subset N_{2\delta_0}(r)$ in F. We do not know if the converse holds. Suppose the lemma is not true. Then there are F_i leaves of $\widetilde{\mathcal{F}}$, r_i rays of $\widetilde{\Lambda}^s_{F_i}$ and p_i in r^*_i so that $B_{2i}(p_i)$ (in F_i) does not intersect r_i. There is one side of r^*_i in F_i so that r_i goes around that side, see fig. 8, a. Let q_i inside a half disk of $B_{2i}(p_i)$ with $B_i(q_i)$ tangent to r^*_i and $\partial B_{2i}(p_i)$, see fig. 8, a.

As usual up to subsequence there are g_i in $\pi_1(M)$ with $g_i(q_i)$ converging to q_0 in L leaf of $\widetilde{\mathcal{F}}$ and so that the geodesic segments ζ_i from $g_i(q_i)$ to $g_i(p_i)$ in F_i converge to a geodesic ray s in L. Choose two markers with points u_0, u_1 in $\partial_\infty L$ very close to s_∞ and on opposite sides of it. The markers are associated to the side of L where the $g_i(F_i)$ are limiting to. Let s_0, s_1 be the geodesic rays of L starting at q_0 and with ideal points u_0, u_1. For i big enough $g_i(F_i)$ is ϵ close to both s_0 and s_1 and so these two rays flow (under $\widetilde{\Phi}$) to curves s'_0, s'_1 in $g_i(F_i)$. The ideal points u'_0, u'_1 of s'_0, s'_1 are in the markers above.

For i big enough the ray $g_i(r_i)$ has a subray which goes around $g_i(B_i(q_i))$ in $g_i(F_i)$ and has ideal point in the small segment of $\partial_\infty g_i(F_i)$ defined by u'_0, u'_1, see fig. 8, b. Since s'_0, s'_1 flows back to L this figure flows back to L producing a ray l_i of $\widetilde{\Lambda}^s_L$ which goes around a big disk in L centered at q_0 and has ideal point in the small segment bounded by u_0, u_1, see fig. 8, c. As i goes to infinity, these l_i escape to infinity in L because bigger and bigger disks in $g_i(F_i)$ flow to L. This implies that there is no ideal point of a ray of $\widetilde{\Lambda}^s_L$ outside the small segment of $\partial_\infty L$ bounded by u_0, u_1. This contradicts the previous proposition that such ideal points are dense in $\partial_\infty L$.

This finishes the proof of the lemma. □

Lemma 5.5. *The limit points of rays of $\widetilde{\Lambda}^s_F$ vary continuously in $\partial_\infty F$ except for the non Hausdorffness in the leaf space of $\widetilde{\Lambda}^s_F$.*

Proof. Suppose that p_i converges to p in F, with respective rays r_i converging to the ray r of $\widetilde{\Lambda}^s_F$. Let l be the leaf of $\widetilde{\Lambda}^s_F$ through p. Up to subsequence assume the r_i are all in the same sector of l defined by p and that they form a nested sequence of rays. Then the ideal ponts $(r_i)_\infty$ form a monotone sequence in $\partial_\infty F$. Perhaps some ideal points are the same. If $(r_i)_\infty$ does not converge to r_∞ there is an interval v in $\partial_\infty F$, between the limit and r_∞. Since the ideal points are dense in $\partial_\infty F$, there is w leaf of $\widetilde{\Lambda}^s_F$ with w_∞ in v. Therefore there is l' not separated from l with r_i converging to l' as well. In this fashion we can go from l to l'. This shows that if there is no leaf of $\widetilde{\Lambda}^s_F$ non separated from l in that side and in the direction the rays r_i go, then the limit points vary continuously.

We analyse a bit further the non Hausdorffness. In the setup above there are subrays of r_i with points converging to a point in l' and we can restart the analysis with l' instead of l. If there are finitely many leaves non separated from l and l' we can assume that l, l' are consecutive. Then they have subrays which share an ideal point. If m is the last leaf non separated from l, l' in the direction the rays r_i go to, then there is a ray ζ of m so that there are subrays of r_i with points converging to a point in ζ and $(r_i)_\infty$ converges to ζ_∞. If there are infinitely many such leaves non separated from l, then we can order them as $\{l_j\}, j \in \mathbf{N}$ all in the direction the rays r_i go to. The ideal points of l_j form a monotone sequence in $\partial_\infty F$ which converge to a point u in $\partial_\infty F$. The arguments above show that $(r_i)_\infty$ converges to u. □

Our next goal is to analyse the non Hausdorffness in the leaf space of $\widetilde{\Lambda}^s_F$ and identification of ideal points. We want to understand when can two ideal points of the same leaf of $\widetilde{\Lambda}^s_F$ be identified. A *Reeb annulus* is an annulus A with a foliation so that the boundary components are leaves and every leaf in the interior is a topological line which spirals towards the two boundary components in the same direction. In the universal cover the lifted foliation does not have Hausdorff leaf space. The lifted foliation to the universal cover is called a *Reeb band.* A *spike region* in a leaf F of $\widetilde{\Lambda}^s$ is a closed $\widetilde{\Lambda}^s_F$ saturated set $\mathcal{E}$ so that there are finitely many boundary leaves which are line leaves of $\widetilde{\Lambda}^s_F$. The ideal points of consecutive rays in the boundary are the same, otherwise they are distinct (like an ideal polygon). In addition the region is a bounded distance from the ideal polygon with these vertices. The bound is not universal in $\widetilde{\mathcal{F}}$. There is an ideal point z of $\mathcal{E}$ so that every leaf in the interior of $\mathcal{E}$ has both ideal points equal to z. In addition the leaves in the interior are nested.

Finally, the finitely many leaves in the boundary are all non separated from each other and they are limits of the interior leaves.

Proposition 5.6. *Let E be a leaf in $\widetilde{\mathcal{F}}$ and v a slice of a leaf v_0 of $\widetilde{\Lambda}^s_E$. Suppose that both ideal points of v are the same. Then v is contained in a spike region B of E. In addition either B projects to a Reeb annulus in a leaf of $\mathcal{F}$ or for any two consecutive rays in ∂B, the region between them projects to a set asymptotic to a Reeb annulus in a leaf of M.*

Proof. Let v be a slice as above with ideal point x^* in $\partial_\infty E$. Let C be the region bounded by v in E which only limits in x. We may assume that v is a line leaf of v_0, since any prong of v_0 which enters C will have ideal point x. We will show that the region C as it approaches x, projects to a set in M which limits to a Reeb annulus in a leaf of $\mathcal{F}$. The process will be done in a series of steps. The proof of this proposition is very long with several intermmediate results and lemmas.

Choose z_0 in v and let e_1, e_2 be the rays of v defined by z_0. Let ζ^* be the geodesic ray of E starting at z_0 and with ideal point x. Then ζ^* is in contained in the $2\delta_0$ neighborhood of e_1 or e_2, where δ_0 is the constant of lemma 5.4. It follows that we can choose p_i, q_i in e_1, e_2 respectively with p_i, q_i converging to x in $E \cup \partial_\infty E$ and also $d_E(p_i, q_i) < 4\delta_0$. Let e^i_1 be the subray of e_1 starting at p_i and e^i_2 the subray of e_2 starting at q_i. As usual up to subsequence there are g_i in $\pi_1(M)$ with $g_i(p_i)$, $g_i(q_i)$ converging to p_0, q_0 respectively, where p_0, q_0 are points in a leaf F of $\widetilde{\mathcal{F}}$. Then $g_i(E)$ converges to F and perhaps other leaves as well.

For i big enough the flowlines of $\widetilde{\Phi}$ through $g_i(p_i), g_i(q_i)$ go through to u_i and v_i in F. Also $u_i \to p_0, v_i \to q_0$. If the leaf of $\widetilde{\Lambda}^s_F$ through p_0 contains q_0 then for i big enough the arcs in leaves of $\widetilde{\Lambda}^s_F$ from u_i to v_i will have bounded length and bounded diameter. The same will happen for the arcs of of $g_i(v)$ between $g_i(p_i)$ and $g_i(q_i)$, contradiction. Hence p_0, q_0 are not in the same leaf of $\widetilde{\Lambda}^s_F$. Let l be the leaf of $\widetilde{\Lambda}^s_F$ through p_0 and r be the one through q_0. Let L, R leaves of $\widetilde{\Lambda}^s$ containing l and r respectively. Since the intersection of a leaf of $\widetilde{\Lambda}^s$ with F is connected, then L and R are distinct and also are not separated from each other in the leaf space of $\widetilde{\Lambda}^s$.

The first goal is to show that we can choose l, r line leaves of $\widetilde{\Lambda}^s_F$ as above so that they also share an ideal point. Let β_i be a ray in the leaf of $\widetilde{\Lambda}^s_F$ through u_i starting at u_i and containing points in the flowlines through to the ray $g_i(e^i_1)$. Similarly let γ_i be a subray in the same leaf starting at v_i and associated to the ray $g_i(e^i_2)$. Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be the collection of line leaves of $\widetilde{\Lambda}^s_F$ that β_i (resp. γ_i) converges to, including the ray of l (resp. r). Let $\mathcal{C}$ be the collection of all line leaves of $\widetilde{\Lambda}^s_F$ which are non separated from l, r. Then $\mathcal{C}$ contains $\mathcal{C}_1$ and $\mathcal{C}_2$. For any element τ in $\mathcal{C}$ it is contained in a leaf $B(\tau)$ of $\widetilde{\Lambda}^s$. All of the $B(\tau)$ are not separated from each other, and they are in the set of leaves $\mathcal{B}$ of $\widetilde{\Lambda}^s$ non separated from both L, R. By theorem 2.6, the set $\mathcal{B}$ has a linear order, making it order isomorphic to either **Z** or a finite set. This induces an order in $\mathcal{C}$ where we can choose this so that an arbitrary element of $\mathcal{C}_1$ is bigger than any element in $\mathcal{C}_2$.

If there are finitely many elements in $\mathcal{C}_1$ let l' be the last one and let ξ_1 be the ideal point of the ray of l' corresponding to the direction of the rays β_i. Otherwise the ideal points of the leaves in $\mathcal{C}_1$ form a weakly monotone sequence in $\partial_\infty F$ and let ξ_1 be the limit of this sequence. Similarly define ξ_2 associated to r, see fig. 9, a. The first thing to prove is the following:

Lemma 5.7. $\xi_1 = \xi_2$.

Proof. Suppose by way of contradiction that this is not true. Choose 2 markers very near ξ_1 bounding an interval J_1 in $\partial_\infty F$ with ξ_1 in the interior and similarly choose markers near ξ_2 and interval J_2 so that J_1, J_2 are disjoint. Let W_1 be the wedge of F centered at a point x_0 with ideal set J_1 and W_2

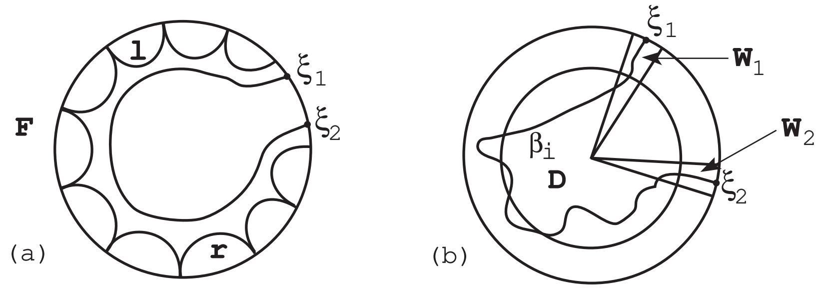

Figure 9: *a. Non Hausdorffness in the limit, b. Showing* $\xi_1 = \xi_2$.

the wedge of F centered also at x_0 with ideal set J_2. For i big enough both boundaries of W_1 and W_2 flow into $g_i(E)$.

Suppose first that there is a last leaf l' in $\mathcal{C}_1$. Then l' has a ray which is eventually contained in a strictly smaller subwedge W'_1 of W_1 – since its ideal point is ξ_1. Now choose a big disk D of F centered in x_0 . Let N_1 be the closure of $W_1 - D$. Choose D big enough so that l' enters N_1 through ∂D and is then entirely in W'_1. For i big enough β_i will be close to l' for a long distance. By lemma 5.5 the ideal points of β_i converge to ξ_1 as i converges to infinity, since l' is the last leaf non separated from l in that side. The ideal point is in the limit set of the subwedge W'_1. If the rays β_i keep exiting W_1 then since they are trapped by l' and β_{i_0} (for some i_0), they will have to intersect a compact part of ∂W_1. Then the sequence $\{\beta_i\}$ has additional limits besides the leaves in $\mathcal{C}_1$, contradiction. Therefore for big enough i, the β_i enters N_1 through ∂D and stays in N_1 from then on.

We want to get the same result when $\mathcal{C}_1$ is infinite. In that case let $\{\nu_j,\ j \in \mathbf{N}\}$ be the leaves in $\mathcal{C}_1$ ordered with same ordering as in $\mathcal{C}_1$ and $\nu_1 = l$. Since these leaves are non separated from each other then they cannot accumulate anywhere in F and the leaves ν_j escape compact sets as j grows. The ideal points of ν_j are also converging to ξ_1. By density of ideal points of $\widetilde{\Lambda}^s_F$ in $\partial_\infty F$ the leaves ν_j cannot be getting closer to non trivial intervals in $\partial_\infty F$. This implies that there is j_0 so that for

$$j \geq j_0, \quad \nu_j \ \text{ is very close to } \ \xi_1 \ \text{ in } F \cup \partial_\infty F$$

and so contained in W'_1. Now an argument entirely similar as in the case $\mathcal{C}_1$ finite implies that for i big enough then β_i has subrays entirely contained in N_1. The same holds for γ_i producing subrays entirely contained in the corresponding set N_2 – the disk D may need to be bigger to satisfy all these conditions.

There is $a_1 > 0$ and i_0 so that for $i \geq i_0$ then except for the initial segment of length a_1 then β_i is entirely contained in N_1 and similarly for γ_i and N_2. Choose k_0 big enough so that D is ϵ close to $g_k(E)$ for any $k \geq k_0$. Then D flows in $g_k(E)$ and so do $\partial W_1, \partial W_2$. For i bigger than both i_0, k_0 the ray β_i flows into the ray $g_i(e^i_1)$ (notice these do not have singularities). The ray $g^i(e^i_1)$ has to be in the generalized wedge which is bounded by the image of ∂W_1 in $g_i(E)$. Similarly for γ_i. This argument is done in lemma 5.2. These two generalized wedges have disjoint ideal sets in $\partial_\infty g_i(E)$. Therefore $g_i(e^i_1)$ and $g_i(e^i_2)$ do not have the same ideal points. This is a contradiction because e_1, e_2 have the same ideal point in $\partial_\infty E$.

This proves that $\xi_1 = \xi_2$. □

Continuation of the proof of proposition 5.6

The fact $\xi_1 = \xi_2$ implies that the ideal points of β_i, γ_i are all the same and equal to ξ_1. Let $\xi = \xi_1$. Let μ be the geodesic ray in F starting at x_0 with ideal point ξ. Since $(\beta_i)_\infty = (\gamma_i)_\infty = \xi$, then lemma 5.4 implies that for z in μ far enough from p_0, there are $b_i(z)$ in β_i and $c_i(z)$ in γ_i both of which are less than $2\delta_0$ away from z in F. This is for any i in $\mathbf{N}$. So up to subsequence we assume $b_i(z)$ converges to $b(z)$ and similarly $c_i(z)$ converges to $c(z)$. By definition of $\mathcal{C}_1$ the point $b(z)$ has to be in one of the leaves of $\mathcal{C}_1$ and similarly for $c(z)$.

