Geometry of foliations and flows: asymptotic behavior, almost transverse pseudo-Anosov flows and circle maps

Sérgio R. Fenley *†

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Abstract

Let $\mathcal{F}$ be a Reebless, finite depth foliation in a closed 3-manifold with negatively curved fundamental group. Such foliations exist whenever the second homology is non trivial. We show that the leaves in the universal cover extend continuously to the sphere at infinity, hence the limit sets are continuous images of the circle. This follows from a more general general result, which proves the continuous extension property whenever a foliation in such 3-manifolds is almost transverse to a quasigeodesic pseudo-Anosov flow. This applies to other classes of foliations, including a large class of foliations where all leaves are dense and infinitely many examples with one sided branching. One key technical tool is a detailed understanding of asymptotic properties of almost pseudo-Anosov singular 1-dimensional foliations in the leaves of $\mathcal{F}$ lifted to the universal cover.

We also analyse general properties of such flows and prove that given a general pseudo-Anosov flow in a closed 3-manifold, then the orbit space in the universal cover is a plane which can be naturally compactified to a closed disk. The boundary is called the ideal boundary of the pseudo-Anosov flow (negatively curved fundamental group not needed for this). If the fundamental group is negatively curved and the flow is quasigeodesic then any section of the orbit map extends to a continuous map of the closed disk. The map restricted to the ideal boundary is a group invariant Peano curve. If in addition there is a foliation which is almost transverse to the quasigeodesic pseudo-Anosov flow, then the ideal map of the pseudo-Anosov flow encodes all the maps of the individual circles at infinity of leaves of the foliation. It parametrizes the limit set of every leaf of the foliation, giving a global description of all limit sets. Finally we show that all such ideal maps are finite to one and completely charaterize points which have the same image under these maps.

1 Introduction

A 2-dimensional foliation in a 3-manifold is called Reebless if it does not have a Reeb component: a foliation of the solid torus so that the boundary is a leaf and the interior is foliated by plane leaves spiralling towards the boundary. As such the boundary leaf does not inject in the fundamental group level and is compressible. Novikov [No] showed that Reebless foliations and the underlying manifolds have excellent topological properties. This result was extended by Rosenberg [Ros], Palmeira [Pa] and many others.

The goal of this article is to analyse geometric properties of foliations. Let $\mathcal{F}$ be a Reebless foliation in M^3 with negatively curved fundamental group. Reebless implies that M is irreducible [Ros]. In this article we will not make use of Perelman's fantastic results [Pe1, Pe2, Pe3], which if confirmed imply that the manifold is hyperbolic. Reebless foliations exist for instance whenever M

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is irreducible, orientable and the second homology of M is not trivial [Ga1, Ga3]. They also exist in much more generality by work of Roberts [Ro1, Ro2, Ro3], Thurston [Th5] and many others.

Let M^3 be closed, irreducible with negatively curved fundamental group. The universal cover is canonically compactified with a sphere at infinity (denoted by S^2_∞), with compactification a closed ball [Be-Me]. The covering transformations act by homeomorphisms in the compactified space. Let $\widetilde{\mathcal{F}}$ be the lifted foliation to the universal cover $\widetilde{M}$. The leaves of $\widetilde{\mathcal{F}}$ are topological planes [No] and they are properly embedded. Hence they only limit in the sphere at infinity. For hyperbolic manifolds, the relationship between objects in hyperbolic 3-space (isometric to $\widetilde{M}$) and their limit sets in the sphere at infinity is central to the theory of such manifolds [Th1, Th2, Mar]. The same is true if $\pi_1(M)$ is negatively curved. There is a metric in M so that all leaves of $\mathcal{F}$ are hyperbolic (that is constant curvature -1) [Ca] and so the universal cover of each leaf of $\mathcal{F}$ is isometric to the hyperbolic plane ($\mathbf{H}^2$). The *continuous extension question* asks whether these leaves extend continuously to the sphere at infinity, that is: given the inclusion map from a leaf F of $\widetilde{\mathcal{F}}$ to $\widetilde{M}$ is there a continuous extension to a map $F \cup \partial_\infty F$ to $\widetilde{M} \cup S^2_\infty$? Here $\partial_\infty F$ is the ideal boundary of F which is homeomorphic to a circle. In that case the restriction of the map to $\partial_\infty F$ expresses the limit set of F as the continuous image of a circle, showing it is locally connected.

In a seminal work, Cannon and Thurston [Ca-Th] proved that such is the case when $\mathcal{F}$ is a fibration over the circle. Previously Thurston had showed that the manifold is hyperbolic when the monodromy of the fibration is pseudo-Anosov [Th1, Th3, Th4]. Since the fundamental group of a leaf of $\mathcal{F}$ is a normal subgroup of the fundamental group of M, then every limit set of a leaf of $\widetilde{\mathcal{F}}$ is the whole sphere. In this way they produced many examples of group invariant Peano curves.

We now describe an extremely important class of foliations. A foliation is *proper* if the leaves never limit on themselves $-$ this is in the foliation sense and it means that a sufficiently small transversal to a given leaf only meets the leaf in a single point. In particular leaves are not dense. In a proper foliation there are compact leaves which are said to have *depth* 0. The depth of a leaf is inductively defined to be i (for finite i) if $i-1$ is the maximum of the depths of leaves in the (foliation) limit set of the leaf. A foliation has *finite depth* if it is proper and there is a finite upper bound to the depths of all leaves.

Gabai proved that whenever a compact 3-manifold M is irreducible, orientable and the second homology group $H_2(M, \partial M, \mathbf{Z})$ is not trivial, then there is a Reebless finite depth, foliation associated to each non trivial homology class [Ga1, Ga3]. The foliation is directly associated to a hierarchy of the manifold and as such is strongly connected with the topological structure of the manifold. These results had several fundametal consequences for the topology of 3-manifolds [Ga1, Ga2, Ga3].

Subsequently Gabai and Mosher showed [Mo3] that any Reebless finite depth foliation in a closed, atoroidal 3-manifold admits a pseudo-Anosov flow Φ which is almost transverse to it. Roughly a flow is *pseudo-Anosov* if it has transverse hyperbolic dynamics $-$ even though it may have finitely many singularities. It has stable and unstable two dimensional foliations which in general are singular. The term *almost transverse* means that one may need to blow up one singular orbit (or more) into a finite collection of joined annuli to make the flow transverse to the foliation. See detailed definitions and comments in section 2. Under the atoroidal condition Thurston [Th1, Th3] proved that M is in fact hyperbolic.

These pseudo-Anosov flows almost transverse to finite depth foliations in hyperbolic 3-manifolds are *quasigeodesic* [Fe-Mo]. This means that flow lines are uniformly efficient in measuring distance in relative homotopy classes, or equivalently, uniformly efficient in measuring distance in the universal cover. This was first proved by Mosher [Mo1, Mo2] for a class of flows transverse to some examples of depth one foliations obtained by handle constructions. A foliation (perhaps singular) is *quasi-isometric* if its leaves are uniformly efficient in measuring distance in the universal cover. There are no non singular 2 dimensional quasi-isometric foliations in closed 3-manifolds with negatively curved fundamental group [Fe2]. As for singular foliations the situation is quite different and there are examples. The stable/unstable singular foliations of the quasigeodesic flows above may be quasi-isometric [Fe8] and may not [Mo3, Fe8]. If both the stable and unstable foliations are quasi-isometric and the flow is actually transverse (as oppposed to being almost transverse) to the finite depth foliation then we proved [Fe8] that $\mathcal{F}$ has the continuous extension property. We also showed that some depth one foliations satisfy both of these requirements.

Our first result proves the continuous extension property for all Reebless finite depth foliations in hyperbolic 3-manifolds. There are no restrictions on the depth of the foliation, or about transversality of the flow or quasi-isometric behavior of the pseudo-Anosov foliations.

Theorem A – Let $\mathcal{F}$ be a Reebless finite depth foliation in M^3 closed hyperbolic. Then $\mathcal{F}$ has the continuous extension property. In particular the limit sets of the leaves are all locally connected.

This shows that any hyperbolic 3-manifold with non trivial second homology has such a foliation with the continuous extension property. Notice that conjecturally any closed, hyperbolic 3-manifold has a finite cover with positive first Betti number, which would imply there would always be a foliation with the continuous extension property in a finite cover.

The continuous extension property has also been proved for another class of foliations: A foliation is *uniform* if any two leaves in the universal cover are a bounded distance apart – the bound depends on the individual leaves. Thurston [Th5] proved that uniform foliations are very common. If in addition $\pi_1(M)$ is negatively curved, then Thurston [Th5] proved that there is a pseudo-Anosov flow transverse to $\mathcal{F}$. From this it is easy to prove that the flow has quasi-isometric stable/unstable foliations. In this case it also easily implies that the foliation $\mathcal{F}$ has the continuous extension property. The arguments are a very clever generalization of the fibering situation.

Theorem A above is an immediate consequence of the following much more general result:

Theorem B – Let $\mathcal{F}$ be a Reebless foliation in M^3 closed, with negatively curved fundamental group. Suppose that $\mathcal{F}$ is almost transverse to a quasigeodesic pseudo-Anosov flow. Then $\mathcal{F}$ has the continuous extension property. Therefore the limits sets of leaves of $\widetilde{\mathcal{F}}$ are locally connected.

This also implies the following:

Corollary C – There are infinitely many examples of foliations with all leaves dense which have the continuous extension property. Many of these have one sided branching. These are not uniform foliations.

Foliations with all leaves dense can be obtained for example starting with finite depth foliations and doing small perturbations – keeping it still almost transverse to the same pseudo-Anosov flow. They are constructed carefully by Gabai [Ga3], providing infinitely many examples with dense leaves to which theorem C applies. In fact whenever a foliations $\mathcal{F}$ satisfies the hypothesis of theorem B, then any $\mathcal{F}'$ sufficiently close to $\mathcal{F}$ will also be transverse to the same flow. By theorem B again, $\mathcal{F}'$ will have the continuous extension property.

A foliation is **R**-*covered* if the leaf space of $\widetilde{\mathcal{F}}$ is homeomorphic to the real numbers. Equivalently this leaf space is Hausdorff. A foliation which is not **R**-covered has *branching*, that is there are non separated points in the leaf space. This leaf space is oriented (being a simply connected, perhaps non Hausdorff 1-manifold) and there is a notion of branching in the positive or negative directions. If it branches only in one direction the foliation is said to have one sided branching. Foliations with one sided branching with all leaves dense and transverse to suspension pseudo-Anosov flows (which are quasigeodesic) were constructed by Meigniez [Me]. This provides infinitely many examples to which theorem C applies.

Theorem B can potentially be widely applicable because of the abundance of pseudo-Anosov flows almost transverse to foliations: Thurston proved this for fibrations [Th4]. It is also true for $\mathbf{R}$-covered foliations [Fe9, Cal1] and Calegari proved it for foliations with one sided branching [Cal2], minimal foliations [Cal4] and many other foliations [Cal4]. One main problem is to analyse the geometry of these pseudo-Anosov flows, in particular to decide whether they are quasigeodesic. By theorem B this would imply the continuous extension property for the corresponding foliations.

The quasigeodesic property of Φ is definitely weaker than the stable and unstable foliations being quasi-isometric, but it still has useful properties. In order to prove theorem B, one analyses the topological structure of the pseudo-Anosov flow. Let Φ_1 be the original pseudo-Anosov flow almost transverse to $\mathcal{F}$. To make the flow transverse to $\mathcal{F}$ one needs in general to blow up a collection of singular orbits into a collection of flow saturated annuli so that each boundary is a closed orbit of the new flow Φ. The blown up flow is called an *almost pseudo-Anosov flow.* If $\widetilde{\Phi}$ is the lifted flow to the universal cover $\widetilde{M}$ and $\mathcal{O}$ is its orbit space, then $\mathcal{O}$ is homeomorphic to the plane $\mathbf{R}^2$ [Fe-Mo] – this is true for pseudo-Anosov and almost pseudo-Anosov flows. When one blows up some singular orbits into a collection of joined annuli, the stable/unstable singular foliations also blow up. The two new singular foliations Λ^s, Λ^u are everywhere transverse to each other except at the singularities and the blown up annuli. The blown up annuli are part of both singular foliations. Since $\mathcal{F}$ is transverse to the blown up foliations, then the stable/unstable foliations Λ^s, Λ^u induce singular 1-dimensional foliations in leaves of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. The behavior of this is described in the following result:

Theorem D – Let $\mathcal{F}$ be a Reebless foliation in M^3 closed. Let Φ_1 be a pseudo-Anosov flow almost transverse to $\mathcal{F}$ and let Φ be a corresponding almost pseudo-Anosov flow transverse to $\mathcal{F}$. Let Λ^s, Λ^u be the stable/unstable 2-dimensional foliations of Φ and $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ the lifts to $\widetilde{M}$. Given F leaf of $\widetilde{\mathcal{F}}$, let $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ be the induced singular 1-dimensional foliations in F. Then for every ray l in a leaf of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$, it limits in a single point of $\partial_\infty F$. If the stable/unstable foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ of Φ have Hausdorff leaf space, then the leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ are uniform quasigeodesics in F, the bound is independent of the leaf. In general the leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ are not quasigeodesic but any non Hausdorffness (of say $\widetilde{\Lambda}^s_F$) is associated to a Reeb annulus in a leaf of $\mathcal{F}$ and when projected to M it either projects to or spirals to a Reeb annulus. The set of ideal points of leaves of $\widetilde{\Lambda}^s_F$ is dense in $\partial_\infty F$ and similarly for $\widetilde{\Lambda}^u_F$. Finally if two rays of the same leaf of $\widetilde{\Lambda}^s_F$ limit to the same ideal point in $\partial_\infty F$ then the leaf is not singular and the region in F bounded by the leaf projects in M to a set in a leaf of $\mathcal{F}$ which is either contained in or asymptotic to a Reeb annulus.

For this result one does not need negatively curved fundamental group or any metric properties of the flow. Theorem D is one of the main technical results used in the proof of theorem B. We stress that in all the previous results concerning the continuous extension property this was also a crucial property on which the whole analysis hinged. In these other situations, the analysis of leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ was either trivial or substantially simpler. The proof here works in complete generality. It uses the denseness of contracting directions for foliations as proved by Thurston [Th6] when he introduced the universal circle for foliations – even though we do not directly use the universal circle here. The basic idea is: if any ray does not limit in a single point then it limits in a non trivial interval of $\partial_\infty F$ and we zoom into this interval and analyse the situation in the limit. This is actually the easiest statement to prove in theorem D. The facts about rays with same ideal point and non Hausdorffness are much trickier, but they will be essential in the analysis of theorem B. The results of theorem D are also used in other contexts, for example to analyse rigidity of pseudo-Anosov flows almost transverse to a given foliation. This will be explored in a future article.

The proof of theorem B has 2 parts: given a leaf F of $\widetilde{\mathcal{F}}$, one first constructs an extension to the ideal boundary and then show it is continuous. To define the extension, one uses the foliations $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ as they hopefully define a basis neighborhood of an ideal point p of F. The best situation is that the corresponding leaves of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ define basis neighborhoods of unique points in S^2_∞, hence defining the image of p. There are several difficulties here: first the leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ are not quasigeodesics, so much more care is needed. Another problem is that the foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^s$ in general do not have Hausdorff leaf space. This keeps recurring throughout the proof. A further difficulty is that if intersections with a leaf F of $\widetilde{\mathcal{F}}$ escape, it does not mean that the corresponding stable/unstable leaves in $\widetilde{M}$ escape compact sets. Consequently there are several cases to be analysed.

Another fact that is important for the analysis of theorems B and D is the following: Let Θ be the projection map from $\widetilde{M}$ to $\mathcal{O}$. A leaf of $\widetilde{\mathcal{F}}$ intersects an orbit of $\widetilde{\Phi}$ at most once defining an injective projection of F to $\Theta(F)$. The projection $\Theta(F)$ is equal to $\mathcal{O}$ if and only if the foliation is $\mathbf{R}$-covered. An important problem here is to determine the boundary $\Theta(F)$ as a subset of $\mathcal{O}$. This turns out to be a collection of subsets of stable/unstable leaves in $\mathcal{O}$. This result is different than what happens for pseudo-Anosov flows transverse to foliations and its proof is much more delicate.

The second part of the article deals with global issues relating the overall structure of pseudo-Anosov flows with the geometry of the foliation. Given a general pseudo-Anosov flow Φ with orbit space $\mathcal{O}$ we first show there is a natural compactification of $\mathcal{O}$ with an ideal circle:

Theorem E – Let Φ be a pseudo-Anosov flow in M^3 closed, with orbit space $\mathcal{O}$ of the lifted flow $\widetilde{\Phi}$ in $\widetilde{M}$. Then there is a natural compactification of $\mathcal{O}$ with an ideal boundary $\partial\mathcal{O}$ which is homeomorphic to a circle and so that the union $\mathcal{D} = \mathcal{O} \cup \partial\mathcal{O}$ is homeomorphic to a closed disk. The fundamental group $\pi_1(M)$ acts naturally on this compactification. The same holds for almost pseudo-Anosov flows: if Φ^* is an almost pseudo-Anosov flow associated to Φ and $\mathcal{O}^*$ its orbit space, then $\mathcal{O}$ is a natural retraction of $\mathcal{O}^*$ and $\partial\mathcal{O}^*$ is naturally homeomorphic to $\partial\mathcal{O}$.

In this result we do not assume that $\pi_1(M)$ is negatively curved or metric properties of flow lines.

The singular foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in $\widetilde{M}$ induce one dimensional foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O}$ and these are used to define the ideal points of $\mathcal{O}$. An ideal point is defined to be a sequence of chains of segments and rays in $\mathcal{O}^s, \mathcal{O}^u$. The chains have to be what is called "convex" and for technical purposes we require the chains to be nested. There is an extended analysis of ideal points and the topology of $\mathcal{D}$ – the proof that $\mathcal{D}$ is homeomorphic to a closed disk is quite involved. We notice that Calegari and Dunfield [Ca-Du] previously showed that if Φ is a pseudo-Anosov flow, then $\pi_1(M)$ acts nontrivially on a circle. This is a very important results with fundamental consequences for the existence question for pseudo-Anosov flows in 3-manifolds [Ca-Du]. Their construction is very different than ours. They consider the space of ends of the leaf space of say $\widetilde{\Lambda}^s$. They show that the space of ends is circularly ordered and then maps injectively to a circle. By collapsing complementary intervals one gets an action in $\mathbf{S}^1$. This is much weaker than producing a nice, natural compactification of $\mathcal{O}$. For example, consider sequences of $\mathcal{O}$, escaping compact sets in $\mathcal{O}$ and so that all points are in the same stable leaf. As seen in the leaf space the points do not go into any end, but they should have a convergent subsequence in a compactification of $\mathcal{O}$. Here we show much more, producing an actual compactification of the orbit space as a closed disk. In addition this compactification is naturally associated with the stable and unstable foliations of the flow.

With additional metric hypothesis we obtain the following results:

Theorem F – Let Φ be a quasigeodesic pseudo-Anosov flow in M^3 closed, with negatively curved fundamentalgroup. Then for any section $\sigma : \mathcal{O} \rightarrow \widetilde{M}$ of the orbit map, there is a continuous extension to $\sigma : \mathcal{D} \rightarrow \widetilde{M} \cup S^2_\infty$. The map σ restricted to $\partial\mathcal{O}$ is independent of the section and it is a group invariant Peano curve. The same holds for almost pseudo-Anosov flows: let Φ^* an associated almost pseudo-Anosov flow, with orbit space $\mathcal{O}^*$. Then it is a quasigeodesic flow [Fe-Mo] and the map from $\partial \mathcal{O}^*$ to S^2_∞ is the same as the one defined for Φ.

This generalizes Cannon-Thurston's result for suspensions and pseudo-Anosov flows transverse to uniform foliations. Given the detailed construction of the ideal points of $\mathcal{O}$, the proof of theorem F is very short. It does use some additional properties of quasigeodesic pseudo-Anosov flows. We remark that Calegari [Cal4] has recently proved that if ξ is a quasigeodesic flow in M^3 closed, hyperbolic, then there is a group invariant Peano curve associated to it. His methods are completely different than those used in this article. Our methods in particular give information about the limit sets of leaves of foliations satisfying the hypothesis of theorem B and also on the set of identifications of ideal points of σ and foliations as described below.

If in addition to these metric properties of the flow, there is a transverse foliation to the flow we use this global map to encode information about all ideal maps of leaves. First we have a general result:

Theorem G – Let $\mathcal{F}$ be a foliation in M^3 closed. Suppose that $\mathcal{F}$ is almost transverse to a pseudo-Anosov flow Φ_1 and transverse to a corresponding almost pseudo-Anosov flow Φ. Given leaf F of $\widetilde{\mathcal{F}}$ the projection $\Theta(F)$ to the orbit space of $\widetilde{\Phi}$ has a well defined ideal boundary B_F which is the closure of $\Theta(F)$ in $\mathcal{D}$ intersected with $\partial \mathcal{O}$ – or the set of ideal points of $\Theta(F)$. There is a circularly weakly monotone map c_F from B_F to $\partial_\infty F$, which is surjective and is also injective except for identifying endpoints of complementary intervals of B_F in $\partial \mathcal{O}$. The same is true for almost pseudo-Anosov flows.

This means that the ideal circle of the pseudo-Anosov flow maps surjectively and continuously to the circle at infinity of an arbitrary leaf of $\widetilde{\mathcal{F}}$. Using this we obtain a general description of the ideal maps of leaves of foliations:

Theorem H – Let $\mathcal{F}$ be a Reebless foliation in M^3 closed, with $\pi_1(M)$ negatively curved. Suppose that $\mathcal{F}$ is almost transverse to a quasigeodesic pseudo-Anosov flow Φ_1 and transverse to the corresponding almost pseudo-Anosov flow Φ. For each leaf F of $\widetilde{\mathcal{F}}$ let $\varphi_F : \partial_\infty F \to S^2_\infty$ be the ideal map associated to the continuous extension of F as shown in theorem B. Let $\sigma : \partial \mathcal{O} \to S^2_\infty$ be the ideal map associated to Φ (or Φ_1). Then each ideal map φ_F is encoded by σ in the following way: Let $c_F : B_F \to \partial_\infty F$ be the surjective map described in theorem G. For each q in B_F and $p = c_F(q)$ in $\partial_\infty F$ then

$$\varphi_F(p) \ = \ \varphi_F \circ c_F(q) \ = \ \sigma(q)$$

In other words $\sigma|_{B_F} = \varphi_F \circ c_F$.

In this way the single ideal map σ encodes all the information of the ideal maps φ_F of individual leaves F of $\widetilde{\mathcal{F}}$ and all the limit sets of leaves of $\widetilde{\mathcal{F}}$. This is an extremely nice global picture!