Lemma 5.8. *There is one element ζ of $\mathcal{C}_1$ which has ideal point ξ.*

Proof. If there are finitely elements in $\mathcal{C}_1$ then the last one satisfies this property. Suppose then there are infinitely many elements in $\mathcal{C}_1$. As z varies in μ, then so does $b(z)$. If there are z escaping in μ so that $b(z)$ is in the same element ζ of $\mathcal{C}_1$ then ζ has an appropriate ray with ideal point ξ. In this case we are done.

Otherwise we can find z_k in μ converging to ξ so that $b(z_k)$ are in leaves $\nu_{i(k)}$ of $\mathcal{C}_1$ which are all distinct. We can choose z_k so that the $i(k)$ increases with k. In the same way we have $c(z_k)$ in elements of $\mathcal{C}_2$. Let

$$B_k \ = \ \widetilde{W}^s(b(z_k)), \ \ C_k \ = \ \widetilde{W}^s(c(z_k)), \quad \text{both in } \mathcal{B}$$

Recall that $\mathcal{B}$ is the set of leaves of $\widetilde{\Lambda}^s$ non separated from both L, R. Since the length from $b(z_k)$ to $c(z_k)$ in F is bounded by $4\delta_0$, then up to subsequence assume $\pi(b(z_k)), \pi(c(z_k))$ converge in M. For i, k big enough there is h_{ik} covering tranlation so that $h_{ik}(b(z_i))$ is very close to $b(z_k)$ and $h_{ik}(c(z_i))$ is very close to $c(z_k)$. Suppose $i >> k$, let $h = h_{ik}$ for simplicity. Then B_k has a point $b(z_k)$ very close to $h(b(z_i)) \in h(B_i)$ and similarly for $c(z_k)$ in C_k very close to $h(c(z_i)) \in h(C_i)$. Since B_k is non separated from C_k and similarly for $h(B_i), h(C_i)$, then the only way this can happen is that

$$h(B_i) \ = \ B_k, \quad h(C_i) \ = \ C_k$$

This implies that h sends the set of leaves non separated from B_i, C_i to itself, that is h acts on the set $\mathcal{C}$ and therefore acts on $\mathcal{B}$ as well. Notice that $B_k < B_i$ in the order of $\mathcal{B}$ because $i > k$ and $C_k \geq C_i$ (the C_k could be all the same, but if they are not then they decrease in the order). Since $h(B_i) = B_k$ then h acts as a decreasing translation in the ordered set $\mathcal{B}$. But since $h(C_i) = C_k$ then h acts as a non decreasing translation. These two facts are incompatible.

This implies that we have to have at least one element in $\mathcal{C}_1$ with ideal point ξ. The same happens for $\mathcal{C}_2$. This finishes the proof of the lemma. $\square$

Since β_i also converges to ζ we can rename the objects and assume that $l = \zeta$ and p_0 is a point in l. This changes the points p_i in the ray e_1. Similarly do the same thing in the other direction. We state this conclusion:

Conclusion – There are p_i, q_i in e_1, e_2 respectively, escaping these rays, so that $d_E(p_i, q_i) < 4\delta_0$ and there are covering translations g_i so that: $g_i(p_i)$ converges to p_0, $g_i(q_i)$ converges to q_0, both in F and in rays l, r of $\widetilde{\Lambda}^s_F$. Also l, r converge to the same ideal point ξ in $\partial_\infty F$.

We will continue this perturbation approach. We want to show that the region in F "between" l and r projects to a Reeb annulus of $\mathcal{F}$ in M. Let then z_i in l converging to ξ and w_i in r converging to ξ, so that $d_F(z_i, w_i)$ is always less than $4\delta_0$. Up to subsequence assume there are h_i covering translations with

$$h_i(z_i) \to z_0, \quad h_i(w_i) \to w_0$$

Notice that $h_i(L), h_i(R)$ are non separated from each other and $h_i(L) \to \widetilde{W}^s(z_0)$, $h_i(R) \to \widetilde{W}^s(w_0)$. The argument in the previous lemma then implies that $h_i(L) = h_j(L)$, $h_i(R) = h_j(R)$ for all i, j at least equal to some i_0. Discard the first i_0 terms and postcompose h_i with $(h_{i_0})^{-1}$ (that is $(h_{i_0}^{-1} \circ h_i)$), to assume that $h_i(L) = L, h_i(R) = R$ for all i. So the h_i are all in the intersection of the isotropy groups of L and R. This group is generated by a covering translation h. Therefore there are n_i with $h_i = h^{n_i}$. Since $h_i(z_i) \to z_0$ and the $\{z_i, \mid i \in \mathbf{N}\}$ do not accumulate in $\widetilde{M}$ then $|n_i| \to \infty$. In addition since L, R are not separated from each other, then h preserves each individual line leaf, slice and possible lift annulus of L.

Up to subsequence and perhaps taking the inverse of h, assume that n_i converges to ∞. If $h(F) = F$, then since $h(L) = L$ this produces a closed leaf in $\pi(F)$. Similarly $h(R \cap L) = R \cap L$ so produces another closed leaf in F and together bound an annulus with a sequence of leaves converging to the boundary leaves. By Euler characteristic reasons, there can be no singularities inside the annulus, so we conclude that the annulus in $\pi(F)$ has a Reeb foliation.

Let $\mathcal{H}$ be the leaf space of $\widetilde{\mathcal{F}}$. This is a one dimensional manifold, which is simply connected, but usually not Hausdorff [Ba2]. The element h acts on $\mathcal{H}$. An analysis of group actions on simply connected non Hausdorff spaces was done in [Ro-St] or [Fe10]. One possibility is that h acts freely in $\mathcal{H}$. Then h has an axis τ in $\mathcal{H}$ which is invariant under h. In general this axis is not properly embedded, see [Fe10]. Since all the $h^{n_i}(F)$ intersect a common transversal, then F has to be in the axis of h and $h^n(F)$ converges to a collection of non separated leaves. In this case we get that F^* and $h(F^*)$ are non separated from each other.

The other situation is that h has fixed points in $\mathcal{H}$. In general the set of fixed points of $\mathcal{H}$ is not a closed set, but the set of points z in $\mathcal{H}$ so that z and $h(z)$ are not separated in $\mathcal{H}$ is a closed subset Z of $\mathcal{H}$. None of the images of F under h can be in Z, so F is in a component of $\mathcal{H} - Z$. Then h permutes these components. In addition h preserves an orientation in $\mathcal{H}$ – since $\mathcal{F}$ is transversely orientable. Since $h^{n_i}(F)$ all intersect a common transversal then they have all to be in the same component U of $\mathcal{H} - Z$. Let i_0 be the smallest positive integer so that $h^{i_0}(U) = U$. It follows that all n_i are multiples of i_0. The leaf F^* is in the boundary of the component U and $h^{i_0}(F^*) = F^*$.

The only remaining case to be analysed is that h acts freely and $h^n(F)$ converges to F^* with $h(F^*)$ non separated from F^*. In this particular case we prove this is not possible, that is:

Claim – $h(F^*) = F^*$.

Suppose this is not true. The leaves $h(F^*), F^*$ are not separated in $\mathbf{H}^2$. This implies that $\Theta(F^*)$ and $\Theta(h(F^*))$ are disjoint subsets of $\mathcal{O}$, see fig. 10. Therefore there are boundary leaves separating them. But L intersects both F^* and $h(F^*)$ as L intersects F and is invariant under h. Therefore both $\Theta(F^*)$ and $\Theta(h(F^*))$ intersect the same stable leaf $\Theta(L)$.

Suppose that there is a stable boundary component of $\Theta(F^*)$ separating it from $\Theta(h(F^*))$. Then it has to be a slice of $\Theta(L)$ as this set intersects both of them. It would not be a line leaf of $\Theta(L)$. But as remarked before, h leaves invariant all the slices, line leaves and lift annuli of L and this contradicts $\Theta(h(F^*))$ being disjoint from $\Theta(F^*)$. This implies there is an unstable boundary component of $\Theta(F^*)$ separating it from $\Theta(h(F^*))$, see fig. 10.

In the same way $\Theta(R)$ intersects both $\Theta(F^*)$ and $\Theta(h(F^*))$. Let $L_i = \widetilde{W}^s(u_i)$. Recall from the beginning of the proof of proposition 5.6 that u_i, v_i are points in F with u_i converging to p_0 in L and v_i converging to q_0 in R. Then $\Theta(L_i)$ converges to $\Theta(L) \cup \Theta(R)$ (maybe other leaves as well). So $\Theta(L_i)$ intersects $\Theta(F^*)$ and $\Theta(h(F^*))$ for i big enough. The intersection of $\Theta(L_i)$ with at least one of $\Theta(F^*)$ or $\Theta(h(F^*))$ cannot be connected, see fig. 10. This contradicts propostion 3.2. This contradiction implies that $h(F^*) = F^*$ and proves the claim.

So far we have proved the following: in any case there is i_0 a positive integer so that if $f = h^{i_0}$ then $f(F^*) = F^*$. As $f(L) = L$ then $f(F^* \cap L) = F^* \cap L$ and similarly $f(F^* \cap R) = F^* \cap R$. This produces an annulus B in $\pi(F^*)$ with a Reeb foliation. The region of F^* bounded by $F^* \cap R$ and $F^* \cap L$ bounds a band B which is a bounded distance from a geodesic in F^* and projects to a Reeb

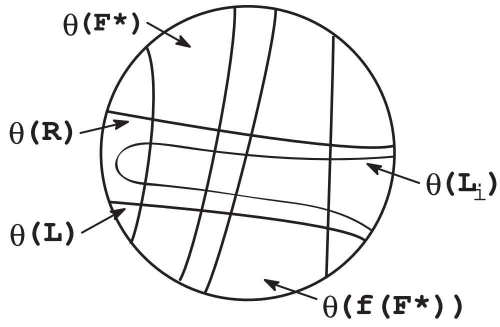

Figure 10: *Contradiction in the orbit space $\mathcal{O}$.*

annulus in a leaf of $\mathcal{F}$.

But to prove proposition 5.6, we really want these facts for F and not just F^*. This turns out to be true: $\pi(E)$ has points converging to $\pi(F)$ and $\pi(F)$ has points converging to an annulus in $\pi(F^*)$. Since the annulus is compact, it turns out the second step is unnecessary. This depends on an analysis of holonomy of the foliation $\mathcal{F}$ near the annulus in $\pi(F^*)$ as explained below.

Claim – The point $\pi(p_0)$ of $\pi(F)$ is in the boundary of a Reeb annulus of $\mathcal{F}$ contained in $\pi(F)$. This implies that $F = F^*$.

The point z_0 is in $F^* \cap L$. Then $\pi(z_0)$ is in $\pi(F^* \cap L) = \alpha$ which is a closed curve since h^{i_0} leaves invariant both F^* and L and their intersection is connected. Previous arguments in the proof imply that for i big enough $h_i(z_i)$ is in the same local sheet of $\widetilde{\Lambda}^s$ as z_0. Hence the points $\pi(z_i)$ are in $W^s(\pi(z_0)) = \pi(L)$ and converge to $\pi(z_0)$. This shows that $\pi(F \cap L)$ is asymptotic to α in the direction corresponding to the projection of the direction of escaping z_i in the ray of $F \cap L$. Namely α has contracting holonomy (of $\mathcal{F}$) in the side the $\pi(z_i)$ are converging to and eventually $\pi(z_i)$ is in the domain of contraction of α.

This means that the direction of F associated to the ideal point ξ is a contracting direction towards F^*. The rays in the leaves $F^* \cap L$, $F^* \cap R$ in F^* are a bounded distance from a geodesic ray in F^* with same ideal point. The contraction above implies that the corresponding rays $F \cap L$, $F \cap R$ of F are also a bounded distance from a ray in F with ideal point ξ.

Now recall the points p_i in E. We have $g_i(p_i)$ very close to p_0 in the leaf l of $\widetilde{\Lambda}^s_F$. Also $\pi(l)$ is eventually in a region contracting towards a Reeb annulus of $\mathcal{F}$. Hence if i is big enough the $g_i(p_i)$ will also be in this region. The leaf through $\pi(p_i)$ will be contracted towards the Reeb annulus in that direction. This implies that the limit of the $\pi(p_i)$ is already in a Reeb annulus, consequently the limit of the $g_i(p_i)$ is already in a Reeb band.

It now follows that $\pi(F) = \pi(F^*)$. That means that the second perturbation procedure (from points in F to points in F^*) in fact does not produce any new leaf. This implies that up to covering translations then the leaf E is asymptotic to F in the direction of the ideal point x^* in $\partial_\infty E$. Let V be the region of E bounded by v with ideal point x^*. Then outside of a compact part it is very near a Reeb band in F and so has no singularity of the foliation $\widetilde{\Lambda}^s_E$. By Euler characteristic reasons it follows that V has no singularities in the compact part also. So far we proved the following:

Conclusion – Let v be a slice of $\widetilde{\Lambda}^s_E$ with two rays converging to the same ideal point x^* of $\partial_\infty E$ and V is the region of E bounded by v. Then v is a line leaf of $\widetilde{\Lambda}^s_E$ in the V side and V has no singularities in the interior. Also $\pi(V)$ is either contained in or asymptotic to a Reeb annulus in a leaf of $\mathcal{F}$ and so E is asymptotic to a Reeb band in a leaf F in the direction x^*.

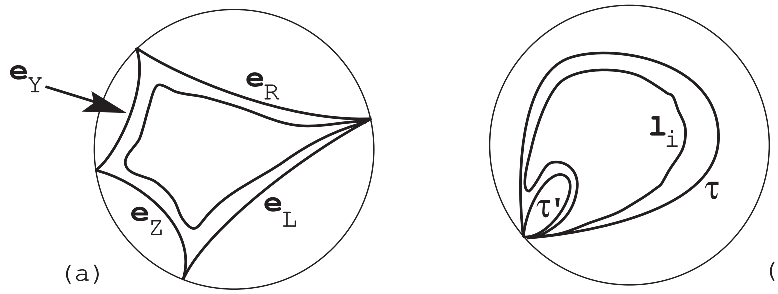

Figure 11: *a. l_i converging to non separated leaves e_L, e_Z, e_Y, e_R of $\widetilde{\Lambda}^s_E$, b. Nested families and identifications of ideal points.*

Continuation of the proof of proposition 5.6.

What we want to prove is that in E itself the region V is contained in the interior of a spike region. Notice it is not true in general that $\pi(V)$ is contained in a Reeb annulus, only that it is asymptotic to a Reeb annulus. For instance start with a leaf of $\mathcal{F}$ having a Reeb annulus and blow that into an I-bundle. Then produce holonomy associated to the core of the Reeb annulus. Then one produces Reeb bands asymptotic to but not contained in Reeb annuli.

Since V is asymptotic to the Reeb band in F, it turns out that (after rearranging by covering translations) that E intersects both L and R leaves of $\widetilde{\Lambda}^s$. Their intersection produces two leaves e_L, e_R of $\widetilde{\Lambda}^s_E$ which are not separated from each other and which have the same ideal point x^*. There are then leaves l_i of $\widetilde{\Lambda}^s_E$ all with ideal point x^* and which converge to $e_L \cup e_R$. This follows from the fact that in F the same is true and E is asymptotic to F in that direction, plus the connectivity of the intersection of E with leaves of $\widetilde{\Lambda}^s$.

Now the sequence l_i can converge to other leaves as well, all of which will be non separated from e_L, e_R. The set of limits is an ordered set and the any other leaf is between e_L and e_R. By theorem 2.6 there are only finitely many of them. We refer to fig. 11, a, where for simplicity we consider there are 4 leaves in the limit: e_L, e_Y, e_Z, e_R contained in leaves L, Z, Y, R of $\widetilde{\Lambda}^s$. These leaves of $\widetilde{\Lambda}^s$ are non separated from each other and form an ordered set. Let ξ be the region of E which is the union of the region bounded by all the l_i plus the boundary leaves, which are non separated from e_L, e_R. Clearly every leaf in the interior has ideal point x^* and has no singularity. We want to show that ξ is a spike region.

Any two consecutive leaves of $\partial\xi$ in this ordering will have rays with same ideal point and leaves l_i converging to them. This situation is important on its own and is analysed in the following proposition:

Proposition 5.9. *Suppose v_1, v_2 are non separated leaves in $\widetilde{\Lambda}^s_G$ for some G leaf of $\widetilde{\mathcal{F}}$. Suppose there are no leaves non separated from v_1, v_2 in between them. Then the corresponding rays of v_1, v_2 have the same ideal point in $\partial_\infty G$. In addition they are a bounded distance from a geodesic ray of G with same ideal point. In M this region either projects to or is asymptotic to a Reeb annulus.*

Proof. We do the essentially the same proof as in the case of leaves of $\widetilde{\Lambda}^s_G$ with same ideal points, except that we go in the direction of the non Hausdorfness. Because there are no non separated leaves in between v_1, v_2, then the corresponding rays have the same ideal point. Choose w_i, y_i in these rays of v_1, v_2 and escaping towards the ideal point and so that $d_G(w_i, y_i)$ is less than $4\delta_0$. We do the limit analysis using $f_i(w_i), f_i(y_i)$ converging in $\widetilde{M}$. Because v_1, v_2 are non separated it follows that $f_i(w_i), f_j(w_j)$ are in the same stable leaf (of $\widetilde{\Lambda}^s$) for i, j big enough. Hence we can readjust so that they are all in the same stable leaf and similarly for $f_i(y_i)$. The same arguments as before show that that region of G between v_1, v_2 projects in M to set in a leaf of $\mathcal{F}$ which is either contained in or asymptotic to a Reeb annulus. The results follow. In general nothing can be said about the other direction in the leaves v_1, v_2: in particular it does not follow at all that the other rays of v_1, v_2 have to have the same ideal point. □

Given this last proposition then for any two consecutive rays in $\partial\mathcal{E}$ it follows that they are a bounded distance from a geodesic ray in E. All that is needed to show that $\mathcal{E}$ is a spike region is to prove that the ideal points of the rays in the boundary are distinct except for consecutive rays.

Suppose there are other identifications of ideal points of leaves in the boundary of $\mathcal{E}$. Then there is at least one line leaf τ in the boundary of $\mathcal{E}$ so that τ has identified ideal points. Our analysis so far shows that τ is in the interior of another region similar to the one constructed above so that all leaves have just one common ideal point. Since the l_i limit on τ, then the ideal point of τ has to be x^*. In addition the leaves in this new region have to be nested. But if the l_i together with τ are a nested family of leaves of $\widetilde{\Lambda}^s_F$, then the τ is outside the l_i hence the region in E bounded by τ enclosed the whole region $\mathcal{E}$, see fig. 11, b. There is at least one other leaf τ' in $\partial\mathcal{E}$. The same arguments we used for τ can be applied to τ'. But it is impossible that the l_i are also nested with the τ', see fig. 11, b.

This shows that the ideal points of $\mathcal{E}$ are distinct except as mandated by consecutive rays. In addition any line leaf in the boundary of $\mathcal{E}$ has distinct ideal points and rays which are a bounded distance from geodesic rays. It follows that the whole leaf is a bounded distance from a geodesic in E. This shows that $\mathcal{E}$ is a spike region. This finishes the proof of proposition 5.6. □

Finally in the case $\widetilde{\Lambda}^s$ has Hausdorff leaf space one can say much, much more about metric properties of leaves of $\widetilde{\Lambda}^s_F$:

Proposition 5.10. *Suppose that Φ is an almost pseudo-Anosov flow transverse to a foliation $\mathcal{F}$ with hyperbolic leaves. Suppose that $\widetilde{\Lambda}^s$ has Hausdorff leaf space. Then there is $k_0 > 0$ so that for any F leaf of $\widetilde{\Lambda}^s$, then the slice leaves of $\widetilde{\Lambda}^s_F$ are uniform k_0 quasigeodesics.*

Proof. If there is a leaf F of $\widetilde{\mathcal{F}}$ and a slice leaf of $\widetilde{\Lambda}^s_F$ with only one ideal point, then the proof of proposition 5.6 shows that there are leaves of $\widetilde{\Lambda}^s$ non separated from each other. This is impossible.

Suppose now that for any integer i, there are x_i in $\widetilde{M}$, x_i in leaves F_i of $\widetilde{\mathcal{F}}$ with x_i in line leaves l_i of $\widetilde{\Lambda}^s_{F_i}$ with distance from x_i to l^*_i in F_i going to infinity. Here l^*_i is the geodesic in F_i with same ideal points as l_i. Up to covering translations assume x_i converges to x. Also assume all x_i are in the same sector of $\widetilde{\Lambda}^s$ defined by x. Since l_i converges to l, the arguments in lemmas 5.5 and 5.2 would show that the ideal points of l are the same. This was just disproved above.

Given that, the line leaves are within some global distance a_0 of the respective geodesics in their leaves. It is well known that these facts imply that the slice leaves of $\widetilde{\Lambda}^s_F$ are uniform quasigeodesics. For a proof of this well known fact see for example [Fe-Mo]. □

6 Continuous extension of leaves

The purpose of this section is to prove the main theorem: the continuous extension property for leaves of foliations which are almost transverse to quasigeodesic singular pseudo-Anosov flows in atoroidal 3-manifolds. As seen before this implies that M has negatively curved fundamental group.

Suppose first that Φ is an almost pseudo-Anosov flow which is transverse to a foliation $\mathcal{F}$ with hyperbolic leaves in a general closed 3-manifold M. Given a leaf F of $\widetilde{\mathcal{F}}$ we introduce geodesic "laminations" in F coming from $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$. We only work with the stable foliation, similar results hold for the unstable foliation. Assume that a leaf l of $\widetilde{\Lambda}^s_F$ is not singular. If both ideal points are the same let l^* be empty. Otherwise let l^* be the geodesic with same ideal points as l. If l is singular, then no line leaves of l have the same ideal point by proposition 5.6. For each line leaf e of l let e^* be the corresponding geodesic and l^* their union. Let now τ^s_F be the union of these geodesics of F. Leaves of $\widetilde{\Lambda}^s_F$ do not have transverse intersections and therefore the same happens for leaves of τ^s_F.

Suppose that $\widetilde{\Lambda}^s_F$ has non separated leaves l, v which are not in the boundary of a spike region. Then there are l_i converging to $l \cup v$ (and maybe other leaves as well), but l^*_i does not converge to l^* or v^*. Notice none of the limit leaves can have identified ideal points, because then they would be in the interior of a spike region (proposition 5.6) and have a neighborhood which is product foliated. Let $\overline{\tau}^s_F$ be the closure of τ^s_F. Then $\overline{\tau}^s_F$ is a geodesic lamination in F. Similarly define τ^u_F, $\overline{\tau}^u_F$. In a complementary region U of $\overline{\tau}^s_F$ associated to non Hausdorffness, there is one boundary component which is added (a leaf of $\overline{\tau}^s_F - \tau^s_F$) and which is the limit of the l^*_i as above. All of the other boundary leaves of the region are associated to the non separated leaves of $\widetilde{\Lambda}^s_F$ and are in τ^s_F.

Lemma 6.1. *The new leaves in $\overline{\tau}^s_F$ (that is those in $\overline{\tau}^s_F$ - τ^s_F) come from non Hausdorfness of $\widetilde{\Lambda}^s_F$.*

Proof. Let e_i in τ^s_F converging to e not in τ^s_F. Then choose l_i line leaves in $\widetilde{\Lambda}^s_F$ with $e_i = l^*_i$. Given u a point in e, there is u_i in l^*_i very close to u. Then there are p_i in l_i which are $2\delta_0$ close to u_i. Up to subsequence assume that p_i converges to p_0 and let l be the line leaf of $\widetilde{\Lambda}^s_F$ that the sequence l_i converges to. Then l^*_i does not converge to l^* so we have a non Hausdorff situation: l_i converging to l and other leaves as well and l^* is the added leaf associated to this non Hausdorfness. This finishes the proof of the lemma. □

Lemma 6.2. *The complementary regions of $\overline{\tau}^s_F$ are ideal polygons associated to singular leaves and non Hausdorff behavior of $\widetilde{\Lambda}^s$. If M is atoroidal then these regions are finite sided ideal polygons.*

Proof. Let x be in a complementary region U of $\overline{\tau}^s_F$. Let e be a leaf in the boundary ∂U. Suppose first that e is an actual leaf of τ^s_F, which comes from a line leaf l of $\widetilde{\Lambda}^s_F$. It may be that l is a singular leaf which is singular on the x side. In that case x is in the region U. Otherwise l is not singular on the side containing x and we may assume there are l_i leaves of $\widetilde{\Lambda}^s_F$ on that side with l_i converging to l. If the ideal points of l_i converge to that of l then eventually l^*_i separates x from e and x is not in the complementary region U – impossible. Hence the ideal points of l_i do not converge to ∂e and there is non Hausdorfness and a complementary region in that side of l. Then x needs to be in this complementary region (which is U) and e is a boundary leaf of U which comes from a line leaf of τ^s_F.

Suppose now that e is an added leaf. There are l_i leaves of $\widetilde{\Lambda}^s_F$ with $e_i = l^*_i$ converging to e on the side opposite to x, otherwise x is not in U. Then l_i converges to more than one leaf of $\widetilde{\Lambda}^s_F$ producing non Hausdorff behavior and a complementary region with e in its boundary. The x is in the region associated to this non Hausdorff behavior, so the complementary region must be U.

If there is a complementary region of $\overline{\tau}^s_F$ with infinitely many sides then it is associated to non Hausdorff behavior and so there are leaves l_i of $\widetilde{\Lambda}^s_F$ converging to infinitely many distinct leaves of $\widetilde{\Lambda}^s_F$. Then there is L leaf of $\widetilde{\Lambda}^s$ which is non separated from infinitely many other leaves. Theorem 2.6 implies that there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$, contradiction. This finishes the proof. □

We now turn to the continuous extension property.

Theorem 6.3. *(Main theorem) Let $\mathcal{F}$ be a foliation in M^3 closed, atoroidal. Suppose that $\mathcal{F}$ is almost transverse to a quasigeodesic, singular pseudo-Anosov flow Φ' and transverse to an associated almost pseudo-Anosov flow Φ. Singular means Φ' is not a topological Anosov flow. Then for any leaf F of $\widetilde{\mathcal{F}}$, the inclusion map $\Psi : F \to \widetilde{M}$ extends to a continuous map*

$$\Psi : \ F \cup \partial_\infty F \ \rightarrow \ \widetilde{M} \cup S^2_\infty$$

The map Ψ restricted to $\partial_\infty F$, gives a continuous parametrization of the limit set of F, which is then locally connected.

Proof. The hypothesis imply that $\pi_1(M)$ is negatively curved. Difficulties in the proof of this result are that $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ may have non Hausdorff leaf space [Mo3, Fe8] and so $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ can have non Hausdorff leaf space. This imples that their leaves cannot be uniform quasigeodesics. The proof is done in two steps: first we define an extension and then we show that it is continuous.

First we need to review some facts about quasigeodesic almost pseudo-Anosov flows. If γ is an orbit of $\widetilde{\Phi}$ then it is a quasigeodesic has unique distinct ideal points γ_- and γ_+ in S^2_∞ corresponding to the positive and negative flow directions [Th1, Gr, Gh-Ha, CDP]. Hence given x in $\widetilde{M}$ define

$$\eta_+(x) \ = \ \gamma_+, \quad \eta_-(x) = \ \gamma_-, \quad \eta_+(x) \ \neq \ \eta_-(x),$$

where γ is the $\widetilde{\Phi}$ flowline through x. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ and a is a limit point of L in S^2_∞, then there is an orbit γ of $\widetilde{\Phi}$ contained in L with either $\gamma_- = a$ or $\gamma_+ = a$, that is, any limit point of L is a limit point of one of its flow lines [Fe8]. Also any such L in $\widetilde{\Lambda}^s$ is Gromov negatively curved [Gr, Gh-Ha, Fe8] and has an intrinsic ideal boundary ∂L consisting of a single forward ideal point and distinct negative ideal points for each flow line [Fe8]. The set $L \cup \partial_\infty L$ is a natural compactification of L in the Gromov sense. The inclusion $\kappa : L \rightarrow \widetilde{M}$ extends to a continuous map $\kappa : L \cup \partial L \rightarrow \widetilde{M} \cup S^2_\infty$. This all follows from the fact that Φ is quasigeodesic. If L is in $\widetilde{\Lambda}^s$ there is a unique distinguished ideal point denoted by L_+ in S^2_∞ which is the forward limit point of any flow line in L. Finally if in addition Λ^s is a quasi-isometric singular foliation, then the extension κ is always a homeomorphism into its image, but this is not true if Λ^s is not quasi-isometric. Similarly for L in $\widetilde{\Lambda}^u$.

Throughout the proof we fix a unique identification of $\widetilde{M} \cup S^2_\infty$ with the closed unit ball in $\mathbf{R}^3$. The Euclidean metric in this ball induces the visual distance in $\widetilde{M} \cup S^2_\infty$. Then $diam(B)$ denotes the diameter in this distance for any subset B of $\widetilde{M} \cup S^2_\infty$. A notation used throughout here is the following: if A is a subset of a leaf F of $\widetilde{\mathcal{F}}$, then $\overline{A}$ is its closure in $F \cup \partial_\infty F$.

We now produce an extension $\Psi : \partial_\infty F \rightarrow S^2_\infty$.

<u>Case 1</u> $-$ Suppose that v in $\partial_\infty F$ is not an ideal point of a ray in $\widetilde{\Lambda}^s_F$ or in $\widetilde{\Lambda}^u_F$.