The map $\sigma : \partial \mathcal{O} \to S^2_\infty$ has image the whole sphere and consequently is not injective. The topological tools developed here completely determine the identifications of this map:

Theorem I – Let Φ be a quasigeodesic pseudo-Anosov flow in M^3 closed, with $\pi_1(M)$ negatively curved. Then the ideal map $\sigma : \partial \mathcal{O} \to S^2_\infty$ of theorem F is finite to one. Suppose p, q are distinct points in $\partial \mathcal{O}$ with same image. Then p, q are ideal points of rays l, r in stable or unstable leaves in $\mathcal{O}$. In addition l, r are connected by a finite chain of leaves so that consecutive leaves are either no separated from each other or have rays which are "asymptotic".

The term asymptotic will be defined in detail and is what is called a "perfect fit" [Fe4, Fe6]. This result helps us to determine the ideal point indetifications for limit sets of foliations.

Theorem J – Let $\mathcal{F}$ be a Reebless foliation in M^3 closed, with $\pi_1(M)$ negatively curved. Suppose that $\mathcal{F}$ is transverse to a quasigeodesic almost pseudo-Anosov flow Φ. Let F be a leaf of $\widetilde{\mathcal{F}}$, φ_F the ideal map of F and σ the ideal map of Φ. Then $\varphi_F : \partial_\infty F \to S^2_\infty$ is a finite to one map. In addition if u, v in $\partial_\infty F$ with same image under φ_F, then there are p, q in $\partial \mathcal{O}$ with $c_F(p) = u, c_F(q) = v$ and $\varphi(p) = \varphi(q)$. The points u, v are ideal points of rays in leaves of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$ which are connected by a chain of perfect fits in F.

In many situations, for example if there are no freely homotopic closed orbits of Φ, or the leaf spaces of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ are Hausdorff, then there no asymptotic rays or perfect fits. In that case the only identifications of φ_F come from being ideal points in rays of the same leaf of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$ and similarly for σ. This gives much more information.

We expect similar results to theorems F, G, H, I and J to hold in the setting of the universal circle of the foliation. This will be pursued in a future article [Fe12].

The article is organized as follows: In the next section we present basic definitions and results concerning pseudo-Anosov flows and almost pseudo-Anosov flows. In section 3 we analyse the set $\Theta(F)$ for leaves in the universal cover. In sections 4 and 5 we analyse the singular foliations $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ and asymptotic properties of their leaves, proving theorem D. In section 6 we prove the continuous extension property, theorem B. In the next section we produce a compactfication of the orbit space $\mathcal{O}$ as a closed disk – theorem E. In section 8 we produce the group invariant Peano curves, theorem F. In section 9 we show the global encoding of the limit sets using the Peano curves, proving theorem H. In section 10 we analyse the identification of ideal points of leaves or the Peano map proving theorems I and J. In the final section we comment on general relationships between foliations and Kleinian groups.

2 Preliminaries: Pseudo-Anosov flows and almost pseudo-Anosov flows

Let Φ be a flow on a closed, oriented 3-manifold M. We say that Φ is a *pseudo-Anosov flow* if the following are satisfied:

- For each $x \in M$, the flow line $t \to \Phi(x,t)$ is C^1, it is not a single point, and the tangent vector bundle $D_t\Phi$ is C^0.

- There is a finite number of periodic orbits $\{\gamma_i\}$, called *singular orbits*, such that the flow is "topologically" smooth off of the singular orbits (see below).

- The flowlines are tangent to two singular transverse foliations Λ^s, Λ^u which have smooth leaves off of γ_i and intersect exactly in the flow lines of Φ. These are like Anosov foliations off of the singular orbits. This is the topologically smooth behavior described above. A leaf containing a singularity is homeomorphic to $P \times I/f$ where P is a p-prong in the plane and f is a homeomorphism from $P \times \{1\}$ to $P \times \{0\}$. In a stable leaf, f contracts towards towards the prongs and in an unstable leaf it expands away from the prongs. We restrict to p at least 2, that is, we do not allow 1-prongs.

- In a stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backwards asymptotic.

Basic references for pseudo-Anosov flows are [Mo1, Mo3].

Notation/definition: The singular foliations lifted to $\widetilde{M}$ are denoted by $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$. If $x \in M$ let $W^s(x)$ denote the leaf of Λ^s containing x. Similarly one defines $W^u(x)$ and in the universal cover $\widetilde{W}^s(x), \widetilde{W}^u(x)$. Similarly if α is an orbit of Φ define $W^s(\alpha)$, etc... Let also $\widetilde{\Phi}$ be the lifted flow to $\widetilde{M}$.

We review the results about the topology of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ that we will need. We refer to [Fe6, Fe8] for detailed definitions, explanations and proofs. The orbit space of $\widetilde{\Phi}$ in $\widetilde{M}$ is homeomorphic to the plane $\mathbf{R}^2$ [Fe-Mo] and is denoted by $\mathcal{O} \cong \widetilde{M}/\widetilde{\Phi}$. Let $\Theta : \widetilde{M} \to \mathcal{O} \cong \mathbf{R}^2$ be the projection map. If L

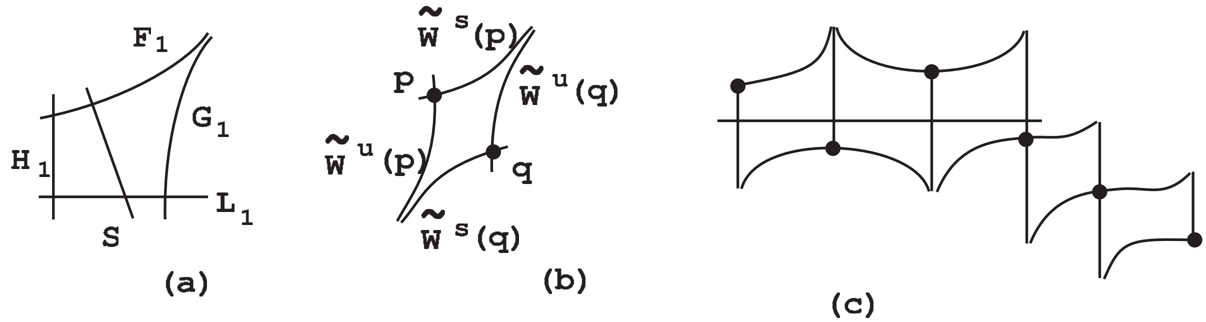

Figure 1: *a. Perfect fits in $\widetilde{M}$, b. A lozenge, c. A chain of lozenges.*

is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then $\Theta(L) \subset \mathcal{O}$ is a tree which is either homeomorphic to $\mathbf{R}$ if L is regular, or is a union of p-rays all with the same starting point if L has a singular p-prong orbit. The foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ induce 1-dimensional foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O}$. Its leaves are $\Theta(L)$ as above. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then a *sector* is a component of $\widetilde{M} - L$. Similarly for $\mathcal{O}^s, \mathcal{O}^u$. If B is any subset of $\mathcal{O}$, we denote by $B \times \mathbf{R}$ the set $\Theta^{-1}(B)$. The same notation $B \times \mathbf{R}$ will be used for any subset B of $\widetilde{M}$: it will just be the union of all flow lines through points of B.

Definition 2.1. *Let L be a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. A slice of L is $l \times \mathbf{R}$ where l is a properly embedded copy of the reals in $\Theta(L)$. For instance if L is regular then L is its only slice. If a slice is the boundary of a sector of L then it is called a line leaf of L. If a is a ray in $\Theta(L)$ then $A = a \times \mathbf{R}$ is called a half leaf of L. If ζ is an open segment in $\Theta(L)$ it defines a* flow band *L_1 of L by $L_1 = \zeta \times \mathbf{R}$. Same notation for the foliations $\mathcal{O}^s, \mathcal{O}^u$ of $\mathcal{O}$.*

If $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$ then F and G intersect in at most one orbit. Also suppose that a leaf $F \in \widetilde{\Lambda}^s$ intersects two leaves $G, H \in \widetilde{\Lambda}^u$ and so does $L \in \widetilde{\Lambda}^s$. Then F, L, G, H form a *rectangle* in $\widetilde{M}$ and there is no singularity in the interior of the rectangle [Fe8]. There will be two generalizations of rectangles: 1) perfect fits = rectangle with one corner removed and 2) lozenges = rectangle with two opposite corners removed. We will also denote by rectangles, perfect fits, lozenges and product regions the projection of these regions to $\mathcal{O} \cong \mathbf{R}^2$.

Definition 2.2. *([Fe6, Fe8]) Perfect fits - Two leaves $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$, form a perfect fit if $F \cap G = \emptyset$ and there are half leaves F_1 of F and G_1 of G and also flow bands $L_1 \subset L \in \widetilde{\Lambda}^s$ and $H_1 \subset H \in \widetilde{\Lambda}^u$, so that the set*

$$\overline{F}_1 \cup \overline{H}_1 \cup \overline{L}_1 \cup \overline{G}_1$$

separates M and forms an a rectangle R with a corner removed: The joint structure of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in R is that of a rectangle with a corner orbit removed. The removed corner corresponds to the perfect of F and G which do not intersect.

We refer to fig. 1, a for perfect fits. There is a product structure in the interior of R: there are two stable boundary sides and two unstable one. An unstable leaf intersects one stable boundary side (not in the corner) if and only if it intersects the other stable boundary side (not in the corner). We also say that the leaves F, G are *asymptotic.*

Definition 2.3. *([Fe6, Fe8]) Lozenges - A lozenge is a region of $\widetilde{M}$ whose closure is homeomorphic to a rectangle with two corners removed. More specifically two points p, q form the corners of a lozenge if there are half leaves A, B of $\widetilde{W}^s(p), \widetilde{W}^u(p)$ defined by p and C, D half leaves of $\widetilde{W}^s(q), \widetilde{W}^u(q)$ so* *that A and D form a perfect fit and so do B and C. The region bounded by the lozenge is R and it does not have any singularities. The sides are not contained in the lozenge, but are in the boundary of the lozenge. See fig. 1, b.*

There are no singularities in the lozenges, which implies that R is an open region in $\widetilde{M}$. There may be singular orbits on the sides of the lozenge and the corner orbits.

Two lozenges are *adjacent* if they share a corner and there is a stable or unstable leaf intersecting both of them, see fig. 1, c. Therefore they share a side. A *chain of lozenges* is a collection $\{\mathcal{C}_i\}, i \in I$, where I is an interval (finite or not) in $\mathbf{Z}$; so that if $i, i+1 \in I$, then $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ share a corner, see fig. 1, c. Consecutive lozenges may be adjacent or not. The chain is finite if I is finite.

Definition 2.4. *Suppose A is a flow band in a leaf of $\widetilde{\Lambda}^s$. Suppose that for each orbit γ of $\widetilde{\Phi}$ in A there is a half leaf B_γ of $\widetilde{W}^u(\gamma)$ defined by γ so that: for any two orbits γ, β in A then a stable leaf intersects B_β if and only if it intersects B_γ. This defines a stable product region which is the union of the B_γ. Similarly define unstable product regions.*

The main property of product regions is the following: for any $F \in \widetilde{\Lambda}^s$, $G \in \widetilde{\Lambda}^u$ so that (i) $F \cap A \neq \emptyset$ and (ii) $G \cap A \neq \emptyset$, then $F \cap G \neq \emptyset$. There are no singular orbits of $\widetilde{\Phi}$ in A.

We abuse convention and call a leaf L of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ is called *periodic* if there is a non trivial covering translation g of $\widetilde{M}$ with $g(L) = L$. This is equivalent to $\pi(L)$ containing a periodic orbit of Φ. In the same way an orbit γ of $\widetilde{\Phi}$ is *periodic* if $\pi(\gamma)$ is a periodic orbit of Φ.

We say that two orbits γ, α of $\widetilde{\Phi}$ (or the leaves $\widetilde{W}^s(\gamma), \widetilde{W}^s(\alpha)$) are connected by a chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$, if γ is a corner of $\mathcal{C}_1$ and α is a corner of $\mathcal{C}_n$.

If $\mathcal{C}$ is a lozenge with corners β, γ and g is a non trivial covering translation leaving β, γ invariant (and so also the lozenge), then $\pi(\beta), \pi(\gamma)$ are closed orbits of $\widetilde{\Phi}$ which are freely homotopic to the inverse of each other.

Theorem 2.5. *[Fe6, Fe8] Let Φ be a pseudo-Anosov flow in M^3 closed and let $F_0 \neq F_1 \in \widetilde{\Lambda}^s$. Suppose that there is a non trivial covering translation g with $g(F_i) = F_i, i = 0, 1$. Let $\alpha_i, i = 0, 1$ be the periodic orbits of $\widetilde{\Phi}$ in F_i so that $g(\alpha_i) = \alpha_i$. Then α_0 and α_1 are connected by a finite chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$ and g leaves invariant each lozenge $\mathcal{C}_i$ as well as their corners.*

A chain from α_0 to α_1 is called *minimal* if all lozenges in the chain are distinct. Exactly as proved in [Fe4] for Anosov flows, it follows that there is a unique minimal chain from α_0 to α_1 and also all other chains have to contain all the lozenges in the minimal chain.

The main result concerning non Hausdorff behavior in the leaf spaces of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ is the following:

Theorem 2.6. *[Fe6, Fe8] Let Φ be a pseudo-Anosov flow in M^3. Suppose that $F \neq L$ are not separated in the leaf space of $\widetilde{\Lambda}^s$. Then F is periodic and so is L. Let F_0, L_0 be the line leaves of F, L which are not separated from each other. Let V_0 be the sector of F bounded by F_0 and containing L. Let α be the periodic orbit in F_0 and H_0 be the component of $(\widetilde{W}^u(\alpha) - \alpha)$ contained in V_0. Let g be a non trivial covering translation with $g(F_0) = F_0$, $g(H_0) = H_0$ and g leaves invariant the components of $(F_0 - \alpha)$. Then $g(L_0) = L_0$. This produces closed orbits of Φ which are freely homotopic in M. Theorem 2.5 then implies that F_0 and L_0 are connected by a finite chain of lozenges $\{A_i\}, 1 \leq i \leq n$, consecutive lozenges are adjacent. They all intersect a common stable leaf C. There is an even number of lozenges in the chain, see fig. 2. In addition let $\mathcal{B}_{F,L}$ be the set of leaves non separated from F and L. Put an order in $\mathcal{B}_{F,L}$ as follows: Put an orientation in the set of orbits of C contained in the union of the lozenges and their sides. If $R_1, R_2 \in \mathcal{B}_{F,L}$ let α_1, α_2 be the respective periodic orbits in R_1, R_2. Then $\widetilde{W}^u(\alpha_i) \cap C \neq \emptyset$ and let $a_i = \widetilde{W}^u(\alpha_i) \cap C$. We define $R_1 < R_2$ in $\mathcal{B}_{F,L}$ if a_1 precedes a_2 in the orientation of the set of orbits of C. Then $\mathcal{B}_{F,L}$ is either order isomorphic to*

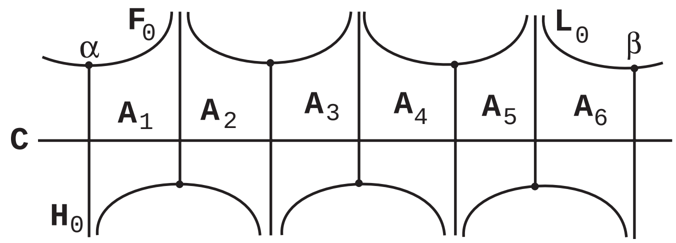

Figure 2: *The correct picture between non separated leaves of* $\widetilde{\Lambda}^s$.

$\{1, ..., n\}$ *for some* $n \in \mathbf{N}$*; or* $\mathcal{B}_{F,L}$ *is order isomorphic to the integers* $\mathbf{Z}$*. In addition if there are* $Z, S \in \widetilde{\Lambda}^s$ *so that* $\mathcal{B}_{Z,S}$ *is infinite, then there is an incompressible torus in* M *transverse to* Φ*. In particular* M *cannot be atoroidal. Also if there are* F, L *as above, then there are closed orbits* α, β *of* Φ *which are freely homotopic to the inverse of each other. Finally up to covering translations, there are only finitely many non Hausdorff points in the leaf space of* $\widetilde{\Lambda}^s$.

Notice that $\mathcal{B}_{F,L}$ is a discrete set in this order. For detailed explanations and proofs, see [Fe6, Fe8].

Theorem 2.7. *([Fe8]) Let* Φ *be a pseudo-Anosov flow. Suppose that there is a stable or unstable product region. Then* Φ *is topologically conjugate to a suspension Anosov flow. In particular* Φ *is non singular.*

Proposition 2.8. *Let* φ *be a (topological) Anosov flow so that every leaf of its stable foliation* $\widetilde{\Lambda}^s$ *intersects every leaf of its stable foliations* $\widetilde{\Lambda}^u$*. Then* φ *is topologically conjugate to a suspension Anosov flow. In particular* M *fibers over the circle with fiber a torus and Anosov monodromy.*

Proof. This result is proved by Barbot [Ba1] when φ is a smooth Anosov flow. That means it is C^1 and it has also strong stable/unstable foliations and contraction on the level of tangent vectors along the flow. Here we only have the weak foliations and orbits being asymptotic in their leaves. With proper understanding all the steps carry through to the general situation.

Lift to a finite cover where Λ^s, Λ^u are transversely orientable. A cross section in the cover projects to a cross section in the manifold (after cut and paste following Fried [Fr]) and so we can prove the result in the cover.

First, the flow φ is expanding: there is $\epsilon > 0$ so that no distinct orbits are always less than ϵ away from each other. Inaba and Matsumoto then proved that this flow is a topological pseudo-Anosov flow [In-Ma]. The main thing is the existence of a Markov partition for the flow. This implies that if F is a leaf of $\widetilde{\Lambda}^s$ which is left invariant by g, then there is a closed orbit of φ in $\pi(F)$ and all orbits are asymptotic to this closed orbit. Similarly for $\widetilde{\Lambda}^u$.

What this means is the following: consider the action of $\pi_1(M)$ in the leaf space of $\widetilde{\Lambda}^s$ which is the reals. Hence we have a group action in $\mathbf{R}$. Let g in $\pi_1(M)$ which fixes a point. There is L in $\widetilde{\Lambda}^s$ with $g(L) = L$. So there is orbit γ of $\widetilde{\varphi}$ with $g(\gamma) = \gamma$. Let U be the unstable leaf of $\widetilde{\varphi}$ with γ contained in U. Then $g(U) = U$. If g is associated to the positive direction of γ then g acts as a contraction in the set of orbits of U with γ as the only fixed point. Since every leaf of $\widetilde{\Lambda}^u$ intersects every leaf of $\widetilde{\Lambda}^s$ then the set of orbits in U is equivalent to the set of leaves of $\widetilde{\Lambda}^s$. This implies the important fact:

Conclusion - If g is in $\pi_1(M)$ has a fixed point in the leaf space of $\widetilde{\Lambda}^s$ then it is of hyperbolic type and has a single fixed point.

Using this topological characterization Barbot [Ba1] showed that $G = \pi_1(M)$ is metabelian, in fact he showed that the commutator subgroup $[G, G]$ is abelian. In particular $\pi_1(M)$ is solvable.

This used only an action by homeomorphisms in $\mathbf{R}$ satisfying the conclusion above. Barbot [Ba1] also proved that the leaves of Λ^s, Λ^u are dense in M.

Plante [Pl1], showed that if $\mathcal{F}$ a minimal foliation in $\pi_1(M)$ solvable then $\mathcal{F}$ is transversely affine: there is a collection of charts $f_i : U_i \to \mathbf{R}^2 \times \mathbf{R}$, so that the transition functions are affine in the second coordinate. Using this Plante [Pl1, Pl2] constructs a homomorphism

$$C : \pi_1(M) \quad \to \quad \mathbf{R}$$

which measures the logarithm of how much distortion there is along an element of $\pi_1(M)$. This is a cohomology class. Every closed orbit γ of φ has a transversal fence which is expanding - this implies that $C(\gamma)$ is positive. Plante then refers to a criterion of Fried [Fr] to conclude that φ has a cross section and therefore it is easily seen that φ is topologically conjugate to a suspension Anosov flow. This finishes the proof of the proposition. $\square$

We now describe almost pseudo-Anosov flows.

Definition 2.9. *Given a pseudo-Anosov flow Φ_1 in a closed 3-manifold, then Φ is an almost pseudo-Anosov flow associated to Φ_1 if Φ is obtained from Φ_1 by blowing some singular orbits of Φ_1 into a collection of flow annuli. Specifically if γ is such a singular orbit of Φ_1, then it blows up into a connected collection of annuli $\{A_i, 1 \leq i \leq n\}$, each of which is flow invariant. The collection is embedded and the annuli have disjoint interiors. In each annulus the boundary components are closed orbits of Φ isotopic to γ as oriented orbits. In the interior of each annulus all orbits are forward asymptotic to one boundary component and backwards asymptotic to the other one. There is a blow down map $\xi : M \to M$, homotopic to the identity and isotopic to the identity in the complement of the A_i and sending each connected collection of A_i into a periodic orbit of Φ_1. The map ξ sends orbits of Φ to orbits of Φ_1 preserving orientation.*

The reason for considering almost pseudo-Anosov flows is as follows. All of the constructions of pseudo-Anosov flows transverse to foliations are in fact constructions of a pair of laminations $-$ stable and unstable $-$ which are transverse to each other and to the foliation [Th4, Mo3, Fe9, Cal1, Cal2]. The intersection of the laminations is oriented producing a flow in this intersection. One then collapses the complementary regions to the laminations to produce transverse singular foliations and a pseudo-Anosov flow.

The transversality problem occurs in this last step, the blow down of complementary regions. In certain situations, for example for finite depth foliations, one cannot guarantee total transversality after the blow down. We briefly explain a possible problem. Mosher's construction [Mo3] of flows (almost) transverse to foliations is done inductively on the depth of the leaves (starting with the top depth leaves), associated to a sutured manifold hierarchy and the ensuing foliations construction of Gabai. At each step there is a foliation which is partially tangent/transverse to the boundary and also two laminations (stable/unstable) which are transverse to each other and to the foliation. There is a flow in the intersection of the laminations and a flow direction in "periodic" leaves, since all orbits in say a stable leaf are forward asymptotic. The next step topologically involves glueing two subsurfaces in the boundary in the construction of the foliation and laminations/flow.

One of the problems that can easily happen is the following. Suppose the glueing is done along surface S and after the glueing there are closed orbits α, β of the flow, which are oriented isotopic to the same simple closed curve of S and are in opposite sides of S, see fig. 3.