Since $\pi_1(M)$ is negatively curved, then complementary regions of $\overline{\tau}^s_F$ are finite sided ideal polygons. Hence there are e_i in $\overline{\tau}^s_F$ so that $\{e_i \cup \partial e_i\}$, $i \in \mathbf{N}$ define a neighborhood basis of v (in $F \cup \partial_\infty F$) and $\{e_i\}$ forms a nested sequence. Here ∂e_i are the ideal points of e_i in $\partial_\infty F$. We say that the $\{e_i\}$ define a neighborhood basis at v. Assume that no two e_i share an ideal point $-$ possible because of hypothesis. If e_i is in $\overline{\tau}^s_F - \tau^s_F$ then it is the limit of leaves in τ^s_F and by adjusting the sequence above we can assume that e_i is always in τ^s_F. Let l_i in $\widetilde{\Lambda}^s_F$ with $l^*_i = e_i$ and L_i leaves of $\widetilde{\Lambda}^s$ with $l_i \subset L_i$.

Similarly there are c_i in $\overline{\tau}^u_F$ defining a neighborhood basis of v. Up to subsequence we may assume that e_1, c_1, e_2, c_2, etc.. are nested and none of them have any common ideal points (in $F \cup \partial_\infty F$) and c_i is in τ^u_F. Let b_i in $\widetilde{\Lambda}^u_F$ with $b^*_i = c_i$ and B_i leaves of $\widetilde{\Lambda}^u$ with $b_i \subset B_i$.

<u>Claim</u> $-$ Both L_i and B_i escape in $\widetilde{M}$.

Notice $e_i \cap c_j = \emptyset$ for any i, j. If $l_i \cap b_j$ is non empty with $j > i$, then the nesting property above implies that $b_{i+1}, b_{i+2}, ..., b_j$ all have to intersect. Since there is a global upper bound on the number of prongs of leaves of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$, this can happen for only finitely many times. Up to taking a further subsequence we may assume that all the l_i, b_j are disjoint. At this point we need the following result:

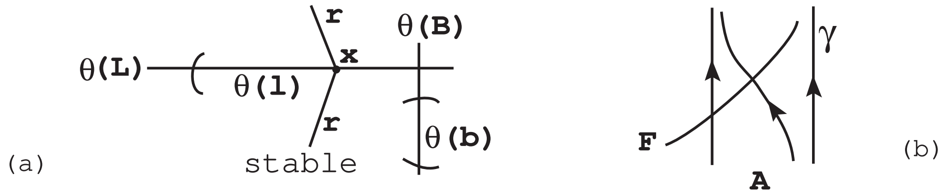

Figure 12: *a. Obstruction to intersections of leaves, b. The case of F escaping up.*

Lemma 6.4. *Let L leaf of $\widetilde{\Lambda}^s$, B leaf of $\widetilde{\Lambda}^u$ and F leaf of $\widetilde{\mathcal{F}}$ so that F intersects both L and B: $l = L \cap F$, $b = F \cap B$. Suppose that b and l are disjoint in F. Then L does not intersects B in $\widetilde{M}$.*

Proof. Suppose not. Recall that $\Theta(L), \Theta(B)$ are finite pronged, non compact trees and they intersect in a compact subtree. The union is also a finite pronged tree. In addition $\Theta(L \cap B)$ is connected. The sets $\Theta(l), \Theta(b)$ are disjoint in this union. Let x be a boundary point of $\Theta(l)$ which is either in $\Theta(L \cap B)$ or separates $\Theta(L \cap B)$ from $\Theta(l)$ in this union, see fig. 12, a.

Let $\gamma = x \times \mathbf{R}$, an orbit of $\widetilde{\Phi}$. The first possibility is that F escapes up as $\Theta(F)$ approaches x. Then γ is a repelling orbit with respect to the $\Theta(l)$ side, see fig. 12, b and γ is in the boundary of a lift annulus A. This means that $\Theta(l)$ is a generalized unstable prong from the point of view of x. By proposition 3.1 there is a stable slice r of $\mathcal{O}^s(x)$ with r contained in $\partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches r, see fig. 12, a. The two sides of r are the closest generalized prongs to $\Theta(l)$ on either side of $\Theta(l)$. This implies that r separates $\Theta(b)$ from $\Theta(F)$ see fig. 12, a. Then $\Theta(b)$ cannot be contained in $\Theta(F)$, contradiction.

The second option is that F escapes down as $\Theta(F)$ approaches x along $\Theta(l)$. Here there is a slice r of $\mathcal{O}^u(x)$ with r contained in $\partial\Theta(F)$ and the closest to $\Theta(l)$ on both sides of $\Theta(l)$. Either $\Theta(b) \subset r$ or r separates $\Theta(b)$ from $\Theta(F)$. In any case $\Theta(b)$ does not intersect $\Theta(F)$, again a contradiction. This finishes the proof of the lemma. □

The lemma shows that $L_i \cap B_j$ is empty for any i, j, and they form nested sequences of leaves in $\widetilde{M}$. Suppose that the sequence $\{L_i\}$ does not escape compact sets. Then there is L in $\widetilde{\Lambda}^s$ which is a limit of L_i (and possibly other leaves as well). Let α be an orbit in L which is not in a lift annulus. Then $\widetilde{W}^u(\alpha)$ is transverse to L in α and hence intersects L_i for i big enough. Since the L_i, B_j are nested this would force $\widetilde{W}^u(\alpha)$ to intersect B_j for j big enough, contradiction. It follows that both L_i and B_j escape compact sets as $i, j \to \infty$.

Let r be a geodesic ray in F with ideal point v. For each i, there is a subray of r contained in the component of $F - l_i$ which is in a small neighborhood of v. Hence $\Psi(r)$ has a subray which is contained in the corresponding component V_i of $\widetilde{M} - L_i$. These components V_i form a nested sequence. The ray $\Psi(r)$ can only limit in the limit set of V_i. We need the following lemma which will be a key tool throughout the proof.

Lemma 6.5. *(basic lemma) Let $\{Z_i\}$ be a sequence of leaves or line leaves or slices or any flow saturated sets in leaves of either in $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ (not all leaves Z_i need to be in the same singular foliation). If the sets Z_i escape compact sets in $\widetilde{M}$, then up to taking a subsequence $\overline{Z}_i$ converges to a point in S^2_∞.*

Proof. Let Y_i be the leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ which contains Z_i. Up to subsequence assume $Y_i \in \widetilde{\Lambda}^s$. The statement is equivalent to $diam(Z_i)$ converges to 0. Otherwise up to subsequence we can assume $diam(Z_i) > a_0$ for some a_0 and all i and hence no subsequence can converge to a single point in S^2_∞. Then there is p_i in Z_i with visual distance from p_i to $(Y_i)_+$ is bigger than $a_0/2$. Notice that $(Y_i)_+$

is a point in $\overline{Z}_i$. Let γ_i the orbit of $\widetilde{\Phi}$ through p_i. If $(\gamma_i)_-$ is very close to $(\gamma_i)_+ = (Y_i)_+$ then the geodesic with these ideal points has very small visual diameter. Since γ_i is a global bounded distance from this geodesic [Gr, Gh-Ha, CDP], the same is true for γ_i contradiction to the choice of p_i. Hence the geodesic above intersects a fixed compact set in $\widetilde{M}$ and so does γ_i. This contradicts the fact that Z_i escape compact sets in $\widetilde{M}$ and finishes the proof. $\square$

We claim that the limit sets of V_i above shrink to a single point in S^2_∞. The limit sets form a weakly monotone decreasing sequence, because the L_i are nested and so are the V_i. If the limit set does not have diameter going to zero, then there are points in the limit set of L_i which are at least $2\delta_1$ apart for some fixed $\delta_1 > 0$. By the previous lemma the L_i cannot escape compact sets in $\widetilde{M}$, contradiction. Since the limit sets of V_i shrinks to a point in S^2_∞, let $\Psi(v)$ be this point. Clearly $\Psi(r)$ limits to this point and so does $\Psi(r')$ for any other geodesic ray r' in F with ideal point v.

<u>Case 2</u> $-$ Suppose that v is an ideal point of a leaf of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$.

Let l be a ray in say $\widetilde{\Lambda}^s_F$ which limits on v and r a geodesic ray on F with ideal point v. Then l is contained in L leaf of $\widetilde{\Lambda}^s$. Either $\Theta(l)$ escapes in $\Theta(L)$ or limits to a point x in $\Theta(L)$.

Consider the first case. Then in the intrinsic geometry of L, the ray l converges to the positive ideal point of L, hence in $\widetilde{M} \cup S^2_\infty$, the image $\Psi(l)$ converges to L_+. In the other option let $\beta = x \times \mathbf{R}$, an orbit of $\widetilde{\Phi}$. As l escapes in F then in L it either escapes up or down. If it escapes down then it converges to the negative ideal point of β in $L \cup \partial_\infty L$ and hence $\Psi(l)$ converges to β_-. Otherwise l escapes up in L as $\Theta(l)$ approaches x. In this case β is in the boundary of a lift annulus and l converges to the positive ideal point in $L \cup \partial_\infty L$ and so $\Psi(l)$ converges to L_+ again. Let $\Psi(v)$ be the limit point in any case.

Every point in r it is $2\delta_0$ close to a point in l in F, hence the limit of $\Psi(r)$ in $\widetilde{M} \cup S^2_\infty$ is the same as that of l. If l' is another ray of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$ converging to p, then it will have points boundedly close to r which escape in l' and therefore $\Psi(l')$ has the same ideal point in S^2_∞. Therefore $\Psi(v)$ is well defined.

This finishes the construction of the extension of Ψ to $\partial_\infty F$.

<u>Proof of continuity of the extension</u> $-$

<u>Case 1</u> $-$ v is not an ideal point of a ray in $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$.

Let r be a geodesic ray in F with ideal point v. Recall the extension construction. There are l_i in $\widetilde{\Lambda}^s_F$ shrinking to v in $F \cup \partial_\infty F$ and similarly b_i in $\widetilde{\Lambda}^u_F$, assumed to be nested with the l_i. Let $\{l^*_i\}$ define a neighborhood basis of v in $F \cup \partial_\infty F$. Let L_i in $\widetilde{\Lambda}^s$ with $l_i \subset L_i$, and $b_i \subset B_i \in \widetilde{\Lambda}^u$ as in the construction case 1. Then as seen in the construction, the L_i, B_i escape in $\widetilde{M}$. Let U_i be the component of $F - l_i$ containing a subray of r and V_i the component of $\widetilde{M} - L_i$ containing U_i. Notice that $\Psi(U_i) \subset V_i$. Let now z in $\overline{U}_i$ with the closure taken in $F \cup \partial_\infty F$ and $\overline{V}_i$ the closure of V_i in $\widetilde{M} \cup S^2_\infty$. Then $\overline{U}_i$ is a neighborhood of v in $F \cup \partial_\infty F$. If z is in $\Psi(\overline{U}_i)$ then using either of the constructions in the extension part shows that z is a limit of points in $\Psi(U_i) \subset V_i$. As seen in the construction arguments the diameter of $\overline{V}_i$ in the visual distance is converging to 0. Hence we obtain continuity of Ψ at v. This finishes the proof in this case.

<u>Case 2</u> $-$ v is an ideal point of a ray of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$.

This case is considerably more complicated, with several possibilities.

<u>Case 2.1</u> $-$ v is an ideal point of $\widetilde{\Lambda}^s_F$ but not of $\widetilde{\Lambda}^u_F$ (or vice versa).

Suppose the first option occurs. There is l ray in $\widetilde{\Lambda}^s_F$ with ideal point v. We may assume that l is not in a leaf of $\widetilde{\Lambda}^s_F$ with same ideal points. Otherwise we can choose l to be one of the boundary leaves of the corresponding spike region. Since v is not an ideal point of $\widetilde{\Lambda}^s_F$, there are g_i line leaves in $\widetilde{\Lambda}^u_F$ defining a basis neighborhood system at v. Let g_i be contained in G_i leaves of $\widetilde{\Lambda}^u$. Let L in $\widetilde{\Lambda}^s$ containing l. If G_i escapes in $\widetilde{M}$ as $i \to \infty$, then as seen in case 1, we are done. Let then G_i converge to the finite set of leaves

$$\mathcal{V} \ = \ H_1 \cup H_2 .... \cup H_m \quad \text{leaves of} \ \ \widetilde{\Lambda}^u$$

We can assume that $G_i \cap l$ is not empty for all i.

<u>Case 2.1.1</u> – Suppose that L intersects $\mathcal{V}$, say $L \cap H_1 \neq \emptyset$.

Then l escapes down as $\Theta(l)$ approaches $\Theta(L \cap H_1)$. Otherwise $L \cap H_1$ is in the boundary of a lift annulus A and l has a subray contained in this lift annulus. But then A is also contained in the unstable leaf $\widetilde{W}^u(L \cap H_1)$ and so G_i cannot intersect l, contradiction. As l escapes down in L, then the ideal point of $\Psi(l)$ is $(L \cap H_1)_-$ which is equal to $(H_1)_-$, the negative ideal point of H_1.

Since the values of $\Psi(p)$ for p in $\partial_\infty F$ are obtained as limits of values in $\Psi(F)$, then we only need to show that if z_k is in F and z_k converges to p as $k \to \infty$, then $\Psi(z_k)$ converges to $\Psi(p)$. Suppose this is not the case.

By taking a subray if necessary, we may assume that l does not intersect a lift annulus and hence it is transverse to the unstable foliation $\widetilde{\Lambda}^u_F$ in F. Parametrize the leaves of $\widetilde{\Lambda}^u_F$ intersected by l as $\{g_t, \ t \in \mathbf{R}_+\}$, contained in $G_t \in \widetilde{\Lambda}^u$ (by an abuse of notation think of the G_i as a discrete subcollection of the $G_t, t \in \mathbf{R}_+$). Let

$$\mathcal{U} \ = \ \bigcup_{t>0} G_t$$

No g_t (or leaf of $\widetilde{\Lambda}^u_F$) has ideal point v in $\partial_\infty F$. This implies that g_t escapes compact sets in F as $t \to \infty$ and the ideal points of g_t converge to v on either side of v. If ideal points do not converge to v then since ideal points of leaves of $\widetilde{\Lambda}^u_F$ are dense in $\partial_\infty F$, there will be leaf g in the limit of the g_t. Then since $\pi_1(M)$ is negatively curved there can only be finitely many leaves in the limit and consecutive leaves share an ideal point, because of the denseness again. It would then follow that some limit leaf has to have ideal point v, contradiction.

Up to subsequence assume that all of the elements of the sequence $\{z_k\}$ are either entirely contained in $\mathcal{U}$ or disjoint from $\mathcal{U}$.

<u>Situation 1</u> – Suppose that z_k is not in $\mathcal{U}$ for any k.

Since z_k is very close to p in $F \cup \partial_\infty F$ and g_t converges to v in $F \cup \partial_\infty F$ when $t \to \infty$, then there are t, s with z_k between g_t and g_s (in F). Notice z_k is not in any of them. Now there is a unique time t_k so that exactly at that time $\Psi(z_k)$ switches from being in one side of G_t in $\widetilde{M}$ to the other (equivalently compare the z and g_t in F). In particular, either there is a line leaf L_{t_k} of G_{t_k} which separates $\Psi(z_k)$ from all the other G_t, see fig. 13, a, or there is a leaf L_{t_k} non separated from G_{t_k} with $\Psi(z_k)$ either in L_{t_k} or L_{t_k} separates $\Psi(z_k)$ from all G_t, see fig. 13, b. This can be seen in the leaf space of $\widetilde{\Lambda}^u$, which is a non Hausdorff tree [Fe10, Ga-Ka, Ro-St].