In the resulting pseudo-Anosov flow, α, β will be (oriented) freely homotopic to each other. By theorem 2.5 when lifted to $\widetilde{M}$ they are connected by a finite chain of lozenges. This forces the existence of another closed orbit α_1 which is freely homotopic to the inverse of α (in the opposite

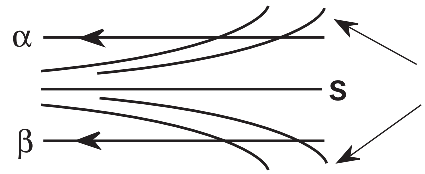

Figure 3: *Obstruction to transversality.*

corner of a lozenge in $\widetilde{M}$). The problem is there is no guarantee such an orbit α_1 will be produced in the inductive process. In order to fix that, then in the collapsing step Mosher collapses α and β into a single orbit. This allows for the collapsed flow to be pseudo-Anosov. Unfortunately the transversality is lost locally near this region of S. There may be more collapsing forced by the inductive process. In order to recover the transversality, in this particular case one blows up the collapsed orbit into an embedded annulus, with boundaries α, β and puts a flow going from one orbit to the other, crossing S in the correct direction. Since other collapsings may be forced we may have a collection of annuli which are joined together and collapse to a single periodic orbit.

We still denote by Λ^s, Λ^u the stable/unstable laminations of an almost pseudo-Anosov flows. They are transverse to each other except at the blown up annuli. The same notation is used for $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$, etc..

The objects perfect fits, lozenges, product regions, etc.. all make sense in the setting of almost pseudo-Anosov flows: they are just the blow ups of the same objects for the corresponding pseudo-Anosov flows. Since the interior of these objects does not have singularities, the blow up operation does not affect these interiors. There may be singular orbits in the boundary which get blown into a collection of annuli. All the results in this section still hold for almost pseudo-Anosov flows, with the blow up operation. For example if F, L in $\widetilde{\Lambda}^s$ are not separated from each other, then they are connected by an even number of lozenges all intersecting a common stable leaf. Since parts of the boundary of these may have been blown into annuli, there is not a product structure in the closure of the union of the lozenges, but there is still a product structure of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in the interior.

3 Projections of leaves of $\widetilde{\mathcal{F}}$ to the orbit space

Let Φ be an almost pseudo-Anosov flow which is transverse to a foliation $\mathcal{F}$. An orbit of $\widetilde{\Phi}$ intersects a leaf of $\widetilde{\mathcal{F}}$ at most once $-$ because the leaves of $\widetilde{\mathcal{F}}$ are properly embedded and $\widetilde{\Phi}$ is transverse to $\widetilde{\mathcal{F}}$. Hence the projection $\Theta : F \to \Theta(F)$ is injective. We want to determine the set of orbits a leaf of $\widetilde{\mathcal{F}}$ intersects $-$ in particular we want to determine the boundary $\partial\Theta(F)$. As it turns out, $\partial\Theta(F)$ is composed of a disjoint union of slice leaves in $\mathcal{O}^s, \mathcal{O}^u$.

Since Φ is transverse to $\mathcal{F}$, there is $\epsilon > 0$ so that if a leaf F of $\widetilde{\mathcal{F}}$ intersects an orbit of $\widetilde{\Phi}$ at p then it intersects every orbit of $\widetilde{\Phi}$ which passes ϵ near p and the intersection is also very near p. To understand $\partial\Theta(F)$ one main ingredient is that when considering pseudo-Anosov flows, then flow lines in the same stable leaf are forward asymptotic. So if F intersects a given orbit in a very future time then it also intersects a lot of other orbits in the same stable since in future time they converge. In the limit this produces a stable boundary leaf of $\Theta(F)$. The blow up operation disturbs this: it is not true that orbits in the same stable leaf of an almost pseudo-Anosov flow are forward asymptotic: when they pass arbitrarily near a blow up annulus the orbits are distorted and their distance can increase enormously. This is the key difficulty in this section. Hence we first analyse the blow up operation more carefully.

<u>Notation</u> $-$ Given Φ an almost pseudo-Anosov flow, let Φ_1 be a corresponding pseudo-Anosov flow

associated to Φ. The term $\widetilde{W}^s(x)$ will denote the stable leaf of $\widetilde{\Phi}$ or $\widetilde{\Phi}_1$, where the context will make clear which one it is.

Recall that $\pi : \widetilde{M} \to M$ denotes the universal covering map.

We will start with Φ_1 and understand the blow up procedure. The blown up annuli come from singular orbits. The *lift annuli* are the lifts of blown up annuli to $\widetilde{M}$. Their projections to $\mathcal{O}$ are called *blown segments*. If L is a blown up leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ the components of L minus the lift annuli are called the *prongs*. A *quarter* associated to an orbit γ of $\widetilde{\Phi}_1$ is the closure of a connected component of $\widetilde{M} - (\widetilde{W}^u(\gamma) \cup \widetilde{W}^s(\gamma))$. Its boundary is a union of γ and half leaves in the stable and unstable leaves of γ. We will be interested in a neighborhood V of γ in this quarter which projects to M near the closed orbit $\pi(\gamma)$. We will understand the blow up in the projection of a quarter. Glueing up different quarter gives the overall picture of the blow up operation. In the projected quarter $\pi(V)$ in M there is a cross setion to the flow Φ_1. The orbits across the cross section are determined by which stable and unstable leaf they are in. The return map on the stable direction is a contraction and an expansion in the unstable direction. Any contraction is topologically conjugate to say $x \to x/2$ and an expansion is conjugate to $x \to 2x$. Hence the local return map is topologically conjugate to

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

a linear map. The whole discussion here is one of topological conjugacy. The flow is conjugate to $(x, y, 0) \;\to\; (2^{-t}x, 2^t y, t)$. Think of the blow up annulus as the set of unit tangent vectors to γ associated to the quarter region. The flow in the annulus is given by the action of DV_t on the tangent vectors. It has 2 closed orbits (the boundary ones corresponding to the stable and unstable leaves). The other orbits are asymptotic to the stable closed orbit in negative time and to the unstable closed orbit in positive time. This makes it into a continuous flow in this blown up part. See detailed explanation in [Fr] or [Ha-Th] (Fried, Handel-Thurston). For future reference recall this fact that in a blow up annulus the boundary components are orbits of the flow and in the interior the flow lines go from one boundary to the other without a Reeb annulus picture (there is a cross section to the flow in the annulus). Do this for each quarter region that is blown up. One can then glue up the 2 sides of the appropriate annuli because they are all of the same topological picture (using the standard model above). This describes the blown up operation in a quarter. There is clearly a blow down map which sends orbits of the blown up flow Φ to orbits of Φ_1 and collapses connected unions of annuli into a single p-prong singular orbit.

We quantify these: let ϵ very small so that any two orbits of Φ_1 which are always less than ϵ apart in forward time, then they are in the same stable leaf. Let $\mathcal{Z}$ the union of the singular orbits of Φ_1 which are blown up. Let $\epsilon' << \epsilon$ and let U be the ϵ' tubular neighborhood of $\mathcal{Z}$. Let U' (resp. U) be the $\epsilon'/2$ (resp. ϵ') tubular neighborhood of $\mathcal{Z}$. Choose the blow up map to be the identity in the complement of U', that is the blown up annuli are also contained in U'. The blow down map is then an isometry of the Riemannian metric outside U'. Choose the blow down to move points very little in U'. Isotope $\mathcal{F}$ so that it is transverse to the flow Φ. We are now ready to analyse $\partial\Theta(F)$.

Proposition 3.1. *Let F in $\widetilde{\mathcal{F}}$. Then $\Theta(F)$ is an open subset of $\mathcal{O}$. Any boundary component of $\Theta(F)$ is a slice of a leaf of $\mathcal{O}^s$ or $\mathcal{O}^u$. If it is a slice of $\mathcal{O}^s$, then as $\Theta(F)$ approaches l, the corresponding points of F escape in the positive direction. Similarly for unstable boundary slices.*

Proof. First notice that since F is transverse to $\widetilde{\Phi}$ then $\Theta(F)$ is an open set. Hence $\partial\Theta(F)$ is disjoint from $\Theta(F)$. The important thing is to notice that the metric is the same outside the small neighborhood U' of the blown up annuli. If two points are in the same stable leaf, then their orbits under the blow down flow Φ_1 are asymptotic in forward time. The same is true for Φ, for big enough time if the point is outside U. This is because the points of the corresponding orbits of Φ_1 will be both outside U' – this is the reason for the construction of two neighborhoods U', U. The following setup will be used in all cases.

Setup – Let v in $\partial\Theta(F)$ and v_i in $\Theta(F)$ with v_i converging to v. Let p_i in F with $\Theta(p_i) = v_i$ and let w in $\widetilde{M}$ with $\Theta(w) = v$. Let D be any small disk in $\widetilde{M}$ transverse to $\widetilde{\Phi}$ with w in the interior of D. For i big enough v_i is in $\Theta(D)$ so there are t_i real numbers with $p_i = \widetilde{\Phi}_{t_i}(w_i)$ and w_i are in D. As v is not in $\Theta(F)$, then $|t_i|$ grows without bound. Without loss of generality assume up to subsequence that $t_i \to \infty$. We will prove that there is a slice leaf L of $\widetilde{W}^s(w)$ so that $\Theta(L) \subset \Theta(F)$ and F goes up as it "approaches" L. The stable/unstable leaves here are those of the almost pseudo-Anosov flow and they may have blown up annuli.

Case 1 – Suppose that w is not in a blown up leaf.

First we show that we can assume no w_i is in $\widetilde{W}^s(w)$. Otherwise up to subsequence assume all w_i are in $\widetilde{W}^s(w)$. The orbits through w_i and w start out very close and aside from the time they stay in $\pi^{-1}(U)$ they are always very close. Let B be the component of the intersection of F with the flow band from $\widetilde{\Phi}_{\mathbf{R}}(w_i)$ to $\widetilde{\Phi}_{\mathbf{R}}(w)$ in the stable leaf $\widetilde{W}^s(w)$, which contains p_i. Then B does not intersect $\widetilde{\Phi}_{\mathbf{R}}(w)$ so it has to either escape up or down. If it escapes down it will have to intersect a small segment from w_i to w and hence so does F. For i big enough w_i is arbitrarily near w, so transversality of $\mathcal{F}$ and Φ then implies that F will intersect $\widetilde{\Phi}_{\mathbf{R}}(w)$ near w, contradiction see fig. 4, a.

We now consider the case that B escapes up. If the forward orbit through w is not always in $\pi^{-1}(U)$ then at those times outside of $\pi^{-1}(U)$ it will be arbitrarily close to $\widetilde{\Phi}_{\mathbf{R}}(w_i)$ and transversality implies again that F intersects $\widetilde{\Phi}_{\mathbf{R}}(w)$. If the forward orbit of w always stays in $\pi^{-1}(U)$ the same happens after the blow down so the blow down orbit is in the stable leaf of the singular orbit which is being blown up. This does not happen in case 1.

We can now assume that all v_i are in a sector of $\mathcal{O}^s(v)$ with l the boundary of this sector and $L = l \times \mathbf{R}$, the line leaf of $\widetilde{W}^s(w)$ which is the boundary of this sector.

Let now q in l. We will show that q is in $\partial\Theta(F)$ so $l \subset \partial\Theta(F)$. There is a segment $[q, v]$ contained in l. Choose x in L with $\Theta(x) = q$. Let α be a segment in $\widetilde{W}^s(w)$ transverse to the flow lines and going from x to w. Let x_i converging to x and x_i in $\widetilde{W}^s(w_i)$. We can do that since all w_i are in the same sector of $\widetilde{W}^s(w)$. Choose segments α_i from x_i to w_i in $\widetilde{W}^s(w_i)$ and transverse to the flowlines of $\widetilde{\Phi}$ in $\widetilde{W}^s(w_i)$.

Claim – For every orbit γ of $\widetilde{\Phi}$ intersecting α_i in y then γ intersects F in $\widetilde{\Phi}_s(y)$ where s converges to ∞ as $i \to \infty$.

Suppose there is $a_0 > 0$ so that for some i_0 then

$$\widetilde{\Phi}_{[a_0,t_i]}(w_i) \ \subset \ \pi^{-1}(U) \quad \text{for all} \ \ i \geq i_0$$

Then $\widetilde{\Phi}_{[a_0,\infty)}(w)$ is contained in the closure of $\pi^{-1}(U)$. As seen before this implies that w is in a blown up stable leaf, which is not the hypothesis of case 1. Therefore up to subsequence, there are arbitrary big times s_i between 0 and t_i so that $\widetilde{\Phi}_{s_i}(w_i)$ is not in $\pi^{-1}(U)$. Hence $\widetilde{\Phi}_{\mathbf{R}}(x_i)$ is very close to $\widetilde{\Phi}_{s_i}(w_i)$ and since F cannot escape up or down then F intersects $\widetilde{\Phi}_{\mathbf{R}}(x_i)$. Hence the segment $[\Theta(x_i), v_i]$ of $\mathcal{O}^s(v)$ is contained in $\Theta(F)$ and so $[x, v]$ is contained in the closure of $\Theta(F)$. Also the time s so that $\widetilde{\Phi}_s(y)$ hits F goes to ∞, hence $[x, v]$ cannot intersect $\Theta(F)$ – else there would be bounded times where it intersects F, by transversality. We conclude that $[x, v] \subset \partial\Theta(F)$, hence $l \subset \partial\Theta(F)$ as desired. If there is a sequence z_i in F escaping down with $\Theta(z_i)$ converging to a point

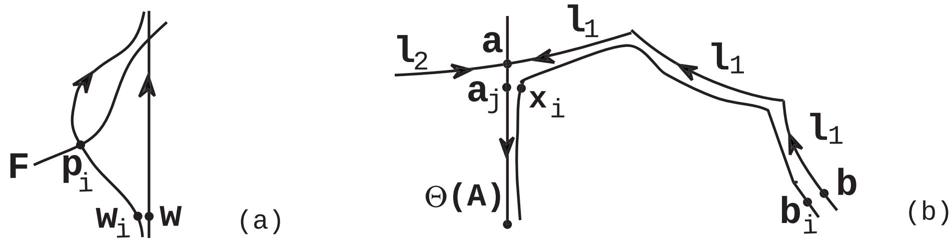

Figure 4: *a. A strangling neck is being forced, b. A slice in a leaf of* $\mathcal{O}^s$ *or* $\mathcal{O}^u$. $x_i = \Theta(z_i)$.

in l, then by connectedness there is one intersecting a compact middle region $-$ this would force an intersection of F with $l \times \mathbf{R}$ which is impossible.

This finishes the proof of case 1. In this case we proved there is a line leaf l of $\Theta(L)$ with $l \subset \partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches l.

Case 2 $-$ w is in a blown up leaf, but F does not intersect a lift annulus in $\widetilde{W}^s(w)$.

Refer to the setup above. As before we first show we can assume w_i are not in $\widetilde{W}^s(w)$. Otherwise, up to subsequence assume all w_i are in $\widetilde{W}^s(w)$. Since F does not intersect lift annuli in $\widetilde{W}^s(w)$, then w_i are all in prongs of $\widetilde{W}^s(w)$. Up to subsequence we can assume they are all in the same prong C of $\widetilde{W}^s(w)$ which has boundary an orbit γ of $\widetilde{\Phi}$. It follows that w is in γ. All the orbits in C are forward asymptotic to γ, even in the blown up situation. The strangling necks analysis of case 1 shows that F will be forced to intersect $\widetilde{\Phi}_{\mathbf{R}}(w)$. This cannot occur.

Hence assume all v_i are in a sector of $\mathcal{O}^s(v)$ bounded by a line leaf l. Let L be $l \times \mathbf{R}$. Let q be a point in l and choose x, α, x_i and α_i as in the proof of case 1. Choose a small disc D which is transverse to $\widetilde{\Phi}$ and has α in its interior. For i big enough then D intersects lift annuli only in $\widetilde{W}^s(x)$. This is because the union of the blown annuli forms a compact set in M, so either α intersects a lift annulus, in which case there is no other lift annulus nearby or D is entirely disjoint from lift annuli. From now on the arguments of case 1 apply perfectly. This shows that $\Theta(L)$ is contained in $\overline{\Theta(F)}$, it is disjoint from $\Theta(F)$ and so it is $\partial\Theta(F)$ and F escapes up as it approaches L. This finishes the proof of case 2.

Now we need to understand what happens when F intersects a lift annulus in general. We separate that in a special case. We need the following facts before addressing this case. A lift annulus W through b is contained in $\widetilde{W}^s(b)$ and $\widetilde{W}^u(b)$ so there is not stable/unstable flow directions in W. However there are still such directions in ∂W, because one attracts nearby orbits of $\widetilde{\Phi}$ in W and the other one repels nearby orbits in W. In this generalized sense the first one is stable and the second one is unstable. In this sense if a is in an endpoint of a blown segment, then all local components of $\mathcal{O}^s(a) - a, \mathcal{O}^u(a) - a$ near a are either generalized stable or unstable. With this understanding there is an even number of such components and they alternate between generalized stable and unstable. Some local components of $\mathcal{O}^s(a) - a$ are also local components of $\mathcal{O}^u(a) - a$ if they are blown segments. One key thing to remember is that generalized stable and unstable alternate.

Case 3 $-$ Suppose that F intersects some lift annulus A contained in $\widetilde{W}^s(u_1)$.

Then F does not intersect both boundary orbits of A. Otherwise we could collapse $\pi(A)$ to a single orbit, still keeping Φ transverse to $\mathcal{F}$. Hence either $F \cap A$ is contained in the interior of A or it intersects only one boundary leaf.

Assume without loss of generality that F escapes up in one direction. This defines an orbit γ of

$\widetilde{\Phi}$ with $a = \Theta(\gamma)$ in $\partial\Theta(F)$. The orbit γ has to be in the boundary of the lift annulus A. This is because an interior orbit is asymptotic to both boundary orbits and hence would intersect F. We now look at the picture in $\mathcal{O}$. Consider the stable leaf $\mathcal{O}^s(a)$. Notice that $\Theta(F)$ intersects $\Theta(A)$. From the point of view of γ, orbits in A move away from γ in future time, that is A is an unstable direction from γ. This means that $\Theta(A)$ is generalized unstable as seen from a. It follows that there are two generalized stable sides of $\mathcal{O}^s(a)$ one on each side of $\Theta(A)$ which are the closest to $\Theta(A)$. Choose one side, start at a and follow along $\mathcal{O}^s(a)$ either through blown segments and eventually into a prong in $\mathcal{O}^s(a)$ so as to produce a piece of a line leaf of $\mathcal{O}^s(a)$ in that direction. This path is regular on the side associated to $\Theta(A)$ and defines a half leaf l_1 of $\mathcal{O}^s(a)$. Similarly define l_2 in the other direction, see fig. 4, b. Let l be the union of l_1 and l_2. Then l is a slice leaf of $\mathcal{O}^s(a)$ but is not a line leaf since $\Theta(A)$ is in $\mathcal{O}^s(a)$ and is not in l.

<u>Claim</u> $-$ l is contained in $\partial\Theta(F)$ and F escapes positively as $\Theta(F)$ approaches l.

Let b in l_1 with b not in blown segment, that is, b in a prong. Choose b_i in $\mathcal{O}^u(b)$, with $b_i \to b$ and in that component of $\mathcal{O} - l$. Let D be an embedded disc in $\widetilde{M}$ which is transverse to $\widetilde{\Phi}$ and projects to $\mathcal{O}$ to a neighborhood of the arc ξ in l_1 from a to b. Let y_i in D with $\Theta(y_i) = b_i$, $y_i \to y$ with $\Theta(y) = b$. Assume that y is not in $\pi^{-1}(U)$. Choose b so that it is not in the unstable leaf of one singular orbit, hence $\widetilde{W}^u(y)$ does not contain lift annuli. In addition choose y_i so that $\widetilde{W}^s(y_i)$ does not contain lift annuli either.

Choose points u_j in $F \cap A$ so that $\Theta(u_j) = a_j$ converges to a. For each j the set $\Theta(F)$ contains a small neighborhood V_j of $\Theta(u_j)$ with V_j converging to a when j converges to infinity. The leaves $\mathcal{O}^s(b_i)$ are getting closer and closer to l_1 and $\Theta(A)$. For j fixed there is i big enough so that $\mathcal{O}^s(b_i)$ intersects V_j. Let

$$z_i \ \in \ F \cap \widetilde{W}^s(y_i) \ \text{ with } \ \Theta(z_i) \in V_j$$

here i depends on j. Let $z_i = \widetilde{\Phi}_{t_i}(r_i)$ with r_i in D. By choosing j and i converging to infinity we get that $\Theta(z_i)$ converges to a and we can ensure that the arc of $D \cap \widetilde{W}^s(y_i)$ between r_i and y_i is converging to an arc η of $\widetilde{W}^s(a) \cap D$ with $\Theta(\eta) = \xi$. We can also choose V_j small enough so that t_i converges to infinity.

The orbits $\widetilde{\Phi}_{\mathbf{R}}(y_i), \widetilde{\Phi}_{\mathbf{R}}(r_i)$ are very close in the forward direction as long as they are outside $\pi^{-1}(U)$. Since $\widetilde{W}^s(y_i)$ does not contain lift annuli then for times s converging to infinity $\widetilde{\Phi}_s(y_i)$ is not in $\pi^{-1}(U)$. Consider the flow band C in $\widetilde{W}^s(y_i)$ between $\widetilde{\Phi}_{\mathbf{R}}(r_i)$ and $\widetilde{\Phi}_{\mathbf{R}}(y_i)$. The leaf F intersects $\widetilde{\Phi}_{\mathbf{R}}(r_i)$ in $\widetilde{\Phi}_{t_i}(r_i)$ with t_i converging to infinity. Then an analysis exactly as in case 1 considering strangling necks and the arcs B in that proof, shows that $F \cap \widetilde{W}^s(y_i)$ cannot escape up down before intersecting $\widetilde{\Phi}_{\mathbf{R}}(y_i)$.

Suppose that F escapes down before intersecting $\widetilde{\Phi}_{\mathbf{R}}(y_i)$. We show that this is impossible. Since $F \cap \widetilde{W}^s(y_i)$ has points z_i in the forward direction from D and points in the backwards direction from D it follows that $F \cap \widetilde{W}^s(y_i)$ must intersect D in at least a point q_i. Up to subsequence we may assume that q_i converges to q in $\widetilde{W}^s(y)$. This will be an iterative process. Let $u_1 = q$. It is crucial to notice that in the flow band of $\widetilde{W}^s(y)$ between $\widetilde{\Phi}_{\mathbf{R}}(y)$ and γ the flow lines tend to go closer to γ, that is, either they project to closed orbits freely homotopic to $\pi(\gamma)$ or they are asymptotic to one of these orbits moving closer to γ. We now consider the component of $F \cap \widetilde{W}^s(y)$ containing u_0 and follow it towards γ. This component does not intersect γ and by the above it can only escape down in $\widetilde{W}^s(y)$. As it escapes down it produces points c_i in $\widetilde{W}^s(r_i)$ and as before produces points c'_i in D, which up to subsequence converge to c in $D \cap F$. By construction c is not u_1 and its orbit is closer to γ. Let $u_2 = c$. We can iterate this process. Notice the u_i cannot accumulate in D, or else all the corresponding points of F are in a compact set of $\widetilde{M}$. On the other hand the process does not terminate. This produces a contradiction.

The contradiction shows that in fact the arc $\Theta(C)$ is in $\Theta(F)$ which implies that $\xi = \Theta(\eta)$ is contained in $\overline{\Theta(F)}$. As the time to hit F from D grows with i, this shows that $\Theta(F)$ does not intersect ξ and hence ξ is contained in $\partial\Theta(F)$. As b is arbitrary this shows that $l \subset \partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches l. This finishes the analysis of case 3.

Case 4 $-$ w is in a blown up stable leaf and F intersects some lift annulus A in $\widetilde{W}^s(w)$.

The difference from case 3 is that in case 3 we obtained a slice boundary l of $\Theta(F)$ $-$ but in our situation we do not yet know if it contains $\Theta(w)$ and whether it is a stable or unstable. Here we prove it is a stable slice and it contains $\Theta(w)$.

Recall the setup: $v = \Theta(w)$ is in $\partial\Theta(F)$ and there are v_i in $\overline{\Theta(F)}$ with v_i converging to v and with p_i in $(v_i \times \mathbf{R}) \cap F$. Also $p_i = \widetilde{\Phi}_{t_i}(w_i)$ with w_i converging to w in $\widetilde{M}$ and t_i converging to infinity. Let ξ be the blown segment $\Theta(A)$.

The analysis of case 3 shows that $\Theta(F)$ contains the interior of $\Theta(A)$. Suppose first that v is in ξ. Then v is in the boundary of ξ and by case 3 again F escapes up or down when $\Theta(F)$ approaches a slice which contains v. If it escapes up, then the slice is a stable slice and we obtain the desired result in this case. We now show that F does not escape down. Let l be the unstable slice in $\partial\Theta(F)$ associated to this. Then l cuts in half a small disk neighborhood of v in $\mathcal{O}$. The set $\Theta(F)$ intersects only one component of the complement, the one which intersects ξ. As F escapes down when $\Theta(F)$ approaches l, then for all points in $\Theta(F)$ near enough v the corresponding point in F is flow backwards from D. This contradicts the fact that t_i is converging to infinity. Therefore F cannot escape down as it approaches l.

We can now assume that v is not in ξ. By changing ξ if necessary assume that ξ is the blown segment in $\mathcal{O}^s(v)$ intersected by $\Theta(F)$ which is closest to v. Let z be the endpoint of ξ separating the rest of ξ from v in $\mathcal{O}^s(v)$.

We first show that z is not in $\Theta(F)$. Suppose that is not the case and let b the intersection point of $z \times \mathbf{R}$ and F. Since ξ is the last blown segment of $\mathcal{O}^s(v)$ between ξ and v intersected by $\Theta(F)$ and $\Theta(F)$ contains an open neighborhood of z, it follows that v is in a prong B of $\mathcal{O}^s(v)$ with endpoint z. Let τ be the component of $F \cap \widetilde{W}^s(b)$ containing b. Since F does not intersect $v \times \mathbf{R}$ then it escapes. As the region between $b \times \mathbf{R}$ and $z \times \mathbf{R}$ is a prong, then F cannot escape up. As seen in the arguments for case 3, F cannot escape down either. This shows that z cannot be in $\Theta(F)$.

It follows that F escapes either up or down as $\Theta(F)$ approaches z. Suppose first that it escapes up. Then we are in the situation of case 3 and we produce a stable slice l in $\partial\Theta(F)$ with F going up as $\Theta(F)$ approaches l. If v is not in l then l separates v from $\Theta(F)$. This contradicts v_i in $\Theta(F)$ with v_i converging to v. Hence v is in l with F escaping up as $\Theta(F)$ approaches l. This is exactly what we want finishing the analysis in this case.

The last situation is F escaping down in A as $\Theta(F)$ approaches z. By case 3 there is a slice leaf l in $\mathcal{O}^u(z)$ with l contained in $\partial\Theta(F)$ and F escaping down as $\Theta(F)$ approaches l. We want to show that this case cannot happen. Notice that the blown segments of $\mathcal{O}^s(z)$ are exactly the same as the blown segments of $\mathcal{O}^u(z)$. The sets $\mathcal{O}^s(z), \mathcal{O}^u(z)$ differ exactly in the prongs and as they go around the collection of blown segments. The collection of all prongs in $\mathcal{O}^s(z), \mathcal{O}^u(z)$ also alternates between stable and unstable as it goes around the union of the blown segments.

Suppose first that v is in l. This contradicts F escaping down and $t_i \to \infty$. Finally suppose that v is not in l. We claim that in this case l separates v from $\Theta(F)$. Let α be the path in $\mathcal{O}^s(v)$ from z to v. If α only intersects l in z, then the separation property follows because l_1 and l_2 contain the local components of $\mathcal{O}^s(z) \cup \mathcal{O}^u(z) - z$ which are closest to $\Theta(A)$. This was part of the construction of l in case 3. Here the ξ is generalized stable at z and l_1, l_2 are generalized unstable at p. The path from z to v in $\mathcal{O}^s(v)$ cannot start in ξ or l_1 or l_2, hence l separates $\Theta(F)$ from v.

If on the other hand $\alpha \cap l = \delta$ is not a single point, then it is a union of blown segments. Let u be the other endpoint of δ. By regularity of l_1 and l_2 on the $\Theta(F)$ side it follows that each blown up segment in δ has flow direction away from z. Hence δ is generalized stable at u. Therefore the closest component of $\mathcal{O}^s(u) \cup \mathcal{O}^u(u) - u$ on the $\Theta(F)$ side is generalized unstable and that is contained in l. In this case it also follows that l separates v from $\Theta(F)$. As seen before this is a contradiction.

This finishes the proof of proposition 3.1 $\square$

This has an important consequence that will be used extensively in this article.

Proposition 3.2. *Let F in $\widetilde{\mathcal{F}}$ and L in $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. Then the intersection $F \cap L$ is connected.*

Proof. By transversality of $\mathcal{F}$ and Φ, the intersection $C = \Theta(F) \cap \Theta(L)$ is open in $\Theta(L)$. Supppose there are 2 disjoint components A, B of C. Then there is v in ∂A with v separating A from B. There are v_i in A with v_i converging to v. By the previous proposition F escapes up or down in $A \times \mathbf{R}$ as $\Theta(F)$ approaches v. Assume wlog that F escapes up. Then there is a slice leaf l of $\mathcal{O}^s(v)$ with $l \subset \partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches l. Since l and $\Theta(F)$ are disjoint then B is disjoint from l. In addition v separates B from A in $\Theta(L)$. It follows from the construction of the slice l as being the closest to A, that l separates A from B. Hence $\Theta(F)$ cannot intersect B, contrary to assumption. This finishes the proof. $\square$

4 Asymptotic properties in leaves of the foliation

Let Φ be an almost pseudo-Anosov flow transverse to a foliation $\mathcal{F}$ with hyperbolic leaves. Let Λ^s, Λ^u be the singular foliations of Φ. Given leaf F of $\widetilde{\mathcal{F}}$ let $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ be the induced one dimensional singular foliations in F. In this section we study asymptotic properties of rays in $\widetilde{\Lambda}^s_F$. First we mention a result of Thurston [Th5] concerning contracting directions, which for convenience we state for 3-manifolds:

Theorem 4.1. *(Thurston) Let $\mathcal{F}$ be a codimension one foliation with hyperbolic leaves in M^3 closed. Then for every x in any leaf F of $\widetilde{\mathcal{F}}$ and every $\epsilon > 0$ there is a dense set of geodesic rays of F starting at x such that: for any such ray r there is a transversal β to $\widetilde{\mathcal{F}}$ starting at x so that any leaf L intersecting β and any y in r, then the distance between y and L is less than ϵ. If there is not a holonomy invariant transverse measure whose support contains $\pi(F)$ then one can show that the directions are actually contracting, that is: if y escapes in r then the distance between y and L converges to 0. Finally if $\pi(F)$ is not closed one can choose the β above to have x in the interior.*

There is a carefully written published version of this result in [Ca-Du]. The directions above where distance to nearby L goes to 0 are called contracting directions. The other ones where distance is bounded by ϵ are called ϵ non expanding directions. We first prove a preliminary result:

Theorem 4.2. *Let Φ be a pseudo-Anosov flow almost transverse to a foliation $\mathcal{F}$ in M^3 closed with $\mathcal{F}$ having hyperbolic leaves. Suppose there is a leaf L of $\widetilde{\mathcal{F}}$ and l a ray in a leaf of $\widetilde{\Lambda}^s_L$ so that l does not limit in a single point in $\partial_\infty L$. Then $\mathcal{F}$ is an $\mathbf{R}$-covered foliation. Similarly for rays of $\widetilde{\Lambda}^u_L$*

Proof. We assume at the start that $\mathcal{F}$ is not $\mathbf{R}$-covered. Let ϵ positive so that if p in $\widetilde{M}$ is less than ϵ from a leaf F of $\widetilde{\mathcal{F}}$, then the flow line through p intersects F less than 2ϵ away from p. Let l be a ray in $\widetilde{\Lambda}^s_L$. Because $\mathcal{F}$ and Φ are transverse, L is properly embedded in $\widetilde{M}$ and leaves of $\widetilde{\Lambda}^s$ are properly embedded, it follows that l is a properly embedded ray in L. Therefore it can only limit in $\partial_\infty L$. Suppose that l limits on 2 points a_0, b_0 in $\partial_\infty L$. Fix p a basepoint in L.

Since l limits in a_0, b_0, there are compact arcs l_i of l with endpoints which converge to a_0, b_0 respectively in $L \cup \partial_\infty L$ and so that the distance from l_i to p in L converges to infinity. Also we can assume that the l_i converges to a segment v in $\partial_\infty L$, where v connects a_0, b_0. This is in the Hausdorff topology of closed sets in $L \cup \partial_\infty L$, which is a closed disk.

The key idea is to bring this situation to a compact part of $\widetilde{M}$. Choose a sequence p_i a bounded distance from points in l_{k_i} so that that p_i converges to a point a in the interior of v. The bound depends on the sequence. Up to subsequence assume that there are convering translations g_i in $\pi_1(M)$ so that $g_i(p_i)$ converges to a point p_0 in $\widetilde{M}$.

We claim that the set of points obtained as above projects to a sublamination of $\mathcal{F}$. Clearly if $g_i(p_i)$ converges to p_0 and q is in the same leaf L_0 of $\widetilde{\mathcal{F}}$ as p, then the distance from p_0 to q is finite and there are q_i in L with $d_L(q_i, p_i)$ bounded and $g_i(q_i)$ converging to q. Also q_i converges to a in $\partial_\infty L$. In addition if a sequence of such limits c_j converges to c_0 then a diagonal process shows that c_0 is also obtained as a single limit. This proves the claim. We extract a minimal sublamination $\mathcal{L}$.

A leaf F of $\widetilde{\mathcal{F}}$ is isometric to the hyperbolic plane. A *wedge* W in F with corner b and ideal set an interval $B \subset \partial_\infty F$ is the union of the rays in F from b with ideal point in B. The angle of the wedge is the angle that the boundary rays of W make at b. For any such sequence p_i as above, then the visual angle at p_i subintended by the arc v in $\partial_\infty L$ grows to 2π. Therefore the angle of wedge with corner p_i and ideal set $\partial_\infty L - v$ converges to 0. This is called the *bad wedge.*

Assume up to subsequence that $g_i(p_i)$ is converging to p_0 in a leaf L_0 of $\widetilde{\mathcal{F}}$ and that the directions of the bad wedges with corners $g_i(p_i)$ in $g_i(L)$ are converging to the direction r_0 of L_0. Let c be the ideal point of r_0 in $\partial_\infty L_0$.

Suppose first that $\pi(L_0)$ is not compact $-$ we shall see briefly that this is in fact always the case. Thurston's theorem shows that the set of two sided contracting directions (or ϵ non expanding directions) in L_0 is dense in $\partial_\infty L_0$. We will use these to transport a lot of the structure of $\widetilde{\Lambda}^s_{L_0}$ to nearby leaves. Choose s_0, s_1 to be rays in L_0 defining contracting directions (or ϵ non expanding directions) very near r_0 so that together they form a small wedge W in L_0 with corner p_0. There is an interval of leaves near L_0 so that any such leaf V is less than ϵ away from s_0, s_1. Then a flow line of $\widetilde{\Phi}$ through any point in s_0 or s_1 intersects V less than 2ϵ away. So s_0 flows to a curve in V, where we can assume it has geodesic curvature very close to 0, if ϵ is sufficiently small. It is therefore a quasigeodesic with a well defined ideal point. The same happens for s_1 and the flow images u_0, u_1 of s_0, s_1 in V define a generalized wedge W' in V. The ideal points e_0, e_1 of u_0, u_1 are close and bound an interval I which is almost all of $\partial_\infty g_i(L)$.

By construction $g_i(l)$ is a ray which limits in an interval of $\partial_\infty g_i(L)$ which contains I in its interior if i is big enough. There are then subarcs τ_j of $g_i(l)$ with endpoints a_j, b_j in u_0, u_1 respectively so that a_j converges to e_0 and b_j converges to e_1 and τ_j converges to I, see fig. 5. Here i is fixed and j varies. Since a_j, b_j are in u_0, u_1 then they flow (by $\widetilde{\Phi}$) to points in L_0. The images in L_0 are in the same leaf of $\widetilde{\Lambda}^s$. By proposition 3.2 these images are in the same leaf of $\widetilde{\Lambda}^s_{L_0}$. Hence the whole segment τ_j flows into L_0.

The point p_0 flows into p' in $g_i(L)$ under the flow. The arc τ_j together with subarcs or u_0, u_1 from a_j, b_j to p' bound a disc D_j in $g_i(L)$. The arguments above show that the boundary of D_j flows into L_0 producing a curve in L_0 bounding a disc B_j. The segments of $\widetilde{\Phi}$ connecting points in ∂D_j to points in ∂B_j produce an annulus C_j. Then $D_j \cup C_j \cup B_j$ is an embedded sphere in $\widetilde{M}$ and hence bounds an embedded ball. Since orbits of $\widetilde{\Phi}$ are properly embedded in $\widetilde{M}$, it follows that all orbits of $\widetilde{\Phi}$ intersecting D_j will also intersect B_j. Hence there is product flow in this ball. Since this is true for all j then the union of the D_j flows into L_0. The union of the D_j is the closure of $g_i(L) - W'$. The image is contained in the closure of $L_0 - W$ in L_0 $-$ call the closure J.

We claim that the image is in fact J. All the τ_j are in the same leaf of $\widetilde{\Lambda}^s$ and hence all their

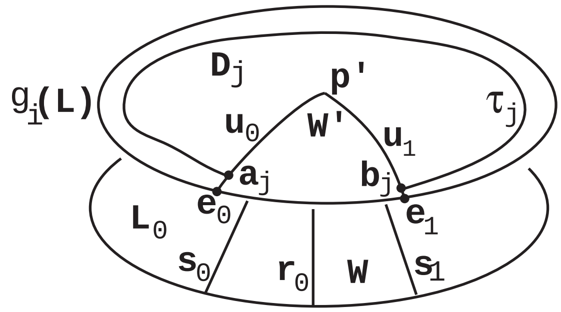

Figure 5: *Transporting the structure between leaves $g_i(L)$ and L_0.*

flow images in L_0 also are. Since rays of $\widetilde{\Lambda}^s_{L_0}$ are properly embedded in L_0 then when j converges to infinity the images of τ_j in L_0 escape compact sets. This shows the claim. Therefore the flow produces a homeomorphism between the closure of $L_0 - W$ and the closure of $g_i(L) - W'$. Clearly the same is true for any leaf in the interval associated to the contracting (non expanding) directions s_0, s_1. In particular we have the following conclusions:

Conclusion − In any limit leaf L_0 with a limit direction r_0 of bad wedges the following happens: Let c be the ideal point of r_0 and A a closed interval of $\partial_\infty L_0 - \{c\}$. Then there is a leaf l of $\widetilde{\Lambda}^s_{L_0}$ with compact subsegments l_i so that the endpoints of l_i converge to the endpoints a, b of A and l_i converges to A. In particular l_i escapes compact sets. There are also subsegments v_i with both endpoints converging to a and so that v_i converges to sets in $\partial_\infty L_0$ which contain A. Finally for sufficiently near leaves there is a wedge in L_0 which forms a product flow region with these nearby leaves.

To get the second assertion above just follow l beyond the endpoint of l_i near b until it returns near a again. As a preliminary step to obtain theorem 4.2 we prove the following:

Lemma 4.3. *Either $\mathcal{F}$ is $\mathbf{R}$-covered or for any limit $g_i(p_i)$ converging to p_0, the distinguished direction of the bad wedge associated to $g_i(p_i)$ converges to a single direction at p_0. In the second case this direction varies continuously with the leaves in $\widetilde{\mathcal{L}}$.*

Proof. Suppose there are subsequences q_i, p_i converging to points in (interior) v with $g_i(p_i), h_i(q_i) \to p_0 \in L_0 \in \widetilde{\mathcal{F}}$, but the directions of the wedges converge to r_0, r_1 distinct geodesic rays in L_0. We will first show that there is an interval of leaves of $\widetilde{\mathcal{F}}$ so that the flow $\widetilde{\Phi}$ is a product flow in this region.

Using the limit direction r_0 we produce a wedge W in L_0 so that the closure of $L_0 - W$ is part of a product flow region with nearby leaves of $\widetilde{\mathcal{F}}$. Using the other limit direction r_1 we produce a flow product region associated to another wedge region W_* disjoint from $W - p_0$. Together they produce a global product structure of the flow in a neighborhood of L_0.

This shows that there is a neighborhood N of L_0 in the leaf space of $\widetilde{\mathcal{F}}$ so that the flow is a product flow in N. In particular there is no non Hausdorffness of $\widetilde{\mathcal{F}}$ in this neighborhood. This is a very strong property as we shall see below. It implies a global product structure of the flow.

Notice that the structure of $\widetilde{\Lambda}^s_{g_i(L)}$ in $g_i(L) - W'$ flows over to L_0. In particular there are many rays of $\widetilde{\Lambda}^s_{L_0}$ which do not have a single limit in $\partial_\infty L_0$. This implies that $\pi(L_0)$ is not compact. This is because Levitt [Le] proved that given any singular foliation with prong singularities in a closed hyperbolic surface R, then the rays of the lift to $\widetilde{R}$ all have unique limit points in the ideal boundary. This shows that the minimal lamination $\mathcal{L}$ is not a compact leaf and hence it has no compact leaves.

Consider the neighborhood N as above. Consider the translates $g(N)$ where g runs through all elements of the fundamental group. Let P be the component of the union containing N. It is easy to see that the set P is precisely invariant: if g is in $\pi_1(M)$ and $g(P)$ intersects P then $g(P)$ is equal to P. In addition $\mathcal{F}$ restricted to P has leaf space homeomorphic to $\mathbf{R}$ because of the product flow property. We are assuming that N is open.

Suppose first that P is not all of $\widetilde{M}$, hence ∂P is a non empty collection of leaves of $\widetilde{\mathcal{F}}$. Let C be the projection of P to M. Then C is open, saturated by leaves of $\mathcal{F}$. Notice that $g(P)$ does not intersect ∂P for any g in $\pi_1(M)$ for otherwise $g(P)$ intersects P and so $g(P) = P$. It follows that $\pi(\partial P)$ is disjoint from C hence C is a proper open, foliated subset of M.

Dippolito [Di] developed a theory of such open, saturated subsets. Let $\overline{C}$ be the metric completion of C. There is an induced foliation in $\overline{C}$, which we will also denote by $\mathcal{F}$. Then

$$\overline{C} \ = \ V \ \cup \ \bigcup_{1}^{n} V_i$$

where V is compact and may be all of $\overline{C}$. Each nonempty V_i is an I-bundle over a non compact surface with boundary, so that $\mathcal{F}$ is a foliation transverse to the I-fibers. Each component of the intersection $\partial V_i \cap V$ is an annulus (or Moebius band) with induced foliation transverse to the I fibers. In our situation with Φ transverse to the flow, if V is not $\overline{C}$, we can choose V big enough so that the flow is transverse to $\mathcal{F}$ in each V_i and induces an I-fibration there.

Consider a component R of ∂C with lift $\widetilde{R}$ a subset of ∂P. Suppose first that R is closed. We show this is impossible, basically using holonomy. Parametrize the leaves of $\widetilde{\mathcal{F}}$ in P as $F_t, 0 < t < 1$ with t increasing with flow direction. A leaf in the boundary of P which is the limit of leaves in P which are limiting from the positive side above has to be the limit of F_t as t goes to 0: Suppose that S is in the boundary of P and there are x_i in F_{t_i} with t_i converging to $t_0 > 0$ and x_i convergint to x in S. Then S and F_{t_0} are not separated from each other. For i big enough the flow line through x_i will intersect S and therefore this flow line will not intersect F_{t_0}. This contradicts the fact that F_{t_o} and F_{t_i} have a flow product structure.

Suppose then that $\widetilde{R}$ is a limit of F_t where t converges to 0. Suppose first that R is compact. Suppose there are t_i converging to 0 so that F_{t_i} are in $\widetilde{\mathcal{L}}$. Then since $\mathcal{L}$ is a closed subset of M it follows that $\widetilde{R}$ is in $\widetilde{\mathcal{L}}$ and so R is in $\mathcal{L}$. But R is closed, contradicting the fact that $\mathcal{L}$ has no closed leaves. There is then $a > 0$ which is the smallest a so that F_a is in $\widetilde{\mathcal{L}}$ – notice that $\widetilde{\mathcal{L}}$ has leaves in P. For any g in $\pi_1(R)$ then $g(N) \cap N$ is not empty hence $g(N) = N$. It follows that $g(F_a) = F_b$ for some b. If b is not a then by taking g^{-1} if necessary we may assume that $b < a$. But as F_b is in $\widetilde{\mathcal{L}}$, this contradicts the definition of a. Hence $g(F_a) = F_a$ for any g in $\pi_1(R)$. This implies that $\pi(F_a)$ is a closed surface, again contradiction.