<u>Claim</u> – In the Gromov-Hausdorff topology of closed sets of $\widetilde{M} \cup S^2_\infty$, the sets $\overline{L}_{t_k}$ converge to $(H_1)_-$ as $k \to \infty$.

If L_{t_k} is a line leaf of G_{t_k}, then $(L_{t_k})_- = (G_{t_k})_-$. If L_{t_k} is not separated from G_{t_k} then also $(L_{t_k})_- = (G_{t_k})_-$. This is because there are E_i leaves of $\widetilde{\Lambda}^u$ with E_i converging to $L_{t_k} \cup G_{t_k}$. So there are $x_i, \ y_i$ in E_i with $x_i \to x, \ y_i \to y$ and $x \in L_{t_k}, \ y \in G_{t_k}$. Then

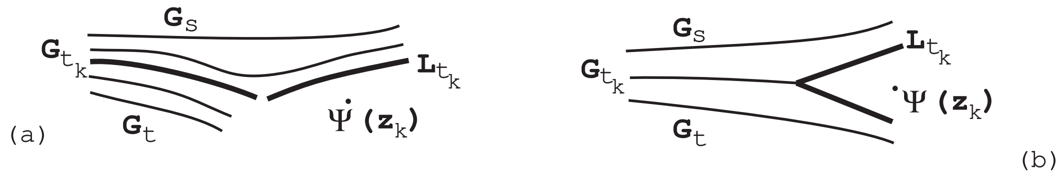

Figure 13: *a. Line leaf separating points, b. Non separated leaf separating points.*

$$\eta_-(x_i) \to \eta_-(x) = \eta_-(L_{t_k}), \quad \eta_-(y_i) \to \eta_-(y) = \eta_-(G_{t_k}) \quad \text{and} \quad \eta_-(x_i) = \eta_-(y_i).$$

The last equality occurs because x_i, y_i are in the same unstable leaf E_i. Therefore $(L_{t_k})_-$ converges to $(H_1)_-$ when $k \to \infty$. Suppose that $\overline{L}_{t_k}$ does not converge to $(H_1)_-$ in $\widetilde{M} \cup S^2_\infty$. Since

$$(L_{t_k})_- \quad \text{converges to} \quad (H_1)_-,$$

then lemma 6.5 shows that L_{t_k} does not escape compact sets in $\widetilde{M}$. Up to subsequence there are u_k in L_{t_k} with u_k converging to u in $\widetilde{M}$. The first possibility is that the L_{t_k} are subsets of the leaves G_{t_k}. This implies that $\widetilde{\Phi}_{\mathbf{R}}(u)$ is in the limit of the sequence of leaves G_{t_k} (in $\widetilde{M}$), so it is contained in $\mathcal{V}$. The second possibility is L_{t_k} non separated from G_{t_k} so L_{t_k} is between $G_{t_{k-1}}$ and $G_{t_{k+1}}$ hence u is again in the limit of the G_t so u is in $\mathcal{V}$. The leaves H_j in $\mathcal{V}$ are non singular in the side the G_t are limiting on, so there is a neighborhood of u on that side of H_j which has no singularities hence the u_k will be in $\mathcal{U}$ for k big enough. This contradicts the hypothesis in this case.

This shows that $\overline{L}_{t_k}$ converges to $(H_1)_-$ in $\widetilde{M} \cup S^2_\infty$. Also L_{t_k} either contains $\Psi(z_k)$ or separates it from a base point in $\widetilde{M}$. It follows that $\Psi(z_k)$ converges to $(H_1)_-$, which is what we wanted to prove. This finishes the analysis in situation 1.

<u>Situation 2</u> – For all k assume that $\Psi(z_k)$ is in $\mathcal{U}$.

Let t_k with $\Psi(z_k)$ in G_{t_k}, hence z_k is in $G_{t_k} \cap F = g_{t_k}$. Then $(\Psi(z_k))_- = (G_{t_k})_-$ converges to $(H_1)_-$. Assume up to taking a subsequence that $\Psi(z_k)$ converges to q different from $(H_1)_-$. As above, up to subsequence assume $\widetilde{\Phi}_{\mathbf{R}}(\Psi(z_k))$ converges to $\widetilde{\Phi}_{\mathbf{R}}(z)$. Since $\Psi(z_k)$ is in G_{t_k} then z is in $\mathcal{V}$, say z is in H_j. Let $p = \Theta(z)$. At this point notice that F does not intersect any leaf H_i in $\mathcal{V}$. If it did, say in w then F intersects the nearby leaves G_t (for any t big enough) near w. This would imply $F \cap G_t = g_t$ does not escape compact sets in F, contradiction. Therefore $\Theta(p)$ is in $\partial\Theta(F)$. Let x_k in $g_{t_k} \cap l$. Then $\Theta(x_k)$ converges to a point in $\Theta(H_1 \cap L)$. There are segments b_k in $F \cap G_{t_k} = g_{t_k}$ from x_k to z_k. Then $\Theta(b_k)$ converges to a ray in $\Theta(H_1)$ and a ray in $\Theta(H_j) \subset \mathcal{O}^u(p)$ and possibly other unstable leaves. Then there is a ray in $\mathcal{O}^u(p)$ contained in $\partial\Theta(F)$. This implies that F escapes down as $\Theta(F)$ approaches this ray of $\Theta(H_j)$. Hence $\Psi(z_k)$ is getting closer to z_- which is $(H_j)_-$, which is also equal to $(H_1)_-$. This is what we wanted to prove anyway.

This finishes the proof of case 2.1.1, that is, when L intersects $\mathcal{V}$.

Lemma 6.6. *Let A in $\widetilde{\Lambda}^u$, B in $\widetilde{\Lambda}^s$ satisfying: there are R_i leaves of $\widetilde{\Lambda}^u$ intersecting B with R_i converging to A and $R_i \cap B$ escaping compact sets in B. Then A_- is equal to B_+.*

Proof. Since R_i converges to A then $(R_i)_-$ converges to A_-. Also R_i intersects B so $(R_i)_- = (R_i \cap B)_-$. As $R_i \cap B$ escapes compact sets in B then in the intrinsic geometry of B, the $R_i \cap B$ converges to the positive ideal point of B. This implies that $(R_i \cap B)_-$ converges to B_+. This implies the result. ∎

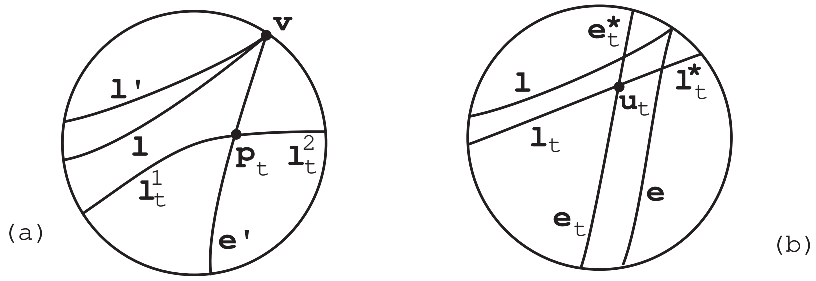

Figure 14: *a. Convergence on one side, b. Case 2.2.1.2 - intersection of leaves.*

Case 2.1.2 $-$ L does not intersect $\mathcal{V}$.

Then $\Theta(l)$ escapes in $\Theta(L)$ and so $\Psi(l)$ converges to L_+. By the previous lemma, this is also equal to $(H_1)_-$. From this point on, the proof is the same as in case 2.1.1. This finishes the proof of case 2.1.

Case 2.2 $-$ v is an ideal point of both $\widetilde{\Lambda}^s_F$ and $\widetilde{\Lambda}^u_F$.

Case 2.2.1 $-$ For any ray l of $\widetilde{\Lambda}^s_F$ and e of $\widetilde{\Lambda}^u_F$ with $l_\infty = e_\infty = v$, then l does not intersect e.

Let l', e' be rays as above. We may assume that l', e' do not have any singularities. Parametrize the leaves of $\widetilde{\Lambda}^s_F$ intersecting e' as $\{l_t, t \geq 0\}$ where $l_t \cap e'$ converges to v in $F \cup \partial_\infty F$ as t converges to infinity.

Since l' limits on v and is disjoint from e', then l' is on a side defined by e'. We will prove continuity of Ψ at v from the other side of e'. The point $p_t = l_t \cap e'$ disconnects l_t. For simplicity we only consider those l_t with $l_t \subset L_t \in \widetilde{\Lambda}^s$ and L_t non singular. Let l^1_t be the component of $(l_t - p_t)$ in the e' side union with p_t. Let l^2_t be the other component of $(l_t - p_t)$ union with p_t, see fig. 14, a.

The l^1_t are rays (here we use L_t non singular - but this is just a technicality) and $(l^1_t)_\infty$ are not equal v by hypothesis. They cannot escape compact sets of F since l' with ideal point v is on that side of e'. Hence as t converges to infinity l^1_t converges to a leaf l of $\widetilde{\Lambda}^s_F$ with a ray (also denoted by l) with ideal point v and maybe some other leaves as well. The leaf l either shares a subray with l' or separates l' from e. Let $e' \subset E$ leaf of $\widetilde{\Lambda}^u$ and $l \subset L$, leaf of $\widetilde{\Lambda}^s$.

Case 2.2.1.1 $-$ l^2_t escapes in F as $t \to \infty$.

Let b_t be the ideal point of l^2_t. Then $b_t \neq v$. Let L^2_t be the union of $\widetilde{\Phi}_{\mathbf{R}}(p_t)$ and the component of $L_t - \widetilde{\Phi}_{\mathbf{R}}(p_t)$ containing l^2_t. If L^2_t escapes in $\widetilde{M}$, then the arguments in case 1 show continuity of Ψ at v in the side of e' not containing l'.

Now assume that L^2_t converges to $R_1 \cup ... \cup R_m$ leaves of $\widetilde{\Lambda}^s$ with union $\mathcal{R}$. Notice F may intersect some of these leaves or not. If $\Theta(\Psi(p_t))$ does not escape in $\Theta(E')$, then one of the R_i, call it R_1, is a leaf intersecting E'. As seen in the arguments for case 2.1.1, F escapes up in this direction so $\Psi(p_t)$ converges to $(R_1)_+$. If $\Theta(\Psi(p_t))$ escapes in $\Theta(E')$, then lemma 6.6 shows that $\Psi(p_t)$ also converges to $(R_1)_+$. This is equal to $(R_j)_+$ for any j.

Suppose there are $t_k \to \infty$ and z_k in $l^2_{t_k}$ with $\Psi(z_k)$ not converging to $(R_1)_+$. Here there is no need to assume that L_{t_k} is non singular. Up to subsequence assume $\Psi(z_k)$ converges to another point q of $\widetilde{M} \cup S^2_\infty$. Then up to subsquence $\widetilde{\Phi}_{\mathbf{R}}(z_k)$ converges to $\widetilde{\Phi}_{\mathbf{R}}(z)$ and hence z is in $\mathcal{R}$, say in R_i. Then $\widetilde{\Phi}_{\mathbf{R}}(z_k)$ are near $\widetilde{\Phi}_{\mathbf{R}}(z)$ and since a ray of $\Theta(R_i)$ is in $\partial\Theta(F)$, then this is stable boundary. So F escapes up as $\Theta(F)$ approaches $\Theta(z)$ and hence $\Psi(z_k)$ converges to $(R_i)_+$. This is equal to $(R_1)_+$. The arguments of Case 2.1.1, situation 1 then show continuity of Ψ at v on this side of e'. This finishes the analysis of case 2.2.1.1.

Case 2.2.1.2 – The l_t^2 limit to r in F as $t \to \infty$.

Choose the leaf r with a ray which has ideal point v. Then the leaves r, l are not separated from each other in the leaf space of $\widetilde{\Lambda}^s_F$. Proposition 5.6 shows that the region bounded by these rays of r, l with ideal point v projects in M to a set asymptotic to a Reeb annulus. It follows that in F this region is a bounded distance from a geodesic ray with ideal point v. Now we restart the process with the ray r of $\widetilde{\Lambda}^s_F$ instead of e' of $\widetilde{\Lambda}^u_F$. Let $\{b_t, t \geq 0\}$ be a parametrization of the leaves of $\widetilde{\Lambda}^u_F$ through the corresponding points x_t of r. If the components of $(b_t - x_t)$ on the side opposite of e' escapes compact sets in F, then the analysis of case 2.2.1.1 shows continuity of Ψ at v in that side of r. Since r and e' are a bounded distance from each other in F, this shows continuity of Ψ at v on that side of e'.

Otherwise this process keeps being repeated. Let $A_0 = L$, A_1 be the leaf of $\widetilde{\Lambda}^s$ containing r. If the process above does not stop, we keep producing A_i in $\widetilde{\Lambda}^s$, so that they all disjoint and A_i is non separated from A_{i+1}. By theorem 2.6 up to covering translations there are only finitely many leaves of $\widetilde{\Lambda}^s$ which are not separated from some other leaf of $\widetilde{\Lambda}^s$. There is then $m > n$ and h covering translation with $h(A_n) = A_m$. Let f be the generator of the joint stabilizer of A_0, A_1. This is non trivial by theorem 2.6. Then f preserves all the prongs of A_1 and therefore leaves invariant all the A_i. Hence $h^{-1}fh(A_n) = A_n$ and so $h^{-1}fh = f^a$ for some integer a. This implies there is a $\mathbf{Z} \oplus \mathbf{Z}$ in $\pi_1(M)$, see detailed arguments in [Fe10]. This is a contradiction.

There is then a last leaf l_y (of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$) obtained from this process. The arguments of case 2.2.1.1 show continuity of Ψ at v on the other side of l_y. The region between e' and l_y is composed of a finite union of regions between non separated rays of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$. They are all a bounded distance from a geodesic ray with ideal point v, so the whole region also satisfies this property. It follows that this region can only limit in $\Psi(v)$ as well and this proves continuity of Ψ at v in that side of e'.

An entirely similar analysis shows continuity of Ψ at v from the side of l' not containing e'.

What remains to be analysed is the region of F between the rays l' and e'. Whenever there is non Hausdorfness involved, this region is a bounded distance (the bound is not uniform) from a geodesic rays with ideal point v. This is not the case a priori if there is no non Hausdorfness involved. In this case the region between l' and e' may not have bounded thickness in F and hence it is unclear whether its image under Ψ can only limit in $\Psi(v)$. We analyse this case now.

In this last case parametrize the leaves of $\widetilde{\Lambda}^u_F$ intersecting the ray l of $\widetilde{\Lambda}^s_F$ as $\{e_t \mid t \geq 0\}$. Since l_t converges to l, then for big enough t, the leaves l_t, e_t intersect – let u_t be their intersection point, see fig. 14, b. Now define l_t^* to be the component of $l_t - u_t$ intersecting e and e_t^* the component of $e_t - u_t$ intersecting l. Since e' is on that side of l, the e_t cannot escape and converge to a leaf e of $\widetilde{\Lambda}^u_F$ with an ideal point v. Let $e \subset E$ leaf of $\widetilde{\Lambda}^u$.

Recall that L_t is the leaf of $\widetilde{\Lambda}^s$ containing l_t^* and similarly let E_t be the leaf of $\widetilde{\Lambda}^u$ containing e_t. Let L_t^* be the component of $L_t - \widetilde{\Phi}_{\mathbf{R}}(u_t)$ containing l_t^* and similarly define E_t^*. In this remaining case the l_t^* escape in F and so do the e_t^*. Hence $\mu_t \ = \ l_t^* \cup \{u_t\} \cup e_t^*$ defines a shrinking neighborhood system of v in $F \cup \partial_\infty F$. Consider the set

$$B_t \ = \ L_t^* \cup \widetilde{\Phi}_{\mathbf{R}}(u_t) \cup E_t^*$$

We want to show that $\overline{B}_t$ converges to L_+ in the topology of closed sets of $\widetilde{M} \cup S^2_\infty$.

First consider $L_t^* \cap E$ which intersects F in $(l_t^* \cap e)$. If $L_t^* \cap E$ does not escape compact sets in E then it limits to an orbit γ contained in a leaf H of $\widetilde{\Lambda}^s$. Then L, H are not separated from each other. But for t big enough then E_t is near enough E and will intersect H as well. This contradicts $E_t \cap L$ is not empty and L, H non separated. Hence $L_t^* \cap E$ escapes in E and similarly $E_t^* \cap L$ escapes in L. Hence L, E form a perfect fit. This implies that $L_+ = E_-$. Also $\Psi(e)$ limits to E_- and $\Psi(l)$

limits to $L_+ = E_-$.

The set $\overline{L}_t^*$ contains $(L_t^* \cap E)_+$ and this converges to E_- when $t \to \infty$. This is because $(L_t^* \cap E)$ escapes in E. If $\overline{L}_t^*$ does not converge to E_- in $\widetilde{M} \cup S^2_\infty$, then we find $t_k \to \infty$ and $x_k \in L^*_{t_k}$ with x_k converging to x not equal to E_-. Since $(x_k)_+ = (L_{t_k})_+$ converges to E_-, then up to subsequence assume $\widetilde{\Phi}_{\mathbf{R}}(x_k)$ converges to $\widetilde{\Phi}_{\mathbf{R}}(z)$ for some z in $\widetilde{M}$. Then z is in a leaf H of $\widetilde{\Lambda}^s$ which is non separated from L.

The leaf H does not intersect F, because l_t^* escapes in F by hypothesis in this final situation. It follows that $\Theta(H)$ has a ray contained in $\partial\Theta(F)$ and so this is stable boundary of $\Theta(F)$. Hence F escapes up as $\Theta(F)$ approaches $\Theta(H)$ and consequently $\Psi(x_k)$ limits to $H_+ = L_+ - = E_-$ – which is what we wanted anyway. This shows that $\overline{L}_t^*$ converges to E_- in $\widetilde{M} \cup S^2_\infty$.

Analysing the sets E_t^* in the same manner we obtain that $\overline{E}_t^*$ converges to L_+ as $t \to \infty$ as well. This implies that $\overline{B}_t$ converges to $L_+ = \Psi(v)$. Since $B_t \cap F = \mu_t$ and the μ_t define a neighborhood basis of v in $F \cup \partial_\infty F$, this shows continuity of Ψ at v. This finishes the proof of case 2.2.1.2 and hence of case 2.2.1.

<u>Case 2.2.2</u> – There are rays l of $\widetilde{\Lambda}^s_F$ and e of $\widetilde{\Lambda}^u_F$ starting at u_0 and having the ideal point v.

We will first prove continuity on the side of e not containing a subray of l. There will be an iteration of steps. Before we start the analysis we want to get rid of some problems as described now. Suppose that there are α_0, β_0 leaves of $\widetilde{\Lambda}^s_F$ (or leaves of $\widetilde{\Lambda}^u_F$) which have non separated rays converging to v in $\partial_\infty F$ and on that side of e. Suppose there are infinitely many of these on that side of e. Let them be α_i, β_i and G_i in $\widetilde{\Lambda}^s$ containing α_i. Each region B between α_0 and any α_i is a bounded distance from a geodesic ray in F with ideal point v. The image $\Psi(B)$ then can only limit in $\Psi(v)$. If the G_i do not escape in $\widetilde{M}$ then they converge to a leaf G of $\widetilde{\Lambda}^s$. Let A be an unstable leaf intersecting G tranversely. For i big enough then A intersects G_i transversely, which is impossible, as it would intersect α_i and β_i and these are not separated. Hence the the G_i escapes in $\widetilde{M}$. Then as seen in case 1, there is continuity of Ψ at v in that side of α_1.

Another situation is when there are leaves α_i in that side of e with two rays with ideal point v. Then they are in the interior of a spike region B with one boundary g with ideal point v. If there are infinitely many of these, where none of the α_i are nested with each other, then let G_i in $\widetilde{\Lambda}^s$ containing α_i. As in the previous paragraph, the G_i have to escape in $\widetilde{M}$ and we have continuity in that side of α_1.

Therefore we can assume there are only finitely many occurrences of spike regions or non separated leaves with ideal point on this side of e. If there is any of these let e_0 be the last ray in that side coming from such occurrences. Otherwise let e_0 be the ray given e by the hypothesis in this case. For simplicity assume that e_0 is a ray in $\widetilde{\Lambda}^u_F$, the other case being similar. Let $e_0 \subset E_0 \in \widetilde{\Lambda}^u$.

Parametrize the ray of e_0 as $\{p_t \mid t \geq 0\}$ with p_t converging to v as $t \to \infty$. Let l_t be the leaf of $\widetilde{\Lambda}^s_F$ through p_t and L_t in $\widetilde{\Lambda}^s$ with $l_t \subset L_t$. If L_t escapes $\widetilde{M}$ as $t \to \infty$ then as seen before we have continuity of Ψ at v in that side of e_0. So now suppose that L_t converges to $A_1 \cup ....A_m$, leaves of $\widetilde{\Lambda}^s$. This case is considerably more involved, with several possibilities.

<u>Claim</u> – $\Psi(e_0)$ converges to $(A_i)_+$ (notice the $(A_i)_+, 1 \leq i \leq m$ are all equal).

If E_0 intersects some A_i, say A_1, then as seen in case 2.1.1, F escapes positively along $\Psi(e_0)$ as $\Theta(F)$ approaches A_1. This implies that $\Psi(e_0)$ converges to $(E_0 \cap A_1)_+ = (A_1)_+$. If E_0 does not intersect any A_i then $\Psi(e_0)$ converges to $(E_0)_- = (A_1)_+$. This proves the claim.

Let l_t^1 be the component of $(l_t - p_t)$ in the side of e_0 we are considering. We are really interested in the behavior for $t \to \infty$, so we may assume p_t is not singular and there is only one such component.

Suppose first that no l_t^1 has a ray with ideal point v and that l_t^1 escapes in F as $t \to \infty$. In this

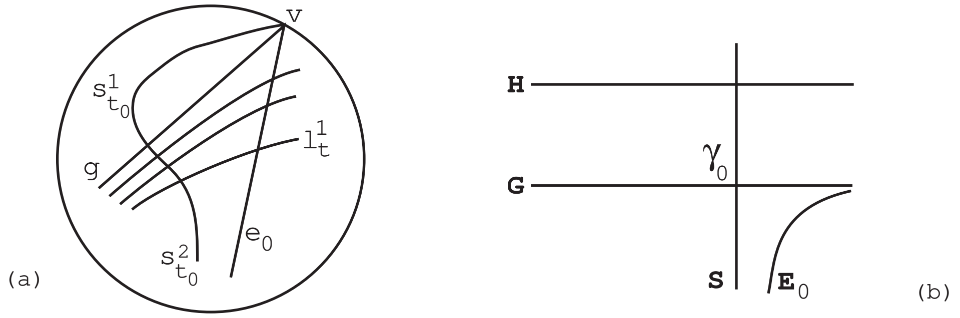

Figure 15: *Some limits in F, b. The picture in $\widetilde{M}$.*

case it is easy to show continuity of Ψ at v and in this side of e_0: Suppose there are x_i in $l^1_{t_i}$ with $t_i \to \infty$ and $\Psi(x_i) \not\to (A_i)_-$. Since $(x_i)_+$ converges to $(A_i)_+$ then up to subsequence assume that $(x_i)_- \to b \neq (A_i)_+$. Up to subsequence $\widetilde{\Phi}_{\mathbf{R}}(x_i) \to \widetilde{\Phi}_{\mathbf{R}}(x)$. Then x is in some A_i say $x \in A_2$. But F escapes positively as $\Theta(F)$ approaches $\Theta(A_2)$, so $\Psi(x_i) \to (A_i)_+$, as we wanted. Then as in case 2.1.1 this implies continuity.

There are 2 other options: 1) There is no t with l^1_t with an ideal point v and l^1_t does not escape in F; and 2) There is t with l^1_t having ideal point v. These two options interact and intercalate in appearance as explained below:

<u>Situation 1</u> – There is no t with l^1_t with ideal point v and l^1_t does not escape in F.

There could be several leaves of $\widetilde{\Lambda}^s_F$ in the limit of l^1_t as $t \to \infty$ but there is a single leaf, call it g with ideal point v. If there is more than one such leaf with ideal point v, then there would have to be one with two rays with ideal point v. This leaf would be in a spike region and it is separated from any other leaf in $\widetilde{\Lambda}^s_F$, contradiction. Let g be contained in a leaf G of $\widetilde{\Lambda}^s$.

Parametrize the ray g as $\{q_t \mid t \geq 0\}$, with $q_t \to v$ as $t \to \infty$. Let s_t be the unstable leaf of $\widetilde{\Lambda}^u_F$ through q_t. Let s^1_t be the component of $(s_t - q_t)$ on the side of g opposite to e_0 and s^2_t the other component. Then s^2_t cannot have ideal point v: for t big enough it intersects l^1_t, see fig. 15, a. Then s^2_t converges to e_0. By hypothesis there are no more occurrences of non separated leaves of $\widetilde{\Lambda}^s_F$ with ideal point v on that side of e_0, which implies that s^1_t cannot limit to a leaf of $\widetilde{\Lambda}^u_F$ at $t \to \infty$ (it would distinct but non separated from e_0). Hence the s^1_t have to escape compact sets in F. If s^1_t does not have an ideal point at v for any t, then the previous analysis shows continuity of Ψ at v in that side of g. As in case 2.2.1.2 if B is the region between g and e_0 then $\Psi(B)$ can only limit in $\Psi(v)$.

Hence assume there is some t_0 so that $s^1_{t_0}$ has ideal point v, see fig. 15, a. Then for t bigger than t_0 all ideal points of s^1_t are v. Let $s^1_{t_0}$ be contained in a leaf S of $\widetilde{\Lambda}^u$ and s_t contained in S_t leaf of $\widetilde{\Lambda}^u$. Since

$$l^1_t \;\to\; g, \quad s^2_t \;\to\; e_0 \quad \text{when} \;\; t \to \infty,$$

$$\text{then} \;\; L_t \;\to\; G, \quad S_t \;\to\; E_0, \quad \text{when} \;\; t \to \infty.$$

It follows that E_0, G form a perfect fit, see fig. 15, b. Hence $(E_0)_- = G_+$. If $\Theta(s^1_{t_0})$ is a ray in $\Theta(S)$ then $\Psi(s^1_{t_0})$ converges to S_-. But $\Theta(s^1_{t_0})$ also converges to

$$\Psi(v) \;=\; (E_0)_- \;=\; G_+ \;=\; (G \cap S)_+.$$

Let $\gamma_0 = G \cap S$, an orbit of $\widetilde{\Phi}$ in G. The above equations imply that

$$(\gamma_0)_+ \;=\; (G \cap S)_+ \;=\; \Psi(v) \;=\; S_- \;=\; (\gamma_0)_-,$$

which is a contradiction. Hence $\Theta(s^1_t)$ is not a ray and has an endpoint x_1 in $\Theta(S)$. Let $\gamma_1 = x_1 \times \mathbf{R}$. Let $H = \widetilde{\Lambda}^s(\gamma_1)$. But F does not intersect H. If F escapes down as $\Theta(F)$ approaches x_1, then $\Psi(v) = (\gamma_1)_-$. But then

$$(\gamma_0)_- \;=\; (\gamma_1)_- \;=\; \Psi(v) \;=\; (\gamma_0)_+$$

contradiction. This implies that F escapes up as $\Theta(F)$ approaches x_1. Hence $\Theta(H)$ has a ray in $\partial\Theta(F)$. Therefore $\Psi(s^1_t)$ limits to $(\gamma_1)_+$. This implies that $(\gamma_0)_+ = (\gamma_1)_+$, where γ_0, γ_1 are distinct orbits of $\widetilde{\Phi}$ in the same unstable leaf S. This is dealt with by the following theorem proved in [Fe7]:

Theorem 6.7. *([Fe7]) Let Φ be a quasigeodesic almost pseudo-Anosov flow in M^3 with $\pi_1(M)$ negatively curved. Suppose there is an unstable leaf V of $\widetilde{\Lambda}^u$ and different orbits β_0, β_1 in V with $(\beta_0)_+ = (\beta_0)_+$. Then $C_0 = \widetilde{\Lambda}^s(\beta_0)$, $C_1 = \widetilde{\Lambda}^s(\beta_1)$ are both periodic, invariant under a nontrivial covering translation f, and the periodic orbits in C_0, C_1 are connected by an even chain of lozenges all intersecting V.*

Remark – This result is case 2 of theorem 5.7 of [Fe7]. In that article the proof is done for quasigeodesic Anosov flows in M^3 with $\pi_1(M)$ negatively curved. The proof goes verbatin to the case of pseudo-Anosov flows. The singularities make no difference. By the blow up operation, the same holds for almost pseudo-Anosov flows.

The theorem implies that G, H are in the boundary of a chain of adjacent lozenges all intersecting S. The first lozenge, call it $\mathcal{C}$ has one stable side contained in G and an unstable side D_1 which makes a perfect fit with G. Suppose first D_1 is in the side of S opposite to E_0, see fig. 16, a. The other unstable side of $\mathcal{C}$ is a leaf D_2 which intersects G on the other side of S. Hence G is some S_c with $c > t_0$. Then $S_c \cap F = s_c$ is a leaf of $\widetilde{\Lambda}^u_F$ and $\Psi(s_c)$ has ideal point $\Psi(v)$. Notice that $\Theta(s_c)$ (which is contained in $\Theta(F)$) escapes in $\Theta(F)$ – otherwise it would produce stable/unstable boundary in $\Theta(F)$ before it hits $\Theta(H)$ and $\Theta(F)$ could not limit on $\Theta(H)$, impossible. Hence $\Psi(s_c)$ limits to $(S_c)_-$ which is equal to $\Psi(v)$. Then

$$(S_c \cap G)_- \;=\; (S_c)_- \;=\; \Psi(v) \;=\; G_+$$

which contradicts the orbit $S_c \cap G$ being a quasigeodesic.

It follows that the perfect fits with G occurs in the E side of S, see fig. 16, b. Here $\Theta(H), \Theta(D_1)$ are contained in the boundary of $\Theta(F)$. We now look at the region B in F bounded by $s_{t_0} = S \cap F$ and $e_0 = E_0 \cap F$.