We conclude that R is not compact, hence it eventually enters some V_i (the point here is that V is not $\overline{C}$). The flow restricted to any component of $\partial V_i \cap \overline{C}$ goes from one component to the other in the annulus. This implies that all $\pi(F_t)$ intersect this annulus. There is then a leaf B of $\mathcal{L}$ which enters V_i. Going deeper and deeper in this non compact I-bundle will produce a limit point which is not in C. This shows the very important fact that $\mathcal{L}$ is not contained in C and therefore

$$\mathcal{E} \ = \ \mathcal{L} \cap (M - C) \ \neq \ \emptyset$$

In addition $\mathcal{E}$ is not equal to $\mathcal{L}$ since $\mathcal{L}$ has leaves in C and $(M - C)$ is closed. Hence $\mathcal{E}$ is a non trivial, proper sublamination of $\mathcal{L}$. This contradicts the fact that $\mathcal{L}$ is a minimal lamination.

This shows that the assumption $P \neq \widetilde{M}$ is impossible. Hence $P = \widetilde{M}$, which implies the flow $\widetilde{\Phi}$ produces a global product picture of $\widetilde{\mathcal{F}}$ and in particular $\mathcal{F}$ is $\mathbf{R}$-covered.

This shows that if $\mathcal{F}$ is not $\mathbf{R}$-covered, then the limits of the bad wedges are unique directions in the limit leaves. It also shows that they vary continuously from leaf to leaf, for otherwise one obtains bad wedges in very near leaves which have definitely separated directions. The same proof above then applies. This finishes the proof of lemma 4.3 □

Continuation of the proof of theorem 4.2

We continue the proof of the theorem, assuming that $\mathcal{F}$ is not $\mathbf{R}$-covered. By the previous lemma we know that limit directions of bad wedges are unique and they vary continuously in leaves of $\widetilde{\mathcal{L}}$. These unique directions are distinguished in their respective leaves.

We first show that any complementary region of $\mathcal{L}$ (if any) is an I-bundle with a product flow.

Lift to a double cover if necessary to assume that M is orientable. Assume this is the original foliation $\mathcal{F}$, flow Φ, etc.. Let Z be a leaf of $\widetilde{\mathcal{L}}$. Since Z has a distinguished ideal point, then the fundamental group of $\pi(Z)$ can be at most $\mathbf{Z}$. Since there is a transverse flow and M is orientable this implies that $\pi(Z)$ is either a plane or an annulus.

Let U be a complementary region of $\mathcal{L}$ with boundary leaves R_1, R_2, R_3, etc.. As explained before the completion of U has a compact thick part and the non compact arms which are in thin, I-bundle regions. Suppose first that R_1 is a plane. There is a big disk D so that $R_1 - D$ is contained in the thin arms and flows across U to another boundary components of U. By connectedness it flows into a single boundary component R_2 of U. Then ∂D flows into a curve γ in R_2 which is null homotopic in M. The flow segments in M produce an annulus C in the completion of U. Since $\mathcal{F}$ is Reebless then γ bounds a disk D' in R_2 and so R_2 is a plane. The union $D \cup C \cup D'$ is an embedded sphere in M which bounds a ball B. Since orbits of $\widetilde{\Phi}$ are properly embedded in $\widetilde{M}$, it follows that the flow has to a product flow in B as well. This shows that flow is a product in the completion of U.

Suppose now that each R_i is an annulus. Let F be a lift of R_1 to $\widetilde{M}$ with F in the boundary of a component $\widetilde{U}$ of $\pi^{-1}(U)$. In R_1 there are two disjoint open annuli A_1, A_2 contained in the thin arms so that $B = R_1 - (A_1 \cup A_2)$ is a closed annulus in the core. Then A_1, A_2 flow into two annuli leaves R_2, R_3 in the boundary of U. Lifting to $F = \widetilde{R}_1$ we see leaves of $\widetilde{\Lambda}^s_F$ limiting in an interval of $\partial_{\infty} F$ with very small complement (near the distinguished ideal point of F). This implies they will have points in the lifts $\widetilde{A}_1, \widetilde{A}_2$ of A_1, A_2 to F. This shows that $\widetilde{A}_1, \widetilde{A}_2$ are in the same leaf of $\widetilde{\mathcal{F}}$. This implies that $R_2 = R_3$. In the same way a half of the infinite strip $\widetilde{B}$ flows into $\widetilde{R}_2$. Since B is compact, then all of B flows into R_2. This implies that the region U is an I-bundle. It is also easy to show that the flow is a product in this I-bundle.

This implies that we can collapse this complementary region along flow lines to completely eliminate it. This is because even in the universal cover we are eliminating product regions of the flow and the asymptotic behavior is stil preserved in the remaining regions. This can be done to all complementary regions and therefore we can assume there are no complementary regions, that is $\mathcal{L} = \mathcal{F}$ or that $\mathcal{F}$ is minimal.

Suppose now that $\mathcal{F}$ is not $\mathbf{R}$-covered. Let F_1, F_2 be leaves of $\widetilde{\mathcal{F}}$ which are not separated from each other. Consider leaves F of $\widetilde{\mathcal{F}}$ which are very close to points in both F_1 and F_2. As stated in the conclusion in the beginning of the proof of this theorem, there is a wedge of F which flows into F_1 and similarly for F_2. Hence there are half planes E_1, E_2 of F which flow into F_1, F_2. As F_1, F_2 are not separated this implies that E_1, E_2 are disjoint. Fix a point w in F and a big enough radius r so that the disk D of radius r around w intersects both E_1, E_2. Again as seen in the conclusion above there is an arc l in a leaf of $\widetilde{\Lambda}^s_F$ so that both endpoints of l are outside D and in E_1 and so that l is entirely outside D and as seen from p the visual measure of l is almost 2π. This implies that l intersects E_2. Since the endpoints of l are in E_1, which flows to F_1, then proposition 3.2 implies that the whole arc l flows into F_1. The points of l in E_2 will also flow to F_2. This is a contradiction.

This contradiction shows that $\mathcal{F}$ has to be $\mathbf{R}$-covered and finishes the proof of theorem 4.2 $\square$

Theorem 4.4. *Let $\mathcal{F}$ be an $\mathbf{R}$-covered foliation and Φ be a pseudo-Anosov flow almost transverse to $\mathcal{F}$. Then Φ is in fact transverse to $\mathcal{F}$. In addition for any leaf F of $\widetilde{\mathcal{F}}$ and for any ray l in $\widetilde{\Lambda}^s_F$ it converges to a unique ideal point in $\partial_\infty F$.*

Proof. If Φ is not transverse to $\mathcal{F}$, let Φ^* be an almost pseudo-Anosov flow which is transverse to $\mathcal{F}$ and is a blow up of Φ. There is flow annulus A of Φ^* with closed orbits γ_1, γ_2 in the boundary, so that A blows down to a single orbit of Φ.

The foliation induced by $\mathcal{F}$ in A has leaves which spiral to at least one boundary component − which they do not intersect. Lifting this picture to the universal cover one obtains an orbit of $\widetilde{\Phi}^*$ which does not intersect every leaf of $\widetilde{\mathcal{F}}$. This means that the flow $\widetilde{\Phi}^*$ is not *regulating* for $\widetilde{\mathcal{F}}$ [Th2, Th4]. We also say that Φ^* does not regulate $\mathcal{F}$. In [Fe11] we analysed a similar situation and proved the following: if Ψ is a pseudo-Anosov flow transverse to an $\mathbf{R}$-covered foliation and Ψ is not regulating, then Ψ is an $\mathbf{R}$-covered Anosov flow. The same arguments work with an almost pseudo-Anosov flow transverse to an $\mathbf{R}$-covered foliation. This shows that Φ^* is an $\mathbf{R}$-covered Anosov flow and has no (topological) singularities. In particular Φ^* is equal to Φ, that is the original flow is already transverse to $\mathcal{F}$. This proves the first assertion of the theorem.

Assume by way of contradiction that there is L' in $\widetilde{\Lambda}^s$ and l in $\widetilde{\Lambda}^s_{L'}$ which does not converge to a single point in $\partial_\infty L'$. As in the proof of theorem 4.2 we construct a minimal sublamination $\mathcal{L}$ of $\mathcal{F}$ such that: for every L in $\widetilde{\mathcal{L}}$ there is an ideal point u in $\partial_\infty L$ so that for every closed segment J in $\partial_\infty L - \{u\}$ there is a ray l of $\widetilde{\Lambda}^s_L$ which has subsegments limiting to J.. As shown in the proof of theorem 4.2, $\mathcal{L}$ cannot be a compact leaf.

Suppose first that every leaf of $\mathcal{F}$ is a plane. Then Rosenberg [Ros] proved that M is the 3-dimensional torus T^3. This manifold is a Seifert fibered space. In this case Brittenham [Br1] proved that an essential lamination is isotopic to one which is either vertical (a union of Seifert fibers) or horizontal (transverse to the fibers). So after isotopy assume $\mathcal{L}$ has one of these types. If $\mathcal{L}$ has a vertical leaf B, then geometrically it is a product of the reals with the circle. Hence it is an Euclidean leaf and in the universal cover it has polynomial growth of area. If $\mathcal{L}$ has a horizontal leaf B, then because the fibration is a product, there is a projection to a T^2 fiber, which distorts distances by a bounded amount. Again the same growth properties hold. But the leaves of $\mathcal{F}$ are hyperbolic, which is a contradiction. We conclude that M cannot be T^3.

Let then F in $\widetilde{\mathcal{L}}$ with $\pi(F)$ not simply connected. Let g in $\pi_1(M)$ non trivial with $g(F) = F$ and ξ be the axis of g in F. At least one ideal point of ξ, call it u, is not the direction of a fixed limit of bad wedges. Then as explained before there is a ray l of $\widetilde{\Lambda}^s_F$ and segments l_i of l, bounded by a_i, b_i both points in ξ, so that l_i escapes compact sets and converges to a non trivial segment in $\partial_\infty F$. We may assume that $l_i \cap \xi = \{a_i, b_i\}$ and also that all l_i are in the same side of ξ. Let e_0 be the translation length of g in F.

If the distance from a_i to b_i along ξ is bigger than e_0 then this produces a contradiction as follows: There is an integer n so that $g^n(a_i)$ is in the open segment (a_i, b_i) of ξ and and $g^n(b_i)$ is outside of the closed segment $[a_i, b_i]$. Since the arc l_i only intersects ξ in a_i, b_i, then l_i, together with $[a_i, b_i]$ bounds a closed disk in F and $g^n(a_i)$ is in (a_i, b_i). But $g^n(b_i)$ is outside and $g^n(l_i)$ is also on this side of ξ, so this produces a transverse self intersection of $\widetilde{\Lambda}^s_F$. If $g^n(l_i)$ is contained in the leaf v which contains l_i, then $g^n(v) = v$ and this produces infinitely many singularities in v, which is impossible. Hence $g^n(l_i)$ is not in v and the transverse intersection is impossible. The same arguments deal with the case that l_i intersects ξ in other points besides a_i, b_i.

We conclude that the distance in ξ from a_i to b_i is bounded. Up to subsequence we may assume there are integers n_i so that $g^{n_i}(a_i)$ converges to a_0 and $g^{n_i}(b_i)$ converges to b_0, both limits in ξ of course. Since the lengths of $g^{n_i}(l_i)$ are converging to infinity, it follows that a_0, b_0 are not in the same leaf of $\widetilde{\Lambda}^s_F$. By proposition 3.2 it follows that a_0, b_0 are not in the same leaf of $\widetilde{\Lambda}^s$. But for each i, the pair of points $g^{n_i}(a_i), g^{n_i}(b_i)$ is in the same leaf of $\widetilde{\Lambda}^s$. This implies that the leaf space of $\widetilde{\Lambda}^s$ is not Hausdorff.

First of all this implies that Φ is regulating for $\mathcal{F}$, for otherwise the aforementioned result from [Fe11] shows that Φ is an $\mathbf{R}$-covered Anosov flow $-$ in particular $\widetilde{\Lambda}^s$ has Hausdorff leaf space. Also by theorem 2.6 the fact that $\widetilde{\Lambda}^s$ has non Hausdorff leaf space implies that there are closed orbits α, β of Φ so that α is freely homotopic to the inverse of β. Let h be a covering translation associated to α and $\widetilde{\alpha}$, $\widetilde{\beta}$ lifts of α, β to $\widetilde{M}$ which are left invariant by h. Without loss of generality assume that h acts in $\widetilde{\alpha}$ sending points forwards. As $\alpha \cong \beta^{-1}$ this implies that h acts on $\widetilde{\beta}$ taking points backwards. But since both of them intersects all leaves of $\widetilde{\mathcal{F}}$ (by the regulating property) then as seen from $\widetilde{\alpha}$ the translation h acts increasingly in the leaf space of $\widetilde{\mathcal{F}}$, with opposite behavior when considering $\widetilde{\beta}$. This is a contradiction, which shows that this cannot happen. This finishes the proof of theorem 4.4. $\square$

Remark - Group invariance and compactness of M are both essential here. For example start with a nicely behaved singular foliation of $\mathbf{H}^2$, so that all rays converge. It could be a foliation by geodesics or for instance the lift of the stable singular foliation associated to a suspension. Fix a base point p. Now rotate the leaves at a distance d of p by an angle d. In this situation all rays limit in all points of $\partial_\infty L$, in fact they spiral indefinitely into it. Another operation is to fix a ray through p and then distort the rest more and more one way and the other way. Here we have the leaves getting closer and closer to segments in $\partial_\infty F$ which are complementary to the ideal point associated to the ray.

5 Properties of leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ and their ideal points

In this section Φ is an almost pseudo-Anosov flow transverse to a foliation $\mathcal{F}$. As in the previous section there is no restriction on M here. In the previous section we proved that for any ray r of a leaf of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$, then it has a unique ideal point in $\partial_\infty F$. The notation for this ideal point will be r_∞. We now analyse further properties of leaves of $\widetilde{\Lambda}^s_F$ and their ideal points. Analogous results hold for $\widetilde{\Lambda}^u_F$.

First we want to show that if E is a fixed leaf of $\widetilde{\Lambda}^s$ (or $\widetilde{\Lambda}^u$) then the ideal points in $\partial_\infty F$ of rays of $E \cap F$ vary continuously with F. In order to do that we first put a topology on the union of ideal boundaries of an interval of leaves. Let p in F leaf of $\widetilde{\mathcal{F}}$ and τ a transversal to $\widetilde{\mathcal{F}}$ with p in the interior. For any L in $\widetilde{\mathcal{F}}$ intersecting τ, the ideal boundary is in 1-1 correspondence with the unit tangent bundle to L at $\tau \cap L$: ideal points correspond to rays in L starting at $L \cap \tau$. This is a homeomorphism. This puts a topology in

$$\mathcal{A} \ = \ \cup \, \{\partial_\infty L \ \mid \ L \cap \tau \neq \emptyset\}$$

making it into an annulus homeomorphic to $\cup \, \{T^1_q \widetilde{\mathcal{F}}, q \in \tau\}$ as a subspace of the unit tangent bundle of M. This topology in $\mathcal{A}$ is independent of the choice of transversal τ. The following definition/result is proved in [Fe9] or [Cal1].

Definition 5.1. *(markers) Given a foliation $\mathcal{F}$ by hyperbolic leaves of M^3 closed, then there is $\epsilon > 0$ so that: Let v be a geodesic ray in a leaf F so that it is associated to a contracting (or ϵ non expanding direction of F. For any leaf L sufficiently near F, then all the points of v flow into L and define a curve denoted by v_L. Then v_L has a unique ideal point denoted by a_L. The union m of the a_L is*

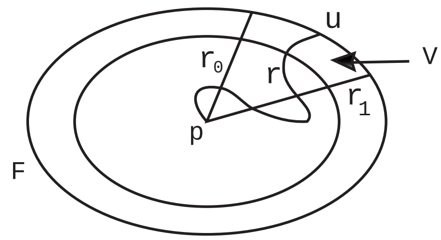

Figure 6: *Leaf in wedge defined by markers.*

called a marker and is a subset of $\mathcal{A} = \cup \{\partial_\infty L\}$. Then m is an embedded curve in $\mathcal{A}$ in the topology defined above.

In addition the markers are dense in $\mathcal{A}$ in the following sense: Let z in $\partial_\infty F$ and a_i, b_i in $\partial_\infty F$ which are in markers associated to contracting (non expanding) directions on a fixed side of F. Suppose that the sequence of open intervals (a_i, b_i) in $\partial_\infty F$ contains z and converges to z as i converges to infinity. Let α_i, β_i be the markers in that side of $\partial_\infty F$ containing a_i, b_i respectively. Let L_i in $\widetilde{\mathcal{F}}$ be a sequnence of leaves converging to F and on that side of F so that $\partial_\infty L_i$ intersects both α_i and β_i. In the annulus $\mathcal{A}$ of circles at infinity, consider the rectangle R_i bounded by (a_i, b_i) in $\partial_\infty F$, the parts of α_i, β_i between $\partial_\infty F$ and $\partial_\infty L_i$ and the small segment in $\partial_\infty L_i$ bounded by $\partial_\infty L_i \cap \alpha_i$ and $\partial_\infty L_i \cap \beta_i$. Then the sets R_i converge to z as i converges to infinity. This is proved in [Fe9].

From now on the ϵ is chosen small enough to also satisfy the conclusions of the definition above and also that any set in $\widetilde{M}$ of diameter less than 10ϵ is in a product box of $\widetilde{\mathcal{F}}$ and $\widetilde{\Phi}$. Given a curve ζ in a leaf F with starting point p and limiting on a unique point q in $\partial_\infty F$, let ζ^* denote the geodesic ray of F with same starting and ideal points.

Lemma 5.2. *Let E be a leaf of $\widetilde{\Lambda}^s$ and p the starting point of the ray r of $E \cap F$. Assume that r does not have any singularity. For any L near F, then $E \cap L$ has a ray r_L which is near r. The ideal points of r_L in $\partial_\infty L$ vary continuously with L in the topology of $\mathcal{A}$ defined above.*

Proof. We do the proof for say the positive side of F. We consider r without singularity or else we would have to check the 2 exterior rays in $\widetilde{\Lambda}^s_F$ emanating from p. We can always get a subray of r which has no singularities.

Let $u = r_\infty$. Choose contracting (or ϵ non expanding) directions in both sides of u, with ideal points very close to u. Let them be defined by geodesic rays r_0, r_1 starting at p. There is τ a small flow segment starting at p and in that side of F so that for any L intersecting τ, then L is asymptotic to F along the r_0, r_1 rays, or at least always $\leq \epsilon$ from F. Hence r_0, r_1 flow along $\widetilde{\Phi}$ to L. Let s_0, s_1 be the flow images in L. The ϵ is also chosen small enough so that s_0, s_1 have geodesic curvature very small (this ϵ depends only on M and $\mathcal{F}$). In particular the curves s_0, s_1 are a small bounded distance (depending only on ϵ) from the corresponding geodesic arcs s^*_0, s^*_1. Let the ideal points of s_0, s_1 in $\partial_\infty L$ be denoted by v_0, v_1 and let J_L be the small closed interval in $\partial_\infty L$ bounded by v_0, v_1. Then v_0, v_1 are in the markers associated to r_0, r_1 respectively and so they vary continuously with L.

Consider $\xi = E \cap L$ and the rays l of ξ starting at $\tau \cap L$ and containing some points which flow back to points in r. It may be that ξ has singularities – even if r does not – but there are only finitely many such rays. We want to prove that the ideal point of any such is in J_L. As the rectangles R_i defined above converge to u in $\mathcal{A}$ this will prove the continuity property of the lemma.

Choose $d > 0$ so that outside of a disk D of radius d in F, then r is in the small wedge W of F defined by r_0, r_1, see fig. 6. Choose τ small enough so that if L intersects τ, then the entire disk D

is ϵ near L. Let V be the closure in F of $W - D$. The boundary ∂V consists of subrays of r_0, r_1 and an arc in ∂D. Therefore all points in ∂V are less than ϵ from L and flow to L under $\widetilde{\Phi}$ with image a curve γ. This curve contains subrays of s_0, s_1 and it is properly embedded in L. Points of F near ∂V also flow to L so there is a unique component U of $L - \gamma$ which has some points flowing back to points in V. We want to show that the ray l is eventually contained in U.

Let r_{init} be the subarc of r between p and the last point c_0 of r in D. As p and c_0 flow into L, then proposition 3.2 shows that the entire arc r_{init} flows into L and let δ be its image in L. As r is singularity free, then so is δ and hence δ is contained in any ray l of $E \cap L$ in that direction. After c_0 the curve r enters V and so l must enter U after δ. If after that the ray l exits U then it must cross $\partial U = \gamma$ in some point, call it c_1. But c_1 flows back to F and one can apply proposition 3.2 again in the backwards direction to show that c_1 has to flow to a point in r. This contradicts the choice of c_0.

This shows that l is eventually entirely contained in U and therefore l_∞ is a point in J_L. This shows the continuity property as desired and finishes the proof of the lemma. ☐

Now we have a property which will be crucial to a lot of our analysis.

Proposition 5.3. *Suppose that $\mathcal{F}$ is not topologically conjugate to the stable foliation of a suspension Anosov flow. Then the set of ideal points of rays of $\widetilde{\Lambda}^s_F$ is dense in $\partial_\infty F$.*

Proof. Suppose that there is F in $\widetilde{\mathcal{F}}$ so that the set of ideal points in $\widetilde{\Lambda}^s_F$ is not dense in $\partial_\infty F$. Let J be an open interval in $\partial_\infty F$ free of such ideal points. Choose p_i in F, p_i converging to a point in J. The visual angle of J as seen from p_i converges to 2π, so the complementary wedge W_i with corner p_i has angle which converges to zero. Up to subsequence assume that $g_i(p_i)$ converges to p_0 in a leaf L of $\widetilde{\mathcal{F}}$ and the small wedges $g_i(W_i)$ converge to a geodesic ray s in L with ideal point z.

Claim – In L all the rays of $\widetilde{\Lambda}^s_L$ converge to z.

Suppose there is x different from z which is an ideal point of a ray r in $\widetilde{\Lambda}^s_L$. Then r is contained in $\widetilde{W}^s(c_0)$ for some c_0 in $\widetilde{M}$ and for $g_i(F)$ sufficiently near L then $\widetilde{W}^s(c_0)$ intersects $g_i(F_0)$. Any ray of $\widetilde{W}^s(c_0) \cap g_i(F)$ which is near r will have ideal point near x in the topology of corresponding annulus $\mathcal{A}$ of ideal circles near $\partial_\infty L$. This is a consequence of the previous lemma. But $g_i(W_i)$ converges to r in this topology of $\mathcal{A}$, so the sets $g_i(\partial_\infty F - J)$ converge to z in $\mathcal{A}$. There are no ideal points of leaves of $\widetilde{\Lambda}^s_{g_i(F)}$ in $g_i(J)$. This contradicts the fact that the ideal points above are very near x and proves the claim.

The proof of the proposition is similar to that of theorem 4.2. As in that theorem consider the set of possible limits $g_i(p_i)$ as above. This projects to a lamination in M and let $\mathcal{L}$ be a minimal sublamination. The claim shows that each leaf of $\widetilde{\mathcal{L}}$ has a distinguished ideal point towards which all rays of $\widetilde{\Lambda}^s_L$ converge. The arguments in the claim also prove that if τ is a transversal to $\widetilde{\mathcal{F}}$, then the ideal points of leaves of $\widetilde{\mathcal{L}}$ intersecting τ vary continuously in the corresponding ideal annulus. Because of the distinguished ideal point property, then each leaf of $\mathcal{L}$ has fundamental group at most $\mathbf{Z}$. If needed lift to a double cover so that all leaves of $\mathcal{F}$ are orientable. Hence a leaf of $\mathcal{L}$ is either a plane or an annulus.

Consider a complementary component U of $\mathcal{L}$ and a boundary leaf A of U. If A is a plane then as in the proof of theorem 4.2, the region U is an I-bundle over A and the flow Φ is a product in U. This region can be collapsed away.

Suppose now that A is an annulus. Assume that flow lines through A flow into U. Again we want to show that U is a product region. As in the proof of theorem 4.2 let A_1, A_2 be two noncompact, disjoint annuli in A with $A - (A_1 \cup A_2)$ a compact annulus and A_1, A_2 contained in the thin, I-bundle

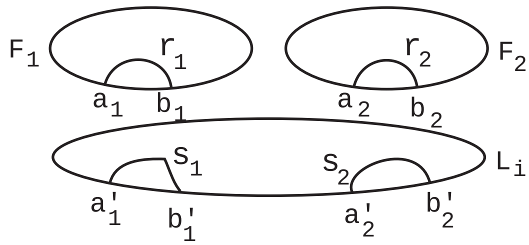

Figure 7: *Pushing ideal points near.*

region. Then A_1, A_2 flow entirely into leaves B and C in ∂U. Suppose first that B, C are different. Lift to the universal cover to produce lifts $\widetilde{U}, \widetilde{A}, \widetilde{A}_1, \widetilde{A}_2, \widetilde{B}, \widetilde{C}$. Then $\widetilde{A}_1, \widetilde{A}_2$ are disjoint half planes of $\widetilde{A}$ which flow positively respectively into $\widetilde{B}$ and $\widetilde{C}$. Let g be the generator of the isotropy group of $\widetilde{A}$, which has fixed points in z, x where z is the distinguished ideal point in $\widetilde{A}$. The argument will show there is a leaf in $\widetilde{\Lambda}^s_{\widetilde{A}}$ which also has ideal point in x, contradiction.

From a point in $\widetilde{A}_1$ draw a geodesic segment of $\widetilde{A}$ to a point in $\widetilde{A}_2$. Let p be the first point of this segment which does not flow positively into $\widetilde{B}$. Then $\Theta(p)$ is in the boundary of $\Theta(\widetilde{B})$. Also points in the segment near p flow to $\widetilde{B}$ in positive time, hence there is a slice leaf l of $\mathcal{O}^s(\Theta(p))$ which is in the boundary of $\Theta(\widetilde{B})$. Notice that every point in l is a limit of points in $\Theta(B)$ on that side. The set $(l \times \mathbf{R})$ intersects $\widetilde{A}$ in at least p: if l is contained in $\Theta(\widetilde{A})$ then it generates a properly embedded copy of the reals in a leaf s of $\widetilde{\Lambda}^s_{\widetilde{A}}$ otherwise the part that is contained in $\Theta(\widetilde{A})$ also does. Every point of s is a limit of points that flow positively into $\widetilde{B}$. Therefore no point in s can flow positively in $\widetilde{C}$ or else we would have points flowing both in $\widetilde{B}$ and $\widetilde{C}$.

This shows that the leaf s of $\widetilde{\Lambda}^s_{\widetilde{A}}$ is a bounded distance from the axis r of g. Iterate s by powers of g acting with z as an expanding fixed point. The iterates $g^n(s)$ with $n > 0$ are all distinct. Either they are all nested or they are disjoint. If they are not nested since they all have to be in a bounded distance neighborhood of the axis of g and have both endpoints in z, then eventually they will have two points which are far along the leaf, but close in $\widetilde{A}$. By Euler characteristic reasons, this would force a center or one prong singularity, which is impossible. Hence they are nested, increasing and they limit to a leaf of $\widetilde{\Lambda}^s_{\widetilde{A}}$ which has ideal limit points in z and x. This is a contradiction. This shows that $B = C$. In fact the same arguments show that all of the points in A flow into B, since that happens for the complement of a compact annulus in A and then the arguments above apply here. Hence U is a product region. Therefore we can collapse $\mathcal{F}$ to a minimal foliation.

As in theorem 4.2 we can then show that $\mathcal{F}$ is $\mathbf{R}$-covered. Suppose this is not the case and let F_1, F_2 be non separated leaves. Let L_i in $\widetilde{\mathcal{F}}$ leaves converging to both F_1, F_2. Let u_1, u_2 be the distinguished ideal points in $\partial_\infty F_1$, $\partial_\infty F_2$ respectively. Let a_1, b_1 be points in $\partial_\infty F_1$ very near u_1 and on opposite sides of u_1 and which are in markers associated to contracting or ϵ non expanding directions in F_1 associated to the L_i side. Let r_1 be the geodesic in F_1 with ideal points a_1, b_1. Similarly for F_2 producing a_2, b_2, r_2. For i big enough L_i is at most ϵ far from all points in r_1, r_2. Therefore r_1 flows (by $\widetilde{\Phi}$) into a curve s_1 in L_i and r_2 flows into s_2. This implies that s_1, s_2 are disjoint in L_i. Also s_1 has ideal points a'_1, b'_1 which are in markers containing a_1, b_1 respectively (this is using a transversal to $\widetilde{\mathcal{F}}$ through a point in F_1). Similarly s_2 has ideal points a'_2, b'_2 in markers containing a_2, b_2 (using transversal to $\widetilde{\mathcal{F}}$ through a point in F_2). As s_1, s_2 are disjoint then a'_1, b'_1 do not link a'_2, b'_2 in $\partial_\infty L_i$, see fig. 7.

The ideal point a'_1 cannot be in a marker to $\partial_\infty F_1$ and to $\partial_\infty F_2$ at the same time since they are non separated leaves. Hence the points a'_1, b'_1, a'_2, b'_2 are all distinct. Let J_1 be the interval of $\partial_\infty L_i$

bounded by a'_1, b'_1 and not containing the other points and similarly define J_2. For simplicity we are ommitting the dependence of J_1, J_2 on L_i (or on i). Now consider E a leaf of $\widetilde{\Lambda}^s$ intersecting F_1. Then $E \cap F_1$ has a ray with ideal point u_1, which is in the interval (a_1, b_1) of $\partial_\infty F_1$. The proof of lemma 5.2 shows that if L_i is close enough to F_1 then the ideal points of the corresponding rays of $(E \cap L_i)$ have to be in J_1. In the same way using F_2 one shows that the distinguished ideal point has to be in J_2. Since J_1, J_2 are disjoint, this is a contradiction. This shows that $\mathcal{F}$ is $\mathbf{R}$-covered.

Since $\mathcal{F}$ is $\mathbf{R}$-covered then theorem 4.4 implies that Φ can be chosen to be a pseudo-Anosov flow.

Also as $\mathcal{F}$ is $\mathbf{R}$-covered we can choose a transversal τ intersecting all the leaves of $\widetilde{\mathcal{F}}$. This shows that the union of all the circles at infinity has a natural topology making it into a cylinder $\mathcal{A}$. This situation of $\mathbf{R}$-covered foliations is carefully analysed in [Fe9]. The fundamental group of M acts in $\mathcal{A}$ by homeomorphisms. The union of the distinguished ideal points of leaves of the distinct leaves of $\widetilde{\mathcal{F}}$ is a continuous curve ζ in $\mathcal{A}$ which is group invariant.

Suppose first that $\mathcal{F}$ admits a holonomy invariant transverse measure. Since $\mathcal{F}$ is minimal then the transverse measure has full support. Under these conditions Imanishi [Im] proved that M fibers over the circle with fiber a closed surface. In addition $\mathcal{F}$ is approximated arbitrarily near by a a fibration. The pseudo-Anosov flow is also transverse to these nearby fibrations and so the same situation occurs for the fibrations: there is a global invariant curve in the cylinder at infinity. Since now there are compact leaves, this is impossible.

We conclude that there is no holonomy invariant transverse measure. Therefore Thurston's theorem shows the existence of contracting directions and not just ϵ non expanding directions. So the markers are associated to contracting directions. If ζ intersects a marker m, that corresponds to a direction in a leaf of $\widetilde{\mathcal{F}}$ which is contracting. Under the flow $\widetilde{\Phi}$ this gets reflected in the contracted leaves nearby, that is the marker is contained in ζ. Since $\mathcal{F}$ is minimal and ζ is $\pi_1(M)$ invariant, this shows that the entire curve ζ is a marker associated to contracting directions. The results from [Fe9] apply here, in particular lemma 3.17 through proposition 3.21 of [Fe9]: they show that no other direction in $\widetilde{\mathcal{F}}$ (outside of ζ) is a contracting direction. By Thurston's theorem again, there would be a holonomy invariant transverse measure, contradiction.

Therefore ζ has no contracting directions. The same analysis of [Fe9] now shows that for any leaf F in $\widetilde{\mathcal{F}}$ and every direction other than the distinguished direction, then it is a contracting direction. In fact it is a contracting direction with any other leaf of the foliation.

This is a very interesting situation. Let a_F be the distinguished ideal point of F leaf of $\widetilde{\mathcal{F}}$. Consider a one dimensional foliation in $\widetilde{M}$ whose leaves are geodesics in leaves F of $\widetilde{\mathcal{F}}$ and which have one ideal point a_F. Let $\widetilde{\xi}$ be the flow which is unit speed tangent to this foliation and moves towards the ideal point a_F.

This is a flow in $\widetilde{M}$. Clearly in each leaf of $\widetilde{\mathcal{F}}$, it is a smooth flow. If q_i in L_i of $\widetilde{\mathcal{F}}$ converge to q in L, then the geodesics of L_i with ideal point a_{L_i} converge to the geodesic through q in L with ideal point a_L. This is because the ideal points a_F vary continuously with F and q_i converges to q $-$ this is the local trivialization of the union of the circles at infinity using the tangent bundles to a transversal. Hence $\widetilde{\xi}$ varies continuously.

Since ζ is group invariant, this induces a flow in M, which is tangent to the foliation $\mathcal{F}$. Clearly it is smooth along the leaves of $\mathcal{F}$ and usually just continuous in the transverse direction.

This flow is a topological Anosov flow: the stable foliation is just the original foliation $\mathcal{F}$. The unstable foliation: Let p in leaf L of $\widetilde{\mathcal{F}}$, let γ be the flow line of $\widetilde{\xi}$ through p. Then γ has positive ideal point a_L and negative ideal point v. As explained above v is in a marker m which is associated to a contracting direction and so that m intersects all ideal circles. For each F in $\widetilde{\mathcal{F}}$, let m_F be the intersection of m and $\partial_\infty F$. Let γ_F be the geodesic in F with ideal points a_F and m_F. Let E_p be the union of these γ_F. Then all orbits of $\widetilde{\xi}$ in E_p are backwards asymptotic by construction. By construction the E_p are either disjoint or equal as p varies in $\widetilde{M}$ and they form a group invariant foliation in $\widetilde{M}$. This is the unstable foliation. Hence ξ is a topologically Anosov flow. Notice that in the universal cover every stable leaf intersects every unstable leaf and vice versa.

By proposition 2.8 it follows that ξ is topologically conjugate to a suspension Anosov flow. The foliation $\mathcal{F}$ is then topologically conjugate to the stable foliation of this flow. This finishes the proof of this proposition. □

Remark – The hypothesis is necessary. Suppose that $\mathcal{F}$ is the stable foliation of a suspension Anosov flow, ξ so that it is transversely orientable. Perturb the flow slightly so that flow lines are still tangent to the original <u>unstable</u> foliation of ξ. The new flow, call it Φ is transverse to $\mathcal{F}$, it has the same unstable foliation as ξ but different stable foliation. The flow Φ is not regulating for $\mathcal{F}$. The intersections of leaves of $\widetilde{\Lambda}^s$ with leaves F of $\widetilde{\mathcal{F}}$ are all horocycles with the same ideal point which is the positive ideal point of flow lines in F. So the ideal points of rays of leaves of $\widetilde{\Lambda}^s_F$ are not dense in $\partial_\infty F$. Notice these leaves are not quasigeodesics in F either. This example is studied in detail in section 7 of [Fe11].

Now we want to study metric properties of slices of leaves of $\widetilde{\Lambda}^s_F$. The best metric property such leaves could have is that they are *quasigeodesic*: this means that length along the curve is at most a bounded multiplicative distortion of length in the leaf F of $\widetilde{\mathcal{F}}$ [Th1, Gr, Gh-Ha, CDP]. If the bound is k then we say the curve is a k-quasigeodesic. Since F is hyperbolic this would imply that such leaves (the non singular ones) are a bounded distance from true geodesics. Very unfortunate for us, this is not true in general. But there are still some good properties.

Let $\mathcal{H}^s$ be the leaf space of $\widetilde{\Lambda}^s$ and $\mathcal{H}^u$ be the leaf space of $\widetilde{\Lambda}^u$. Clearly since $\mathcal{H}^s$ may be non Hausdorff, it could be that some $\widetilde{\Lambda}^s_F$ does not have Hausdorff leaf space. This easily would imply that the slices of $\widetilde{\Lambda}^s_F$ are not uniformly quasigeodesic [Fe2]. This in fact occurs, see Mosher [Mo1, Mo3]. Still it could be that given a ray in $\widetilde{\Lambda}^s_F$, it is a quasigeodesic – with the quasigeodesic constant depending on the particular ray. We are not able to prove this and we cannot conjecture what happens in generality. But we are able to prove a weaker property, which will be enough for our purposes. If r is a ray in a leaf of $\widetilde{\Lambda}^s_F$, recall that r^* is the unique geodesic ray in F with same starting point as r and same ideal point. We would like to prove that r, r^* are a bounded distance apart, but we do not know if that is true. But we can prove the following important property:

Lemma 5.4. *There is $\delta_0 > 0$ so that for any F in $\widetilde{\mathcal{F}}$ and any ray r in a leaf of $\widetilde{\Lambda}^s_F$, then given any segment of length δ_0 in r^*, there is a point in this segment which is less than δ_0 from r in F. That implies that r^* is in the neighborhood of radius $2\delta_0$ of r in F.*

Proof. This means that $r^* \subset N_{2\delta_0}(r)$ in F. We do not know if the converse holds. Suppose the lemma is not true. Then there are F_i leaves of $\widetilde{\mathcal{F}}$, r_i rays of $\widetilde{\Lambda}^s_{F_i}$ and p_i in r^*_i so that $B_{2i}(p_i)$ (in F_i) does not intersect r_i. There is one side of r^*_i in F_i so that r_i goes around that side, see fig. 8, a. Let q_i inside a half disk of $B_{2i}(p_i)$ with $B_i(q_i)$ tangent to r^*_i and $\partial B_{2i}(p_i)$, see fig. 8, a.

As usual up to subsequence there are g_i in $\pi_1(M)$ with $g_i(q_i)$ converging to q_0 in L leaf of $\widetilde{\mathcal{F}}$ and so that the geodesic segments ζ_i from $g_i(q_i)$ to $g_i(p_i)$ in F_i converge to a geodesic ray s in L. Choose two markers with points u_0, u_1 in $\partial_\infty L$ very close to s_∞ and on opposite sides of it. The markers are associated to the side of L where the $g_i(F_i)$ are limiting to. Let s_0, s_1 be the geodesic rays of L starting at q_0 and with ideal points u_0, u_1. For i big enough $g_i(F_i)$ is ϵ close to both s_0 and s_1 and so these two rays flow (under $\widetilde{\Phi}$) to curves s'_0, s'_1 in $g_i(F_i)$. The ideal points u'_0, u'_1 of s'_0, s'_1 are in the markers above.

For i big enough the ray $g_i(r_i)$ has a subray which goes around $g_i(B_i(q_i))$ in $g_i(F_i)$ and has ideal point in the small segment of $\partial_\infty g_i(F_i)$ defined by u'_0, u'_1, see fig. 8, b. Since s'_0, s'_1 flows back to L

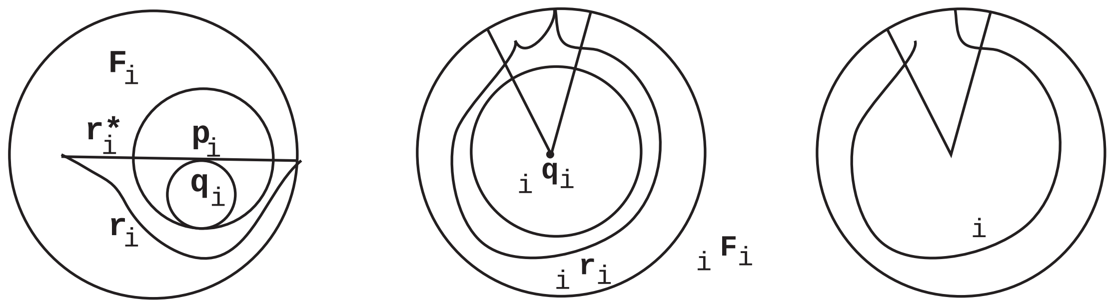

Figure 8: *a. Limits of points, b. Going around disks in F_i, c The picture in L.*

this figure flows back to L producing a ray l_i of $\widetilde{\Lambda}^s_L$ which goes around a big disk in L centered at q_0 and has ideal point in the small segment bounded by u_0, u_1, see fig. 8, c. As i goes to infinity, these l_i escape to infinity in L because bigger and bigger disks in $g_i(F_i)$ flow to L. This implies that there is no ideal point of a ray of $\widetilde{\Lambda}^s_L$ outside the small segment of $\partial_\infty L$ bounded by u_0, u_1. This contradicts the previous proposition that such ideal points are dense in $\partial_\infty L$.

This finishes the proof of the lemma. □

Lemma 5.5. *The limit points of rays of $\widetilde{\Lambda}^s_F$ vary continuously in $\partial_\infty F$ except for the non Hausdorffness in the leaf space of $\widetilde{\Lambda}^s_F$.*

Proof. Suppose that p_i converges to p in F, with respective rays r_i converging to the ray r of $\widetilde{\Lambda}^s_F$. Let l be the leaf of $\widetilde{\Lambda}^s_F$ through p. Up to subsequence assume the r_i are all in the same sector of l defined by p and that they form a nested sequence of rays. Then the ideal ponts $(r_i)_\infty$ form a monotone sequence in $\partial_\infty F$. Perhaps some ideal points are the same. If $(r_i)_\infty$ does not converge to r_∞ there is an interval v in $\partial_\infty F$, between the limit and r_∞. Since the ideal points are dense in $\partial_\infty F$, there is w leaf of $\widetilde{\Lambda}^s_F$ with w_∞ in v. Therefore there is l' not separated from l with r_i converging to l' as well. In this fashion we can go from l to l'. This shows that if there is no leaf of $\widetilde{\Lambda}^s_F$ non separated from l in that side and in the direction the rays r_i go, then the limit points vary continuously.

We analyse a bit further the non Hausdorffness. In the setup above there are subrays of r_i with points converging to a point in l' and we can restart the analysis with l' instead of l. If there are finitely many leaves non separated from l and l' we can assume that l, l' are consecutive. Then they have subrays which share an ideal point. If m is the last leaf non separated from l, l' in the direction the rays r_i go to, then there is a ray ζ of m so that there are subrays of r_i with points converging to a point in ζ and $(r_i)_\infty$ converges to ζ_∞. If there are infinitely many such leaves non separated from l, then we can order them as $\{l_j\}, j \in \mathbf{N}$ all in the direction the rays r_i go to. The ideal points of l_j form a monotone sequence in $\partial_\infty F$ which converge to a point u in $\partial_\infty F$. The arguments above show that $(r_i)_\infty$ converges to u. □

Our next goal is to analyse the non Hausdorffness in the leaf space of $\widetilde{\Lambda}^s_F$ and identification of ideal points. We want to understand when can two ideal points of the same leaf of $\widetilde{\Lambda}^s_F$ be identified. A *Reeb annulus* is an annulus A with a foliation so that the boundary components are leaves and every leaf in the interior is a topological line which spirals towards the two boundary components in the same direction. In the universal cover the lifted foliation does not have Hausdorff leaf space. The lifted foliation to the universal cover is called a *Reeb band.* A *spike region* in a leaf F of $\widetilde{\Lambda}^s$ is a closed $\widetilde{\Lambda}^s_F$ saturated set $\mathcal{E}$ so that there are finitely many boundary leaves which are line leaves of

$\widetilde{\Lambda}^s_F$. The ideal points of consecutive rays in the boundary are the same, otherwise they are distinct (like an ideal polygon). In addition the region is a bounded distance from the ideal polygon with these vertices. The bound is not universal in $\widetilde{\mathcal{F}}$. There is an ideal point z of $\mathcal{E}$ so that every leaf in the interior of $\mathcal{E}$ has both ideal points equal to z. In addition the leaves in the interior are nested. Finally, the finitely many leaves in the boundary are all non separated from each other and they are limits of the interior leaves.

Proposition 5.6. *Let E be a leaf in $\widetilde{\mathcal{F}}$ and v a slice of a leaf v_0 of $\widetilde{\Lambda}^s_E$. Suppose that both ideal points of v are the same. Then v is contained in a spike region B of E. In addition either B projects to a Reeb annulus in a leaf of $\mathcal{F}$ or for any two consecutive rays in ∂B, the region between them projects to a set asymptotic to a Reeb annulus in a leaf of M.*

Proof. Let v be a slice as above with ideal point x^* in $\partial_\infty E$. Let C be the region bounded by v in E which only limits in x. We may assume that v is a line leaf of v_0, since any prong of v_0 which enters C will have ideal point x. We will show that the region C as it approaches x, projects to a set in M which limits to a Reeb annulus in a leaf of $\mathcal{F}$. The process will be done in a series of steps. The proof of this proposition is very long with several intermmediate results and lemmas.

Choose z_0 in v and let e_1, e_2 be the rays of v defined by z_0. Let ζ^* be the geodesic ray of E starting at z_0 and with ideal point x. Then ζ^* is in contained in the $2\delta_0$ neighborhood of e_1 or e_2, where δ_0 is the constant of lemma 5.4. It follows that we can choose p_i, q_i in e_1, e_2 respectively with p_i, q_i converging to x in $E \cup \partial_\infty E$ and also $d_E(p_i, q_i) < 4\delta_0$. Let e^i_1 be the subray of e_1 starting at p_i and e^i_2 the subray of e_2 starting at q_i. As usual up to subsequence there are g_i in $\pi_1(M)$ with $g_i(p_i)$, $g_i(q_i)$ converging to p_0, q_0 respectively, where p_0, q_0 are points in a leaf F of $\widetilde{\mathcal{F}}$. Then $g_i(E)$ converges to F and perhaps other leaves as well.