Claim 1 – The image $\Psi(B)$ can only limit in $\Psi(v)$.

The region $\Psi(B)$ is contained in the region $\mathcal{E}$ of $\widetilde{M}$ which is bounded by E, D_1 (maybe other unstable leaves non separated from D_1 as well), H and S, see fig. 16, b. Notice that F does not intersect D_1 or any leaf non separated from D_1 which is beyond D_1. Otherwise $b_0 = (D_1 \cap F)$ is contained in B and non separated from e_0, so it would have both ideal points v. Then it would be contained in the interior of a spike region and could not be non separated from another leaf – impossible. On the other hand since $\Theta(H)$ has a line leaf in the stable boundary of $\Theta(F)$, then $\Theta(D_1)$ has a line leaf in the unstable boundary of $\Theta(F)$. Hence F escapes down as $\Theta(F)$ approaches $\Theta(D_1)$.

Let z_k in B escaping in F and hence converging to v in $\partial_\infty F$. Suppose that $\Psi(z_k)$ does not converge to $\Psi(v)$. Given that z_k escapes F and the structure of the region $\mathcal{E}$, it follows that up to subsequence either $\widetilde{W}^u(z_k)$ converges to D_1 or $\widetilde{W}^s(z_k)$ converges to H. Suppose that $\widetilde{W}^s(z_k)$

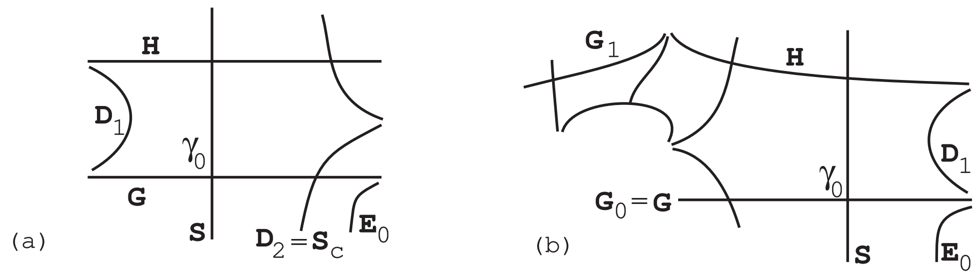

Figure 16: *Perfect fit with G in the side opposite to E_0, b. Perfect fit in the E side.*

converges to H. In that case $(z_k)_+$ converges to $H_+ = \Psi(v)$. Then as seen before if $(z_k)_-$ does not converge to $\Psi(v)$ we can assume up to subsequence $\widetilde{\Phi}_{\mathbf{R}}(z_k)$ converges to $\widetilde{\Phi}_{\mathbf{R}}(z)$. Then z is in a leaf non separated from H and since $\Psi(z_k)$ has to be in $\mathcal{E}$ then z can only be in H. As F escapes up as $\Theta(F)$ approaches $\Theta(H)$ then $\Psi(z_k)$ converges to $H_+ = \Psi(v)$. The case $\widetilde{W}^u(z_k)$ converges to D_1 leads to $\widetilde{\Phi}_{\mathbf{R}}(z_k)$ converging to $\widetilde{\Phi}_{\mathbf{R}}(z)$ with z in unstable leaf non separated from D_1. As F escapes down as $\Theta(F)$ approaches these unstable leaves, then $\Psi(z_k)$ converges to $(D_1)_- = \Psi(v)$. Since this works for any subsequence of z_k, then $\Psi(z_k)$ has to converge to $\Psi(v)$ always. This proves claim 1.

Let $G_0 = G$. Notice that G is periodic and connected to H by an even chain of lozenges. We consider the ray $s_{t_0} = S \cap F$ which has ideal point v. Parametrize it as $\{z_t \mid t \geq 0\}$. Let y_t be the leaf of $\widetilde{\Lambda}^s_F$ through z_t and y^1_t the component of $(y_t - z_t)$ in the side opposite to e_0. The ray s_{t_0} has the same behavior as the original ray e_0. Hence we obtain continuity in that side of s_{t_0} unless y^1_t converges to a leaf μ of $\widetilde{\Lambda}^s_F$ with ideal point v. Let G_1 in $\widetilde{\Lambda}^s$ with $\mu \subset G_1$. Then G_1 is non separated from H, see fig. 16, b and therefore connected to it by a chain of lozenges. It follows that G_1 is connected to G_0 by a chain of lozenges. As in the proof of claim 1, the region B_1 of F between e_0 and $(F \cap G_1)$ has image $\Psi(B_1)$ which can limit only in $\Psi(v)$.

We restart the process with $g_1 = G_1 \cap F$ instead of g. The leaves of $\widetilde{\Lambda}^u_F$ through points of g_1 already converge to the unstable leaf $(D_3 \cap F)$ of $\widetilde{\Lambda}^u_F$ (D_3 is depicted in fig. 16, b). The leaf $(D_3 \cap F)$ cannot be non separated from any other leaf of $\widetilde{\Lambda}^u_F$ in that side of $(D_3 \cap F)$. It follows that the unstable leaves intersected by g_1 escape in F. The only case to be analysed is that some of these unstable leaves have ideal point v. This brings the process exactly to the situation of some s^1_t of $\widetilde{\Lambda}^u_F$ having ideal point v as described before (it was $s^1_{t_0}$). So this would produce H_1 of $\widetilde{\Lambda}^s$ with similar properties as H. This process can now be iterated. As in claim 1 the region of F between g_i and g_{i+1} maps to $\widetilde{M}$ to a region which can only limit in $\Psi(v)$.

We show that this process has to stop. Otherwise produce G_i leaves of $\widetilde{\Lambda}^s$ which are all connected to G_0 by a chain of lozenges. The G_i are all non separated from some other leaf of $\widetilde{\Lambda}^s$, Hence there are G_i, G_j which project to the same stable leaf in M. There is a covering translation h taking G_i to G_j. If f is a generator of the isotropy group of G_0 leaving all sectors invariant, then it leaves invariant all lozenges in any chain starting in G_0 so leaves invariant all the G_i. As before this leads to $h^{-1} f h = f^n$ for some n in $\mathbf{Z}$ and to a $\mathbf{Z} \oplus \mathbf{Z}$ in $\pi_1(M)$. This is disallowed. Therefore the process finishes after say j steps and we obtain continuity of Ψ at v in that side of $g_j = G_j \cap F$. As seen above the region between s_{t_0} and g_j maps by Ψ into a region that can only limit in $\Psi(v)$. This proves continuity of Ψ at v in that side of e_0. This finishes the analysis of situation 1.

Situation 2 $-$ There is $l^1_{t_0}$ with ideal point v.

Recall the setup before the analysis of situation 1. Let $\{u_t \mid t \geq 0\}$ be the collection of unstable leaves intersected by the ray $l^1_{t_0}$. The analysis is extremely similar to the analysis of situation 1, which shows all cases produce continuity in the first step except when u_t converges to a leaf u of $\widetilde{\Lambda}^u_F$ with ideal point v. Then consider the stable leaves intersecting u. The analysis of situation 1 shows continuity unless there is stable leaf with ideal point v. From now on the analysis is exactly the same as in situation 1, with unstable replaced by stable and vice versa.

So far we proved continuity of Ψ at v from the side of e_0 opposite to l. The same works for the other side of l, producing l_0 with similar properties as e_0. We now must consider the regions between e_0 and e, between e and l and between l and l_0.

First consider the region between e and e_0, which occurs only when they are distinct. This implies that the ray e_0 is a bounded distance from a geodesic ray in F with ideal point v. Let $\{\mu_t \mid t \geq 0\}$ be a parametrization of the stable leaves of $\widetilde{\Lambda}^s_F$ through e. Let μ^1_t be the component of $(\mu^1_t - e)$ in the side of e we are considering. If some μ^1_t has ideal point v then both ideal points of μ_t are v and μ_t is inside a spike region. The same is true for e and so e is a bounded distance from a geodesic ray in F with ideal point a. Hence the region between e and e_0 is a bounded distance from a geodesic ray and we are finished in this case.

The remaining case to be analysed here is that μ^1_t has no ideal point v. Then μ^1_t does not escape F as $t \to \infty$, because e_0 is in that side of e. So μ^1_t converges to a leaf μ which has ideal point v. Now consider a parametrization $\{\nu_t \mid t \geq 0\}$ of the unstable leaves intersected by μ. Then ν_t converges to the leaf e. If it converges to some other leaf, then e is a bounded distance from a geodesic ray in F and we are done. Otherwise it must be that some ν_t has ideal point v. Therefore we exactly in the setup analysed in situation 1 above.

This shows continuity of Ψ for the region between e and e_0 and similarly for the region between l and l_0.

Finally we analyse the region B between e and l. First notice there is no singularity in the interior of B. Otherwise there would be a line leaf in B and hence a leaf with both endpoints v. It would have to be part of a spike region and the spike region does not have any singularities in its interior.

Parametrize the leaves of $\widetilde{\Lambda}^u_F$ through l as $\{e_s \mid s \geq 0\}$ and similarly those of $\widetilde{\Lambda}^s_F$ through e as $\{l_t \mid t \geq 0\}$. Let L, L_t leaves of $\widetilde{\Lambda}^s$ with $l \subset L$, $l_t \subset L_t$ and similarly define E, E_t. There are 2 possibilities:

1) Product case – Any l_t intersects every e_s and vice versa.

Equivalently $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ define a product structure in the region B bounded by l_0 and e_0. If the L_t escapes in $\widetilde{M}$ as $t \to \infty$, then there is a stable product region defined by a segment in L_0. But then theorem 2.7 implies that Φ is topologically conjugate to a suspension, contradiction. It follows that the L_t converge to $H_1 \cup ... H_m$ as $t \to \infty$. Since the l_t are stable leaves, it follows that F escapes up as $\Theta(F)$ appraches $\Theta(H_i)$. This implies that $\Psi(e)$ limits to $(H_i)_+$ which is then equal to $\Psi(v)$. Similarly E_s converges to $V_1 \cup ... V_n$ and F escapes down as $\Theta(F)$ approaches $\Theta(V_j)$. Hence $\Psi(l)$ limits to $(V_j)_- = \Psi(v)$. If some H_i intersects some V_j, then

$$(V_j \cap H_i)_+ \; = \; (H_i)_+ \; = \; \Psi(v) \; = \; (V_j)_- \; = \; (V_j \cap H_i)_-,$$

contradiction. Let now $\{z_k\}$ be a sequence in B converging to v. The product structure implies that up to subsequence we may assume that either $\widetilde{W}^s(z_k)$ converges to H_i or $\widetilde{W}^u(z_k)$ converges to V_j. This is analysed carefully in Claim 1 above, which shows that $\Psi(z_k)$ must converge to $\Psi(v)$. This shows continuity of Ψ when restricted to the region B.

2) Non product case.

There are $t, u > 0$ with $l_t \cap e_u = \emptyset$. Consider one such u. Let a be the infimum of t with $l_t \cap e_u = \emptyset$. Now let b be the infimum of u with $l_a \cap e_u = \emptyset$. Then $l_a \cap e_b = \emptyset$, but for any $0 \leq t < a$ and $0 \leq u < b$

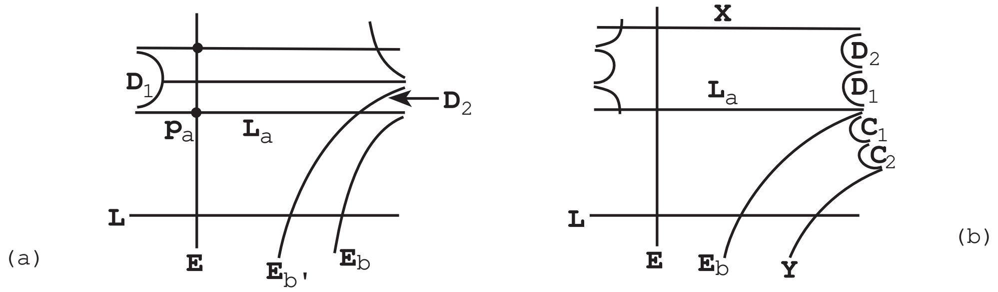

Figure 17: *a. Reaching before, b. Reaching at the exact time.*

one has $l_t \cap e_u \neq \emptyset$. Since $l_a \cap e_b = \emptyset$, then $L_a \cap E_b = \emptyset$. It follows that L_a, E_b form a perfect fit, see fig. 17, a. If $\Theta(l_a)$ does not escape in $\Theta(L_a)$, then there would be unstable boundary of $\Theta(F)$ in the limit and that would keep F from intersecting E_b, contradiction. Hence $\Theta(l_a)$ escapes in $\Theta(L_a)$ and $\Theta(e_b)$ escapes in $\Theta(E_b)$. Hence $\Psi(l_a)$ limits to $(L_a)_+$ and $\Psi(e_b)$ limits to $(E_b)_-$. Also l_a, e_b limit to v in $\partial_\infty F$.

Let $p_t = l_t \cap e$. If $\Theta(p_t)$ escapes in $\Theta(E)$, then $\Psi(e)$ converges to E_-. Notice that $\Psi(e)$ converges to $\Psi(v)$ so:

$$E_- \ = \ \Psi(v) \ = \ (L_a)_+ \ = \ (L_a \cap E)_+$$

contradiction. It follows that $\widetilde{\Phi}_{\mathbf{R}}(p_t)$ converges to $\widetilde{\Phi}_{\mathbf{R}}(x)$ with x in E. Also F has to escape up as $\Theta(F)$ approaches $\Theta(x)$ – same as in Situation 1 above. Hence $\Psi(e)$ limits to x_+. So

$$x_+ \ = \ \Psi(v) \ = \ (L_a)_+ \ = \ (p_a)_+$$

Let $X = \widetilde{W}^s(x)$. Then x, p_a are in 2 distinct orbits of E with the same positive ideal point. Therefore theorem 6.7 implies that L_a, X are connected by an even chain of lozenges all intersecting E. Let $\mathcal{C}$ be the first lozenge. It has a stable side in L_a and one unstable side, call it D_1 which makes a perfect fit with L_a. Suppose first that D_1 is in the component of $\widetilde{M} - E$ opposite to E_b. Then the other unstable side of $\mathcal{C}$, call it D_2 has to intersect L_a in the other side of E. Then D_2 must be some E_t, let it be $E_{b'}$, see fig. 17, a. Then $\Theta(e_{b'})$ has to escape in $\Theta(E_{b'})$ or else one produces stable boundary to $\Theta(F)$ and $\Theta(F)$ cannot limit to $\Theta(x)$ contradiction. Hence $\Psi(e_{b'})$ converges to $\Psi(v)$ and also to $(E_{b'})_-$. But then

$$(E_{b'} \cap L_a)_- \ = \ (E_{b'})_- \ = \ \Psi(v) \ = \ (E_b)_- \ = \ (L_a)_+$$

again a contradiction.

This implies that D_1 is on the side of E containing E_b, see fig. 17, b.