For i big enough the flowlines of $\widetilde{\Phi}$ through $g_i(p_i), g_i(q_i)$ go through to u_i and v_i in F. Also $u_i \rightarrow p_0, v_i \rightarrow q_0$. If the leaf of $\widetilde{\Lambda}^s_F$ through p_0 contains q_0 then for i big enough the arcs in leaves of $\widetilde{\Lambda}^s_F$ from u_i to v_i will have bounded length and bounded diameter. The same will happen for the arcs of of $g_i(v)$ between $g_i(p_i)$ and $g_i(q_i)$, contradiction. Hence p_0, q_0 are not in the same leaf of $\widetilde{\Lambda}^s_F$. Let l be the leaf of $\widetilde{\Lambda}^s_F$ through p_0 and r be the one through q_0. Let L, R leaves of $\widetilde{\Lambda}^s$ containing l and r respectively. Since the intersection of a leaf of $\widetilde{\Lambda}^s$ with F is connected, then L and R are distinct and also are not separated from each other in the leaf space of $\widetilde{\Lambda}^s$.

The first goal is to show that we can choose l, r line leaves of $\widetilde{\Lambda}^s_F$ as above so that they also share an ideal point. Let β_i be a ray in the leaf of $\widetilde{\Lambda}^s_F$ through u_i starting at u_i and containing points in the flowlines through to the ray $g_i(e^i_1)$. Similarly let γ_i be a subray in the same leaf starting at v_i and associated to the ray $g_i(e^i_2)$. Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be the collection of line leaves of $\widetilde{\Lambda}^s_F$ that β_i (resp. γ_i) converges to, including the ray of l (resp. r). Let $\mathcal{C}$ be the collection of all line leaves of $\widetilde{\Lambda}^s_F$ which are non separated from l, r. Then $\mathcal{C}$ contains $\mathcal{C}_1$ and $\mathcal{C}_2$. For any element τ in $\mathcal{C}$ it is contained in a leaf $B(\tau)$ of $\widetilde{\Lambda}^s$. All of the $B(\tau)$ are not separated from each other, and they are in the set of leaves $\mathcal{B}$ of $\widetilde{\Lambda}^s$ non separated from both L, R. By theorem 2.6, the set $\mathcal{B}$ has a linear order, making it order isomorphic to either $\mathbf{Z}$ or a finite set. This induces an order in $\mathcal{C}$ where we can choose this so that an arbitrary element of $\mathcal{C}_1$ is bigger than any element in $\mathcal{C}_2$.

If there are finitely many elements in $\mathcal{C}_1$ let l' be the last one and let ξ_1 be the ideal point of the ray of l' corresponding to the direction of the rays β_i. Otherwise the ideal points of the leaves in $\mathcal{C}_1$ form a weakly monotone sequence in $\partial_\infty F$ and let ξ_1 be the limit of this sequence. Similarly define ξ_2 associated to r, see fig. 9, a. The first thing to prove is the following:

Lemma 5.7. $\xi_1 = \xi_2$.

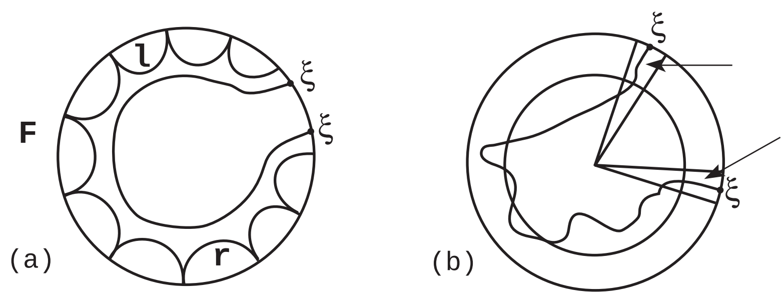

Figure 9: *a. Non Hausdorffness in the limit, b. Showing* $\xi_1 = \xi_2$.

Proof. Suppose by way of contradiction that this is not true. Choose 2 markers very near ξ_1 bounding an interval J_1 in $\partial_\infty F$ with ξ_1 in the interior and similarly choose markers near ξ_2 and interval J_2 so that J_1, J_2 are disjoint. Let W_1 be the wedge of F centered at a point x_0 with ideal set J_1 and W_2 the wedge of F centered also at x_0 with ideal set J_2. For i big enough both boundaries of W_1 and W_2 flow into $g_i(E)$.

Suppose first that there is a last leaf l' in $\mathcal{C}_1$. Then l' has a ray which is eventually contained in a strictly smaller subwedge W_1' of W_1 $-$ since its ideal point is ξ_1. Now choose a big disk D of F centered in x_0 . Let N_1 be the closure of $W_1 - D$. Choose D big enough so that l' enters N_1 through ∂D and is then entirely in W_1'. For i big enough β_i will be close to l' for a long distance. By lemma 5.5 the ideal points of β_i converge to ξ_1 as i converges to infinity, since l' is the last leaf non separated from l in that side. The ideal point is in the limit set of the subwedge W_1'. If the rays β_i keep exiting W_1 then since they are trapped by l' and β_{i_0} (for some i_0), they will have to intersect a compact part of ∂W_1. Then the sequence $\{\beta_i\}$ has additional limits besides the leaves in $\mathcal{C}_1$, contradiction. Therefore for big enough i, the β_i enters N_1 through ∂D and stays in N_1 from then on.

We want to get the same result when $\mathcal{C}_1$ is infinite. In that case let $\{\nu_j, \ j \in \mathbf{N}\}$ be the leaves in $\mathcal{C}_1$ ordered with same ordering as in $\mathcal{C}_1$ and $\nu_1 = l$. Since these leaves are non separated from each other then they cannot accumulate anywhere in F and the leaves ν_j escape compact sets as j grows. The ideal points of ν_j are also converging to ξ_1. By density of ideal points of $\widetilde{\Lambda}^s_F$ in $\partial_\infty F$ the leaves ν_j cannot be getting closer to non trivial intervals in $\partial_\infty F$. This implies that there is j_0 so that for

$$j \geq j_0, \quad \nu_j \quad \text{is very close to} \quad \xi_1 \quad \text{in } F \cup \partial_\infty F$$

and so contained in W_1'. Now an argument entirely similar as in the case $\mathcal{C}_1$ finite implies that for i big enough then β_i has subrays entirely contained in N_1. The same holds for γ_i producing subrays entirely contained in the corresponding set N_2 $-$ the disk D may need to be bigger to satisfy all these conditions.

There is $a_1 > 0$ and i_0 so that for $i \geq i_0$ then except for the initial segment of length a_1 then β_i is entirely contained in N_1 and similarly for γ_i and N_2. Choose k_0 big enough so that D is ϵ close to $g_k(E)$ for any $k \geq k_0$. Then D flows in $g_k(E)$ and so do $\partial W_1, \partial W_2$. For i bigger than both i_0, k_0 the ray β_i flows into the ray $g_i(e^i_1)$ (notice these do not have singularities). The ray $g^i(e^i_1)$ has to be in the generalized wedge which is bounded by the image of ∂W_1 in $g_i(E)$. Similarly for γ_i. This argument is done in lemma 5.2. These two generalized wedges have disjoint ideal sets in $\partial_\infty g_i(E)$. Therefore $g_i(e^i_1)$ and $g_i(e^i_2)$ do not have the same ideal points. This is a contradiction because e_1, e_2 have the same ideal point in $\partial_\infty E$.

This proves that $\xi_1 = \xi_2$. □

<u>Continuation of the proof of proposition 5.6</u>

The fact $\xi_1 = \xi_2$ implies that the ideal points of β_i, γ_i are all the same and equal to ξ_1. Let $\xi = \xi_1$. Let μ be the geodesic ray in F starting at x_0 with ideal point ξ. Since $(\beta_i)_\infty = (\gamma_i)_\infty = \xi$, then lemma 5.4 implies that for z in μ far enough from p_0, there are $b_i(z)$ in β_i and $c_i(z)$ in γ_i both of which are less than $2\delta_0$ away from z in F. This is for any i in $\mathbf{N}$. So up to subsequence we assume $b_i(z)$ converges to $b(z)$ and similarly $c_i(z)$ converges to $c(z)$. By definition of $\mathcal{C}_1$ the point $b(z)$ has to be in one of the leaves of $\mathcal{C}_1$ and similarly for $c(z)$.

Lemma 5.8. *There is one element ζ of $\mathcal{C}_1$ which has ideal point ξ.*

Proof. If there are finitely elements in $\mathcal{C}_1$ then the last one satisfies this property. Suppose then there are infinitely many elements in $\mathcal{C}_1$. As z varies in μ, then so does $b(z)$. If there are z escaping in μ so that $b(z)$ is in the same element ζ of $\mathcal{C}_1$ then ζ has an appropriate ray with ideal point ξ. In this case we are done.

Otherwise we can find z_k in μ converging to ξ so that $b(z_k)$ are in leaves $\nu_{i(k)}$ of $\mathcal{C}_1$ which are all distinct. We can choose z_k so that the $i(k)$ increases with k. In the same way we have $c(z_k)$ in elements of $\mathcal{C}_2$. Let

$$B_k \ = \ \widetilde{W}^s(b(z_k)), \quad C_k \ = \ \widetilde{W}^s(c(z_k)), \quad \text{both in } \ \mathcal{B}$$

Recall that $\mathcal{B}$ is the set of leaves of $\widetilde{\Lambda}^s$ non separated from both L, R. Since the length from $b(z_k)$ to $c(z_k)$ in F is bounded by $4\delta_0$, then up to subsequence assume $\pi(b(z_k)), \pi(c(z_k))$ converge in M. For i, k big enough there is h_{ik} covering tranlation so that $h_{ik}(b(z_i))$ is very close to $b(z_k)$ and $h_{ik}(c(z_i))$ is very close to $c(z_k)$. Suppose $i >> k$, let $h = h_{ik}$ for simplicity. Then B_k has a point $b(z_k)$ very close to $h(b(z_i)) \in h(B_i)$ and similarly for $c(z_k)$ in C_k very close to $h(c(z_i)) \in h(C_i)$. Since B_k is non separated from C_k and similarly for $h(B_i), h(C_i)$, then the only way this can happen is that

$$h(B_i) \ = \ B_k, \quad h(C_i) \ = \ C_k$$

This implies that h sends the set of leaves non separated from B_i, C_i to itself, that is h acts on the set $\mathcal{C}$ and therefore acts on $\mathcal{B}$ as well. Notice that $B_k < B_i$ in the order of $\mathcal{B}$ because $i > k$ and $C_k \geq C_i$ (the C_k could be all the same, but if they are not then they decrease in the order). Since $h(B_i) = B_k$ then h acts as a decreasing translation in the ordered set $\mathcal{B}$. But since $h(C_i) = C_k$ then h acts as a non decreasing translation. These two facts are incompatible.

This implies that we have to have at least one element in $\mathcal{C}_1$ with ideal point ξ. The same happens for $\mathcal{C}_2$. This finishes the proof of the lemma. □

Since β_i also converges to ζ we can rename the objects and assume that $l = \zeta$ and p_0 is a point in l. This changes the points p_i in the ray e_1. Similarly do the same thing in the other direction. We state this conclusion:

Conclusion $-$ There are p_i, q_i in e_1, e_2 respectively, escaping these rays, so that $d_E(p_i, q_i) < 4\delta_0$ and there are covering translations g_i so that: $g_i(p_i)$ converges to p_0, $g_i(q_i)$ converges to q_0, both in F and in rays l, r of $\widetilde{\Lambda}^s_F$. Also l, r converge to the same ideal point ξ in $\partial_\infty F$.

We will continue this perturbation approach. We want to show that the region in F "between" l and r projects to a Reeb annulus of $\mathcal{F}$ in M. Let then z_i in l converging to ξ and w_i in r converging to ξ, so that $d_F(z_i, w_i)$ is always less than $4\delta_0$. Up to subsequence assume there are h_i covering translations with

$$h_i(z_i) \to z_0, \quad h_i(w_i) \to w_0$$

Notice that $h_i(L), h_i(R)$ are non separated from each other and $h_i(L) \rightarrow \widetilde{W}^s(z_0),\ h_i(R) \rightarrow \widetilde{W}^s(w_0)$. The argument in the previous lemma then implies that $h_i(L) = h_j(L),\ h_i(R) = h_j(R)$ for all i, j at least equal to some i_0. Discard the first i_0 terms and postcompose h_i with $(h_{i_0})^{-1}$ (that is $(h_{i_0}^{-1} \circ h_i)$), to assume that $h_i(L) = L, h_i(R) = R$ for all i. So the h_i are all in the intersection of the isotropy groups of L and R. This group is generated by a covering translation h. Therefore there are n_i with $h_i = h^{n_i}$. Since $h_i(z_i) \rightarrow z_0$ and the $\{z_i, \mid i \in \mathbf{N}\}$ do not accumulate in $\widetilde{M}$ then $|n_i| \rightarrow \infty$. In addition since L, R are not separated from each other, then h preserves each individual line leaf, slice and possible lift annulus of L.

Up to subsequence and perhaps taking the inverse of h, assume that n_i converges to ∞. If $h(F) = F$, then since $h(L) = L$ this produces a closed leaf in $\pi(F)$. Similarly $h(R \cap L) = R \cap L$ so produces another closed leaf in F and together bound an annulus with a sequence of leaves converging to the boundary leaves. By Euler characteristic reasons, there can be no singularities inside the annulus, so we conclude that the annulus in $\pi(F)$ has a Reeb foliation.

Let $\mathcal{H}$ be the leaf space of $\widetilde{\mathcal{F}}$. This is a one dimensional manifold, which is simply connected, but usually not Hausdorff [Ba2]. The element h acts on $\mathcal{H}$. An analysis of group actions on simply connected non Hausdorff spaces was done in [Ro-St] or [Fe10]. One possibility is that h acts freely in $\mathcal{H}$. Then h has an axis τ in $\mathcal{H}$ which is invariant under h. In general this axis is not properly embedded, see [Fe10]. Since all the $h^{n_i}(F)$ intersect a common transversal, then F has to be in the axis of h and $h^n(F)$ converges to a collection of non separated leaves. In this case we get that F^* and $h(F^*)$ are non separated from each other.

The other situation is that h has fixed points in $\mathcal{H}$. In general the set of fixed points of $\mathcal{H}$ is not a closed set, but the set of points z in $\mathcal{H}$ so that z and $h(z)$ are not separated in $\mathcal{H}$ is a closed subset Z of $\mathcal{H}$. None of the images of F under h can be in Z, so F is in a component of $\mathcal{H} - Z$. Then h permutes these components. In addition h preserves an orientation in $\mathcal{H}$ $-$ since $\mathcal{F}$ is transversely orientable. Since $h^{n_i}(F)$ all intersect a common transversal then they have all to be in the same component U of $\mathcal{H} - Z$. Let i_0 be the smallest positive integer so that $h^{i_0}(U) = U$. It follows that all n_i are multiples of i_0. The leaf F^* is in the boundary of the component U and $h^{i_0}(F^*) = F^*$.

The only remaining case to be analysed is that h acts freely and $h^n(F)$ converges to F^* with $h(F^*)$ non separated from F^*. In this particular case we prove this is not possible, that is:

Claim $-$ $h(F^*) = F^*$.

Suppose this is not true. The leaves $h(F^*), F^*$ are not separated in $\mathbf{H}^2$. This implies that $\Theta(F^*)$ and $\Theta(h(F^*))$ are disjoint subsets of $\mathcal{O}$, see fig. 10. Therefore there are boundary leaves separating them. But L intersects both F^* and $h(F^*)$ as L intersects F and is invariant under h. Therefore both $\Theta(F^*)$ and $\Theta(h(F^*))$ intersect the same stable leaf $\Theta(L)$.

Suppose that there is a stable boundary component of $\Theta(F^*)$ separating it from $\Theta(h(F^*))$. Then it has to be a slice of $\Theta(L)$ as this set intersects both of them. It would not be a line leaf of $\Theta(L)$. But as remarked before, h leaves invariant all the slices, line leaves and lift annuli of L and this contradicts $\Theta(h(F^*))$ being disjoint from $\Theta(F^*)$. This implies there is an unstable boundary component of $\Theta(F^*)$ separating it from $\Theta(h(F^*))$, see fig. 10.

In the same way $\Theta(R)$ intersects both $\Theta(F^*)$ and $\Theta(h(F^*))$. Let $L_i = \widetilde{W}^s(u_i)$. Recall from the beginning of the proof of proposition 5.6 that u_i, v_i are points in F with u_i converging to p_0 in L and v_i converging to q_0 in R. Then $\Theta(L_i)$ converges to $\Theta(L) \cup \Theta(R)$ (maybe other leaves as well). So $\Theta(L_i)$ intersects $\Theta(F^*)$ and $\Theta(h(F^*))$ for i big enough. The intersection of $\Theta(L_i)$ with at least one of $\Theta(F^*)$ or $\Theta(h(F^*))$ cannot be connected, see fig. 10. This contradicts propostion 3.2. This contradiction implies that $h(F^*) = F^*$ and proves the claim.

So far we have proved the following: in any case there is i_0 a positive integer so that if $f = h^{i_0}$

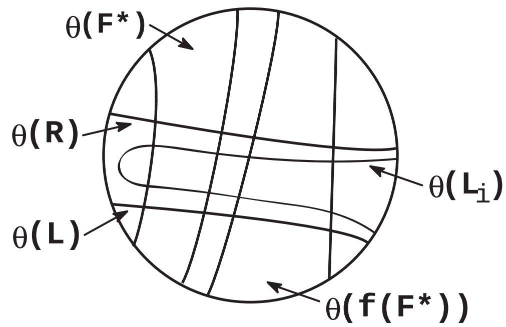

Figure 10: *Contradiction in the orbit space $\mathcal{O}$.*

then $f(F^*) = F^*$. As $f(L) = L$ then $f(F^* \cap L) = F^* \cap L$ and similarly $f(F^* \cap R) = F^* \cap R$. This produces an annulus B in $\pi(F^*)$ with a Reeb foliation. The region of F^* bounded by $F^* \cap R$ and $F^* \cap L$ bounds a band B which is a bounded distance from a geodesic in F^* and projects to a Reeb annulus in a leaf of $\mathcal{F}$.

But to prove proposition 5.6, we really want these facts for F and not just F^*. This turns out to be true: $\pi(E)$ has points converging to $\pi(F)$ and $\pi(F)$ has points converging to an annulus in $\pi(F^*)$. Since the annulus is compact, it turns out the second step is unnecessary. This depends on an analysis of holonomy of the foliation $\mathcal{F}$ near the annulus in $\pi(F^*)$ as explained below.

Claim – The point $\pi(p_0)$ of $\pi(F)$ is in the boundary of a Reeb annulus of $\mathcal{F}$ contained in $\pi(F)$. This implies that $F = F^*$.

The point z_0 is in $F^* \cap L$. Then $\pi(z_0)$ is in $\pi(F^* \cap L) = \alpha$ which is a closed curve since h^{i_0} leaves invariant both F^* and L and their intersection is connected. Previous arguments in the proof imply that for i big enough $h_i(z_i)$ is in the same local sheet of $\widetilde{\Lambda}^s$ as z_0. Hence the points $\pi(z_i)$ are in $W^s(\pi(z_0)) = \pi(L)$ and converge to $\pi(z_0)$. This shows that $\pi(F \cap L)$ is asymptotic to α in the direction corresponding to the projection of the direction of escaping z_i in the ray of $F \cap L$. Namely α has contracting holonomy (of $\mathcal{F}$) in the side the $\pi(z_i)$ are converging to and eventually $\pi(z_i)$ is in the domain of contraction of α.

This means that the direction of F associated to the ideal point ξ is a contracting direction towards F^*. The rays in the leaves $F^* \cap L$, $F^* \cap R$ in F^* are a bounded distance from a geodesic ray in F^* with same ideal point. The contraction above implies that the corresponding rays $F \cap L$, $F \cap R$ of F are also a bounded distance from a ray in F with ideal point ξ.

Now recall the points p_i in E. We have $g_i(p_i)$ very close to p_0 in the leaf l of $\widetilde{\Lambda}^s_F$. Also $\pi(l)$ is eventually in a region contracting towards a Reeb annulus of $\mathcal{F}$. Hence if i is big enough the $g_i(p_i)$ will also be in this region. The leaf through $\pi(p_i)$ will be contracted towards the Reeb annulus in that direction. This implies that the limit of the $\pi(p_i)$ is already in a Reeb annulus, consequently the limit of the $g_i(p_i)$ is already in a Reeb band.

It now follows that $\pi(F) = \pi(F^*)$. That means that the second perturbation procedure (from points in F to points in F^*) in fact does not produce any new leaf. This implies that up to covering translations then the leaf E is asymptotic to F in the direction of the ideal point x^* in $\partial_\infty E$. Let V be the region of E bounded by v with ideal point x^*. Then outside of a compact part it is very near a Reeb band in F and so has no singularity of the foliation $\widetilde{\Lambda}^s_E$. By Euler characteristic reasons it follows that V has no singularities in the compact part also. So far we proved the following:

Conclusion – Let v be a slice of $\widetilde{\Lambda}^s_E$ with two rays converging to the same ideal point x^* of $\partial_\infty E$ and V is the region of E bounded by v. Then v is a line leaf of $\widetilde{\Lambda}^s_E$ in the V side and V has no

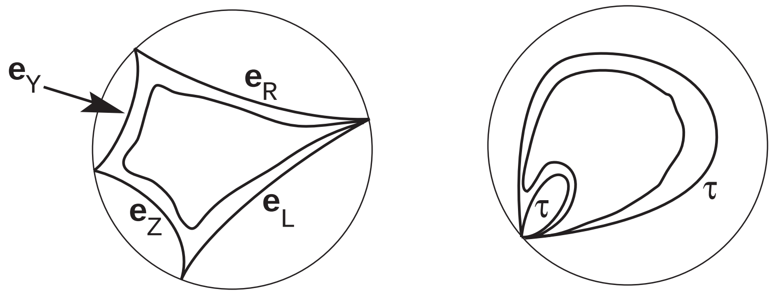

Figure 11: *a. l_i converging to non separated leaves e_L, e_Z, e_Y, e_R of $\widetilde{\Lambda}^s_E$, b. Nested families and identifications of ideal points.*

singularities in the interior. Also $\pi(V)$ is either contained in or asymptotic to a Reeb annulus in a leaf of $\mathcal{F}$ and so E is asymptotic to a Reeb band in a leaf F in the direction x^*.

Continuation of the proof of proposition 5.6.