If there are only 2 lozenges in the chain from L_1 to X, then D_1 also makes a perfect fit with X. Otherwise there are $D_2, ..., D_i$ all non separated from D_1 and so that D_i makes a perfect fit with X and the D_j are all in the boundary of the chain of lozenges. As seen in claim 1 above, F cannot intersect any D_j $(1 \leq j \leq i)$, but all $\Theta(D_j)$ are contained in the unstable boundary of $\Theta(F)$. Also F escapes down as $\Theta(F)$ limits to $\Theta(D_j)$. The set $\Theta(X)$ also has a line leaf which is a stable boundary of $\Theta(F)$ and F escapes up when $\Theta(F)$ approaches $\Theta(X)$.

The same discussion applies to L, so there is y in L, $Y = \widetilde{W}^u(y)$ with $\Theta(Y)$ having a line leaf in the unstable boundary of $\Theta(F)$ and F escapes down accordingly. There are $C_1, ..., C_n$ leaves in $\widetilde{\Lambda}^u$, all non separated from each other and in the boundary of the lozenges in the chain from E_b to Y so that C_1 makes a perfect fit with E_b and C_n makes a perfect fit with Y, see fig. 17, b. Finally $\Theta(C_j)$ has a line leaf in the stable boundary of $\Theta(F)$ and F escapes up accordingly.

Let $\mathcal{E}$ be the region in $\widetilde{M}$ bounded by

$$E,\ L,\ X,\ Y,\ C_1, ..., C_n,\ D_1, ..., D_i$$

Then $\mathcal{E} \cap F$ is exactly the region B bounded by the rays e and l. Let z_k in B escaping to v. Then the region $\mathcal{E}$ shows that up to subsequence one of the following must occur:

1) $\widetilde{W}^s(z_k)$ converges to either X or C_1. The analysis of claim 1 above shows that $\Psi(z_k)$ converges to either X_+ or $(C_1)_+$ both of which are equal to $\Psi(v)$.

2) $\widetilde{W}^u(z_k)$ converges to either Y or D_1. Here $\Psi(z_k)$ converges to either Y_- or $(D_1)_-$ both of which are equal to $\Psi(v)$.

In any case this shows continuity of Ψ in the region B. This finishes the non product case.

This finishes the proof of theorem 6.3, the continuous extension theorem. □

7 Foliations and Kleinian groups

There are many similarities between foliations in hyperbolic 3-manifolds and Kleinian groups. We refer to [Mi, Can, Mar] for basic definitions concerning degenerate and non degenerate Kleinian groups, in particular singly and doubly degenerate groups.

If the foliation is **R**-covered then the limit set of any leaf in $\widetilde{M}$ is the whole sphere [Fe5]. This corresponds to doubly degenerate surface Kleinian groups [Th1, Mi, Can, Mar, Bon]. There is always a pseudo-Anosov flow which is transverse to the foliation [Fe9, Cal1]. If the flow is quasigeodesic then the results of this article imply that the foliation has the continuous extension property.

If the foliation has one sided branching, say branching down, then limit sets of leaves can only have domain of discontinuity "above" [Fe5]. Let F in $\widetilde{\mathcal{F}}$ and Λ_F its limit set. If p is not in Λ_F, the p is said to be *above* F if there is a neighborhood V of p in $\widetilde{M} \cup S^2_\infty$, so that $V \cap \widetilde{M}$ is on the positive side of F. This corresponds to simply degenerate surface Kleinian groups [Th1, Mi, Can]. There are examples of foliations with one sided branching transverse to suspension pseudo-Anosov flows provided by Meigniez [Me]. Suspension flows are always quasigeodesic flows [Ze]. The results of this article show the continuous extension property for such foliations. Under these conditions, the limit sets are locally connected, the continuous extension provides parametrizations of these limit sets.

Finally if there is branching in both directions, then there can be domain of discontinuity above and below leaves. This corresponds to non degenerate Kleinian groups [Th1, Mi, Can]. These occur for example in the case of finite depth foliations, where the depth 0 leaves are not virtual fibers [Fe1].

There are many interesting questions:

Question 1 – Given a foliation $\mathcal{F}$, is it **R**-covered if and only if for every $F \in \widetilde{\mathcal{F}}$ then the limit set Γ_F is S^2_∞?

The forward direction is true. The backwards direction is true if there is a compact leaf [Go-Sh]. In addition if there is one leaf with limit set the whole sphere then all leaves have limit set the whole sphere [Fe5] – whether $\mathcal{F}$ is **R**-covered or not.

Question 2 – Given $\mathcal{F}$ an **R**-covered foliation, is there a quasigeodesic transverse pseudo-Anosov flow?

This is true in the case of slitherings or uniform foliations as defined by Thurston [Th5]. Examples are fibrations, **R**-covered Anosov flows and many others. There is always a transverse pseudo-Anosov flow, the question is whether it is quasigeodesic.

Question 3 – Is there domain of discontinuity of Λ_F only above F if and only if $\mathcal{F}$ has one sided branching in the negative direction?

This occurs for the examples constructed by Meigniez [Me].

Question 4 – Are the pseudo-Anosov flows constructed by Calegari [Cal2] and transverse to one sided branching foliations quasigeodesic?

Question 5 – If $\mathcal{F}$ has 2 sided branching is there always domain of discontinuity above and below? Is there a quasigeodesic pseudo-Anosov flow almost transverse to $\mathcal{F}$?

References

[Ba1] T. Barbot, *Caractérization des flots d'Anosov pour les feuilletages*, Erg. Th. Dyn. Sys. **15** (1995) 247-270.

[Ba2] T. Barbot, *Actions de groups sur les* 1*-variétés non séparées et feuilletages de codimension un*, Ann. Fac. Sci. Toulose Math. **7** (1998) 559-597.

[Be-Me] M. Bestvina and G. Mess, *The boundary of negatively curved groups*, Jour. Amer. Math. Soc. **4** (1991) 469-481.

[Bl-Ca] S. Bleiler and A. Casson, *Automorphims of surfaces after Nielsen and Thurston*, Cambridge Univ. Press, 1988.

[Bon] F. Bonahon, *Bouts des variétés hyperboliques de dimension 3*, Ann. of Math. **124** (1986) 71-158.

[Br1] M. Brittenham, *Essential laminations in,* The boundary of negatively curved groups*, Jour. Amer. Math. Soc.* **4** *(1991) 469-481. Seifert fibered spaces*, Topology **32** (1992) 61-85.

[Cal1] D. Calegari, *The geometry of* **R***-covered foliations*, Geom. Topol. **4** (2000) 457-515 (eletronic).

[Cal2] D. Calegari, *Foliations with one-sided branching*, Geom. Ded. **96** (2003) 1-53.

[Cal3] D. Calegari, *Promoting essential laminations*, eprint math.GT/0210148.

[Ca-Du] D. Calegari and N. Dunfield, *Laminations and groups of homeomorphisms of the circle*, Inven. Math. **152** (2003) 149-204.

[Ca] A. Candel, *Uniformization of surface laminations*, Ann. Sci. Ecole Norm. Sup. **26** (1993) 489-516.

[Can] R. Canary, *Ends of hyperbolic* 3*-manifolds*, Jour. Amer. Math. Soc. **6** (1993) 1-35.

[Ca-Th] J. Cannon and W. Thurston, *Group invariant Peano curves*, preprint, 1987.

[Ca-Co] J. Cantwell and L. Conlon, *Smoothability of proper foliations*, Ann. Inst. Fourier Grenoble **38** (1988) 403-453.

[CDP] M. Coornaert, T. Delzant and A. Papadopoulos, *Géométrie and théorie des groupes, Les groupes hyperboliques de Gromov*, Lec. Notes in Math **1441** Springer (1991).

[Di] P. Dippolito, *Codimension one foliations of closed manifolds*, Ann. of Math. **107** (1978) 403-453.

[Fe1] S. Fenley, *Asymptotic properties of depth one foliations in hyperbolic 3-manifolds*, Jour. Diff. Geom. **36** (1992) 269-313.

[Fe2] S. Fenley, *Quasi-isometric foliations*, Topology **31** (1992) 667-676.

[Fe3] S. Fenley, *Anosov flows in* 3*-manifolds*, Ann. of Math. **139** (1994) 79-115.

[Fe4] S. Fenley, *Quasigeodesic Anosov flows and homotopic properties of flow lines*, Jour. Diff. Geom. **41** (1995) 479-514.

[Fe5] S. Fenley, *Limit sets of foliations*, Topology **37** (1998) 875-894.

[Fe6] S. Fenley, *The structure of branching in Anosov flows of* 3*-manifolds*, Comm. Math. Helv. **73** (1998) 259-297.

[Fe7] S. Fenley, *Local and global properties of Anosov foliations in 3-manifolds with negatively curved fundamental group*, Trans. A.M.S. **350** (1998) 3923-3941.

[Fe8] S. Fenley, *Foliations with good geometry*, Jour. Amer. Math. Soc. **12** (1999) 619-676.

[Fe9] S. Fenley, *Foliations, topology and geometry of 3-manifolds:* **R***-covered foliations and transverse pseudo-Anosov flows*, Comm. Math. Helv. **77** (2002) 415-490.

[Fe10] S. Fenley, *Pseudo-Anosov flows and incompressible tori*, Geom. Ded. **99** (2003) 61-102.

[Fe11] S. Fenley, *Regulating flows, topology of foliations and rigidity*, to appear in Trans. A.M.S.

[Fe12] S. Fenley, in preparation.

[Fe-Mo] S. Fenley and L. Mosher, *Quasigeodesic flows in hyperbolic 3-manifolds*, Topology **222** (2001) 503-537.

[Fr] D. Fried, *Transitive Anosov flows and pseudo-Anosov maps*, Topology **22** (1983) 299-303.

[Ga1] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Diff. Geo. **18** (1983) 445–503.

[Ga2] D. Gabai, *Foliations and the topology of 3-manifolds II*, J. Diff. Geo. **26** (1987) 461–478.

[Ga3] D. Gabai, *Foliations and the topology of 3-manifolds III*, J. Diff. Geo. **26** (1987) 479–536.

[Ga-Oe] D. Gabai and U. Oertel, *Essential laminations and 3-manifolds*, Ann. of Math. **130** (1989) 41-73.

[Ga-Ka] D. Gabai and W. Kazez, *Group negative curvature for 3-manifolds with genuine laminations*, Geom. Top. **2** (1998) 65-77.

[Gh-Ha] E. Ghys and P. De la Harpe, editors, *Sur les groupes hyperboliques d'aprés Mikhael Gromov*, Progress in Math. **83**, Birkhäuser, 1990.

[Go-Sh] S. Goodman and S. Shields, A condition for the stability of R-covered foliations of 3-manifolds, Trans. Amer. Math. Soc. **352** (2000) 4051-4065.

[Gr] M. Gromov, *Hyperbolic groups*, in Essays on Group theory, Springer-Verlag, 1987, pp. 75-263.

[Han] M. Handel, *Global shadowing of pseudo-Anosov homemorphisms*, Ergod. Th. Dyn. Sys. **5** (1985) 373-377.

[Ha-Th] M. Handel and W. Thurston, *Anosov flows in new 3-manifolds*, Inven. Math. **59** (1980) 95-103.

[He] J. Hempel, *3-manifolds*, Ann. of Math. Studies 86, Princeton University Press, 1976.

[Im] H. Imanishi, *On the theorem of Denjoy-Sacsteder for codimension one foliations without holonomy*, J. Math. Kyoto Univ. **14** (1974) 607-634.

[In-Ma] T. Inaba and S. Matsumoto, *Non singular expansive flows on 3-manifolds and foliations with circle prong singularities*, Japan. Jour. Math. **16** (1990) 329-340.

[Le] G. Levitt, *Foliations and laminations in hyperbolic surfaces*, Topology **22** (1983) 119-135.

[Man] B. Mangum, *Incompressible surface and pseudo-Anosov flows*, Topol. Appl. **87** (1998) 29-51.

[Mar] A. Marden, *The geometry of finitely generated Kleinian groups*, Ann. of Math. **99** (1974) 383-462.

[Me] G. Meigniez, *Bouts d'un groupe opérant sur la droite: 2. Applications á la topologie de feuilletages*, Tôhoku Math. Jour. **43** (1991) 473-500.

[Mor] J. Morgan, *On Thurston's uniformization theorem for 3-dimensional manifolds*, in The Smith Conjecture, J. Morgan and H. Bass, eds., Academic Press, New York, 1984, 37-125.

[Mi] Y. Minsky, *On Thurston's ending lamination conjecture*, in Proc. Low-Dimensional topology, May 18-23, 1992, Inter. Press.

[Mo1] L. Mosher, *Dynamical systems and the homology norm of a 3-manifold II*, Invent. Math. **107** (1992) 243–281.

[Mo2] L. Mosher, *Examples of quasigeodesic flows on hyperbolic 3-manifolds*, in Proceedings of the Ohio State University Research Semester on Low-Dimensional topology, W. de Gruyter, 1992.

[Mo3] L. Mosher, *Laminations and flows transverse to finite depth foliations*, Part I: Branched surfaces and dynamics – available from http://newark.rutgers.edu:80/ mosher/, Part II in preparation.

[No] S. P. Novikov, *Topology of foliations*, Trans. Moscow Math. Soc. **14** (1963) 268-305.

[Pa] F. Palmeira, *Open manifolds foliated by planes*, Ann. of Math. **107** (1978) 109-131.

[Pe1] G. Perelman, *The entropy formulat for the Ricci flow and its geometric applications*, eprint math.DG/0211159.

[Pe2] G. Perelman, *Ricci flow with surgery on 3-manifolds*, eprint math.DG/0303109.

[Pe3] G. Perelman, *Finite extinction time for the solutions to the Ricci flow in certain 3-manifolds*, eprint math.DG/0307245.

[Pl1] J. Plante, *Solvable groups acting on the line*, Trans. A.M.S. **278** (1983) 401-414.

[Pl2] J. Plante, *Anosov flows transversely affine foliations and a conjecture of Verjovsky*, Jour. London Math. Soc. **23** (1981) 359-362.

[Ro1] R. Roberts, *Constructing taut foliations*, Comm. Math. Helv. **70** (1995) 516-545.

[Ro2] R. Roberts, *Taut foliations in punctured surface bundles, I*, Proc. London Math. Soc. **82** (2001) 747-768.

[Ro3] R. Roberts, *Taut foliations in punctured surface bundles, II*, Proc. London Math. Soc. **83** (2001) 443-471.

[Ro-St] R. Roberts and M. Stein, *Group actions on order trees*, Topol. Appl. **115** (2001) 175-201.

[Ros] Rosenberg, *Foliations by planes*, Topology **7** (1968) 131-138.

[Th1] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton University Lecture Notes, 1982.

[Th2] W. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. A.M.S. **6** (1982) 357-381.

[Th3] W. Thurston, *Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds that fiber over the circle*, preprint.

[Th4] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. A.M.S. 19 (1988) 417-431.

[Th5] W. Thurston, *3-manifolds, foliations and circles I*, preprint.

[Th6] W. Thurston, *3-manifolds, foliations and circles II: Transverse asymptotic geometry of foliations*, preprint.

[Th7] W. Thurston, *The universal circle for foliations and transverse laminations*, lectures in M.S.R.I., 1997.

[Ze] A. Zeghib, *Sur les feuilletages géodésiques continus des variétés hyperboliques*, Inven. Math. **114** (1993) 193-206.

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