What we want to prove is that in E itself the region V is contained in the interior of a spike region. Notice it is not true in general that $\pi(V)$ is contained in a Reeb annulus, only that it is asymptotic to a Reeb annulus. For instance start with a leaf of $\mathcal{F}$ having a Reeb annulus and blow that into an I-bundle. Then produce holonomy associated to the core of the Reeb annulus. Then one produces Reeb bands asymptotic to but not contained in Reeb annuli.

Since V is asymptotic to the Reeb band in F, it turns out that (after rearranging by covering translations) that E intersects both L and R leaves of $\widetilde{\Lambda}^s$. Their intersection produces two leaves e_L, e_R of $\widetilde{\Lambda}^s_E$ which are not separated from each other and which have the same ideal point x^*. There are then leaves l_i of $\widetilde{\Lambda}^s_E$ all with ideal point x^* and which converge to $e_L \cup e_R$. This follows from the fact that in F the same is true and E is asymptotic to F in that direction, plus the connectivity of the intersection of E with leaves of $\widetilde{\Lambda}^s$.

Now the sequence l_i can converge to other leaves as well, all of which will be non separated from e_L, e_R. The set of limits is an ordered set and the any other leaf is between e_L and e_R. By theorem 2.6 there are only finitely many of them. We refer to fig. 11, a, where for simplicity we consider there are 4 leaves in the limit: e_L, e_Y, e_Z, e_R contained in leaves L, Z, Y, R of $\widetilde{\Lambda}^s$. These leaves of $\widetilde{\Lambda}^s$ are non separated from each other and form an ordered set. Let ξ be the region of E which is the union of the region bounded by all the l_i plus the boundary leaves, which are non separated from e_L, e_R. Clearly every leaf in the interior has ideal point x^* and has no singularity. We want to show that ξ is a spike region.

Any two consecutive leaves of $\partial\xi$ in this ordering will have rays with same ideal point and leaves l_i converging to them. This situation is important on its own and is analysed in the following proposition:

Proposition 5.9. *Suppose v_1, v_2 are non separated leaves in $\widetilde{\Lambda}^s_G$ for some G leaf of $\widetilde{\mathcal{F}}$. Suppose there are no leaves non separated from v_1, v_2 in between them. Then the corresponding rays of v_1, v_2 have the same ideal point in $\partial_\infty G$. In addition they are a bounded distance from a geodesic ray of G with same ideal point. In M this region either projects to or is asymptotic to a Reeb annulus.*

Proof. We do the essentially the same proof as in the case of leaves of $\widetilde{\Lambda}^s_G$ with same ideal points, except that we go in the direction of the non Hausdorfness. Because there are no non separated leaves in between v_1, v_2, then the corresponding rays have the same ideal point. Choose w_i, y_i in these rays of v_1, v_2 and escaping towards the ideal point and so that $d_G(w_i, y_i)$ is less than $4\delta_0$. We do the limit analysis using $f_i(w_i), f_i(y_i)$ converging in $\widetilde{M}$. Because v_1, v_2 are non separated it follows that $f_i(w_i), f_j(w_j)$ are in the same stable leaf (of $\widetilde{\Lambda}^s$) for i, j big enough. Hence we can readjust so that they are all in the same stable leaf and similarly for $f_i(y_i)$. The same arguments as before show that that region of G between v_1, v_2 projects in M to set in a leaf of $\mathcal{F}$ which is either contained in or asymptotic to a Reeb annulus. The results follow. In general nothing can be said about the other direction in the leaves v_1, v_2: in particular it does not follow at all that the other rays of v_1, v_2 have to have the same ideal point. $\square$

Given this last proposition then for any two consecutive rays in $\partial \mathcal{E}$ it follows that they are a bounded distance from a geodesic ray in E. All that is needed to show that $\mathcal{E}$ is a spike region is to prove that the ideal points of the rays in the boundary are distinct except for consecutive rays.

Suppose there are other identifications of ideal points of leaves in the boundary of $\mathcal{E}$. Then there is at least one line leaf τ in the boundary of $\mathcal{E}$ so that τ has identified ideal points. Our analysis so far shows that τ is in the interior of another region similar to the one constructed above so that all leaves have just one common ideal point. Since the l_i limit on τ, then the ideal point of τ has to be x^*. In addition the leaves in this new region have to be nested. But if the l_i together with τ are a nested family of leaves of $\widetilde{\Lambda}^s_F$, then the τ is outside the l_i hence the region in E bounded by τ enclosed the whole region $\mathcal{E}$, see fig. 11, b. There is at least one other leaf τ' in $\partial \mathcal{E}$. The same arguments we used for τ can be applied to τ'. But it is impossible that the l_i are also nested with the τ', see fig. 11, b.

This shows that the ideal points of $\mathcal{E}$ are distinct except as mandated by consecutive rays. In addition any line leaf in the boundary of $\mathcal{E}$ has distinct ideal points and rays which are a bounded distance from geodesic rays. It follows that the whole leaf is a bounded distance from a geodesic in E. This shows that $\mathcal{E}$ is a spike region. This finishes the proof of proposition 5.6. $\square$

Finally in the case $\widetilde{\Lambda}^s$ has Hausdorff leaf space one can say much, much more about metric properties of leaves of $\widetilde{\Lambda}^s_F$:

Proposition 5.10. *Suppose that Φ is an almost pseudo-Anosov flow transverse to a foliation $\mathcal{F}$ with hyperbolic leaves. Suppose that $\widetilde{\Lambda}^s$ has Hausdorff leaf space. Then there is $k_0 > 0$ so that for any F leaf of $\widetilde{\Lambda}^s$, then the slice leaves of $\widetilde{\Lambda}^s_F$ are uniform k_0 quasigeodesics.*

Proof. If there is a leaf F of $\widetilde{\mathcal{F}}$ and a slice leaf of $\widetilde{\Lambda}^s_F$ with only one ideal point, then the proof of proposition 5.6 shows that there are leaves of $\widetilde{\Lambda}^s$ non separated from each other. This is impossible.

Suppose now that for any integer i, there are x_i in $\widetilde{M}$, x_i in leaves F_i of $\widetilde{\mathcal{F}}$ with x_i in line leaves l_i of $\widetilde{\Lambda}^s_{F_i}$ with distance from x_i to l^*_i in F_i going to infinity. Here l^*_i is the geodesic in F_i with same ideal points as l_i. Up to covering translations assume x_i converges to x. Also assume all x_i are in the same sector of $\widetilde{\Lambda}^s$ defined by x. Since l_i converges to l, the arguments in lemmas 5.5 and 5.2 would show that the ideal points of l are the same. This was just disproved above.

Given that, the line leaves are within some global distance a_0 of the respective geodesics in their leaves. It is well known that these facts imply that the slice leaves of $\widetilde{\Lambda}^s_F$ are uniform quasigeodesics. For a proof of this well known fact see for example [Fe-Mo]. $\square$

6 Continuous extension of leaves

The purpose of this section is to prove theorem B: the continuous extension property for leaves of foliations which are almost transverse to quasigeodesic pseudo-Anosov flows in 3-manifolds with negatively curved fundamental group.

To start suppose that Φ is an almost pseudo-Anosov flow which is transverse to a foliation $\mathcal{F}$ in a general closed 3-manifold M. Given leaf F of $\widetilde{\mathcal{F}}$ we first introduce geodesic "laminations" in F coming from $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$. We only work with the stable foliation, similar results hold for the unstable lamination. Assume that a leaf l of $\widetilde{\Lambda}^s_F$ is not singular. If both ideal points are the same let l^* be empty. Otherwise let l^* be the geodesic with same ideal points as l. If l is singular, then no line leaves of l have the same ideal point by proposition 5.6. For each line leaf e of l let e^* be the corresponding geodesic. Let l^* be their union. Let now τ^s_F be the union of these geodesics of F. Leaves of $\widetilde{\Lambda}^s_F$ do not have transverse intersections and therefore the same happens for leaves of τ^s_F.

Suppose that $\widetilde{\Lambda}^s_F$ has non separated leaves l, v which are not in the boundary of a spike region. Then there are l_i converging to $l \cup v$ (and maybe other leaves as well), but l^*_i does not converge to l^* or v^*. Notice none of the limit leaves can have identified ideal points, because then they would be in the interior of a spike region (proposition 5.6) and have a neighborhood which is product foliated. Let $\overline{\tau}^s_F$ be the closure of τ^s_F. Then $\overline{\tau}^s_F$ is a geodesic lamination in F. Similarly define τ^u_F, $\overline{\tau}^u_F$. In a complementary region U of $\overline{\tau}^s_F$ associated to non Hausdorffness, there is one boundary component which is added (a leaf of $\overline{\tau}^s_F - \tau^s_F$) and which is the limit of the l^*_i as above. All of the other boundary leaves of the region are associated to the non separated leaves of $\widetilde{\Lambda}^s_F$ and are in τ^s_F.

Lemma 6.1. *The new leaves in $\overline{\tau}^s_F$ (that is those in $\overline{\tau}^s_F$ - τ^s_F) come from non Hausdorfness of $\widetilde{\Lambda}^s_F$.*

Proof. Let e_i in τ^s_F converging to e not in τ^s_F. Then choose l_i line leaves in $\widetilde{\Lambda}^s_F$ with $e_i = l^*_i$. Given u a point in e, there is u_i in l^*_i very close to u. Then there are p_i in l_i which are $2\delta_0$ close to u_i. Up to subsequence assume that p_i converges to p_0 and let l be the line leaf of $\widetilde{\Lambda}^s_F$ that the sequence l_i converges to. Then l^*_i does not converge to l^* so we have a non Hausdorff situation: l_i converging to l and other leaves as well and l^* is the added leaf associated to this non Hausdorfness. This finishes the proof of the lemma. ∎

Lemma 6.2. *The complementary regions of $\overline{\tau}^s_F$ are ideal polygons associated to singular leaves and non Hausdorff behavior of $\widetilde{\Lambda}^s$. If $\pi_1(M)$ is negatively curved then these complementary regions are finite sided ideal polygons.*

Proof. Let x be in a complementary region U of $\overline{\tau}^s_F$. Let e be a leaf in the boundary ∂U. Suppose first that e is an actual leaf of τ^s_F, which comes from a line leaf l of $\widetilde{\Lambda}^s_F$. It may be that l is a singular leaf which is singular on the x side. In that case x is in the region U. Otherwise l is not singular on the side containing x and we may assume there are l_i leaves of $\widetilde{\Lambda}^s_F$ on that side with l_i converging to l. If the ideal points of l_i converge to that of l then eventually l^*_i separates x from e and x is not in the complementary region U − impossible. Hence the ideal points of l_i do not converge to ∂e and there is non Hausdorfness and a complementary region in that side of l. Then x needs to be in this complementary region (which is U) and e is a boundary leaf of U which comes from a line leaf of τ^s_F.

Suppose now that e is an added leaf. There are l_i leaves of $\widetilde{\Lambda}^s_F$ with $e_i = l^*_i$ converging to e on the side opposite to x, otherwise x is not in U. Then l_i converges to more than one leaf of $\widetilde{\Lambda}^s_F$ producing non Hausdorff behavior and a complementary region with e in its boundary. The x is in the region associated to this non Hausdorff behavior, so the complementary region must be U.

If there is a complementary region of $\overline{\tau}^s_F$ with infinitely many sides then it is associated to non Hausdorff behavior and so there are leaves l_i of $\widetilde{\Lambda}^s_F$ converging to infinitely many distinct leaves of $\widetilde{\Lambda}^s_F$. Then there is L leaf of $\widetilde{\Lambda}^s$ which is non separated from infinitely many other leaves. Theorem 2.6 implies that there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$ and hence $\pi_1(M)$ is not negatively curved, contradiction. This finishes the proof. ∎

We now turn to the continuous extension property. A preliminary analysis of the continuous extension property was done in [Fe8] in the case that Λ^s, Λ^u where quasi-isometric singular foliations, $\mathcal{F}$ is a finite depth foliation, and Φ is a pseudo-Anosov transverse to $\mathcal{F}$. Under these conditions it was shown in [Fe8] that leaves of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ are uniform quasigeodesics in their respective leaves F. Here we are analysing a much more general situation: in particular there are examples where $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ have non Hausdorff leaf space [Mo3, Fe8] and so $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ can have non Hausdorff leaf space, immediately implying that their leaves cannot be uniform quasigeodesics. In addition the results here apply to general foliations, for instance to foliations with dense leaves, foliations with one sided branching, etc.. The main theorem is:

Theorem 6.3. *Let $\mathcal{F}$ be a foliation in M^3 closed, with $\pi_1(M)$ negatively curved. Suppose that $\mathcal{F}$ is almost transverse to a pseudo-Anosov flow Φ' and transverse to an associated almost pseudo-Anosov flow Φ. Then for any leaf F of $\widetilde{\mathcal{F}}$, the inclusion map $\Psi : F \to \widetilde{M}$ extends to a continuous map*

$$\Psi: \ F \cup \partial_\infty F \ \rightarrow \ \widetilde{M} \cup S^2_\infty$$

The map Ψ restricted to $\partial_\infty F$, gives a continuous parametrization of the limit set of F, which is then locally connected.

Proof. The proof is done in two steps: first we define an extension and then we show that it is continuous.

First we need to review some facts about quasigeodesic almost pseudo-Anosov flows. If γ is an orbit of $\widetilde{\Phi}$ then it is a quasigeodesic has unique distinct ideal points γ_- and γ_+ in S^2_∞ corresponding to the positive and negative flow directions [Th1, Gr, Gh-Ha, CDP]. Hence given x in $\widetilde{M}$ define

$$\eta_+(x) \ = \ \gamma_+, \quad \eta_-(x) = \ \gamma_-, \quad \eta_+(x) \ \neq \ \eta_-(x),$$

where γ is the $\widetilde{\Phi}$ flowline through x. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ and a is a limit point of L in S^2_∞, then there is an orbit γ of $\widetilde{\Phi}$ contained in L with either $\gamma_- = a$ or $\gamma_+ = a$, that is, any limit point of L is a limit point of one of its flow lines [Fe8]. Also any such L in $\widetilde{\Lambda}^s$ is Gromov negatively curved [Gr, Gh-Ha, Fe8] and has an intrinsic ideal boundary ∂L consisting of a single forward ideal point and distinct negative ideal points for each flow line [Fe8]. The set $L \cup \partial_\infty L$ is a natural compactification of L in the Gromov sense. The inclusion $\kappa : L \to \widetilde{M}$ extends to a continuous map $\kappa : L \cup \partial L \to \widetilde{M} \cup S^2_\infty$. This all follows from the fact that Φ is quasigeodesic. If L is in $\widetilde{\Lambda}^s$ there is a unique distinguished ideal point denoted by L_+ in S^2_∞ which is the forward limit point of any flow line in L. Finally if in addition Λ^s is a quasi-isometric singular foliation, then the extension κ is always a homeomorphism into its image, but this is not true if Λ^s is not quasi-isometric. Similarly for L in $\widetilde{\Lambda}^u$.

Throughout the proof we fix a unique identification of $\widetilde{M} \cup S^2_\infty$ with the closed unit ball in $\mathbf{R}^3$. The Euclidean metric in this ball induces the visual distance in $\widetilde{M} \cup S^2_\infty$. Then $diam(B)$ denotes the diameter in this distance for any subset B of $\widetilde{M} \cup S^2_\infty$. A notation used throughout here is the following: if A is a subset of a leaf F of $\widetilde{\mathcal{F}}$, then $\overline{A}$ is its closure in $F \cup \partial_\infty F$.

We now produce an extension $\Psi : \partial_\infty F \to S^2_\infty$.

Case 1 $-$ Suppose that v in $\partial_\infty F$ is not an ideal point of a ray in $\widetilde{\Lambda}^s_F$ or in $\widetilde{\Lambda}^u_F$.

Since $\pi_1(M)$ is negatively curved, then complementary regions of $\overline{\tau}^s_F$ are finite sided ideal polygons. Hence there are e_i in $\overline{\tau}^s_F$ so that $\{e_i \cup \partial e_i\}$, $i \in \mathbf{N}$ define a neighborhood basis of v (in $F \cup \partial_\infty F$) and $\{e_i\}$ forms a nested sequence. Here ∂e_i are the ideal points of e_i in $\partial_\infty F$. We say that the $\{e_i\}$ define a neighborhood basis at v. Assume that no two e_i share an ideal point $-$ possible because of hypothesis. If e_i is in $\overline{\tau}^s_F - \tau^s_F$ then it is the limit of leaves in τ^s_F and by adjusting the sequence above we can assume that e_i is always in τ^s_F. Let l_i in $\widetilde{\Lambda}^s_F$ with $l^*_i = e_i$ and L_i leaves of $\widetilde{\Lambda}^s$ with $l_i \subset L_i$.

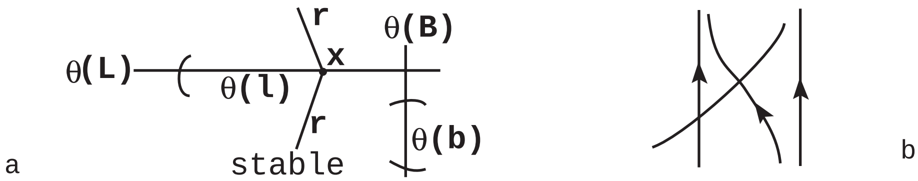

Figure 12: *a. Obstruction to intersections of leaves, b. The case of F escaping up.*

Similarly there are c_i in $\overline{\tau}^u_F$ defining a neighborhood basis of v. Up to subsequence we may assume that e_1, c_1, e_2, c_2, etc.. are nested and none of them have any common ideal points (in $F \cup \partial_\infty F$) and c_i is in τ^u_F. Let b_i in $\widetilde{\Lambda}^u_F$ with $b^*_i = c_i$ and B_i leaves of $\widetilde{\Lambda}^u$ with $b_i \subset B_i$.

Claim $-$ Both L_i and B_i escape in $\widetilde{M}$.

Notice $e_i \cap c_j = \emptyset$ for any i, j. If $l_i \cap b_j$ is non empty with $j > i$, then the nesting property above implies that $b_{i+1}, b_{i+2}, ..., b_j$ all have to intersect. Since there is a global upper bound on the number of prongs of leaves of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$, this can happen for only finitely many times. Up to taking a further subsequence we may assume that all the l_i, b_j are disjoint. At this point we need the following result:

Lemma 6.4. *Let L leaf of $\widetilde{\Lambda}^s$, B leaf of $\widetilde{\Lambda}^u$ and F leaf of $\widetilde{\mathcal{F}}$ so that F intersects both L and B: $l = L \cap F$, $b = F \cap B$. Suppose that b and l are disjoint in F. Then L does not intersects B in $\widetilde{M}$.*

Proof. Suppose not. Recall that $\Theta(L), \Theta(B)$ are finite pronged, non compact trees and they intersect in a compact subtree. The union is also a finite pronged tree. In addition $\Theta(L \cap B)$ is connected. The sets $\Theta(l), \Theta(b)$ are disjoint in this union. Let x be a boundary point of $\Theta(l)$ which is either in $\Theta(L \cap B)$ or separates $\Theta(L \cap B)$ from $\Theta(l)$ in this union, see fig. 12, a.

Let $\gamma = x \times \mathbf{R}$, an orbit of $\widetilde{\Phi}$. The first possibility is that F escapes up as $\Theta(F)$ approaches x. Then γ is a repelling orbit with respect to the $\Theta(l)$ side, see fig. 12, b and γ is in the boundary of a lift annulus A. This means that $\Theta(l)$ is a generalized unstable prong from the point of view of x. By proposition 3.1 there is a stable slice r of $\mathcal{O}^s(x)$ with r contained in $\partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches r, see fig. 12, a. The two sides of r are the closest generalized prongs to $\Theta(l)$ on either side of $\Theta(l)$. This implies that r separates $\Theta(b)$ from $\Theta(F)$ see fig. 12, a. Then $\Theta(b)$ cannot be contained in $\Theta(F)$, contradiction.

The second option is that F escapes down as $\Theta(F)$ approaches x along $\Theta(l)$. Here there is a slice r of $\mathcal{O}^u(x)$ with r contained in $\partial\Theta(F)$ and the closest to $\Theta(l)$ on both sides of $\Theta(l)$. Either $\Theta(b) \subset r$ or r separates $\Theta(b)$ from $\Theta(F)$. In any case $\Theta(b)$ does not intersect $\Theta(F)$, again a contradiction. This finishes the proof of the lemma. $\square$

The lemma shows that $L_i \cap B_j$ is empty for any i, j, and they form nested sequences of leaves in $\widetilde{M}$. Suppose that the sequence $\{L_i\}$ does not escape compact sets. Then there is L in $\widetilde{\Lambda}^s$ which is a limit of L_i (and possibly other leaves as well). Let α be an orbit in L which is not in a lift annulus. Then $\widetilde{W}^u(\alpha)$ is transverse to L in α and hence intersects L_i for i big enough. Since the L_i, B_j are nested this would force $\widetilde{W}^u(\alpha)$ to intersect B_j for j big enough, contradiction. It follows that both L_i and B_j escape compact sets as $i, j \to \infty$.

Let r be a geodesic ray in F with ideal point v. For each i, there is a subray of r contained in the component of $F - l_i$ which is in a small neighborhood of v. Hence $\Psi(r)$ has a subray which is contained in the corresponding component V_i of $\widetilde{M} - L_i$. These components V_i form a nested sequence. The ray $\Psi(r)$ can only limit in the limit set of V_i. We need the following lemma which will be a key tool throughout the proof.

Lemma 6.5. *(basic lemma) Let $\{Z_i\}$ be a sequence of leaves or line leaves or slices or any flow saturated sets in leaves of either in $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ (not all leaves Z_i need to be in the same singular foliation). If the sets Z_i escape compact sets in $\widetilde{M}$, then up to taking a subsequence $\overline{Z}_i$ converges to a point in S^2_∞.*

Proof. Let Y_i be the leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ which contains Z_i. Up to subsequence assume $Y_i \in \widetilde{\Lambda}^s$. The statement is equivalent to $diam(Z_i)$ converges to 0. Otherwise up to subsequence we can assume $diam(Z_i) > a_0$ for some a_0 and all i and hence no subsequence can converge to a single point in S^2_∞. Then there is p_i in Z_i with visual distance from p_i to $(Y_i)_+$ is bigger than $a_0/2$. Notice that $(Y_i)_+$ is a point in $\overline{Z}_i$. Let γ_i the orbit of $\widetilde{\Phi}$ through p_i. If $(\gamma_i)_-$ is very close to $(\gamma_i)_+ = (Y_i)_+$ then the geodesic with these ideal points has very small visual diameter. Since γ_i is a global bounded distance from this geodesic [Gr, Gh-Ha, CDP], the same is true for γ_i contradiction to the choice of p_i. Hence the geodesic above intersects a fixed compact set in $\widetilde{M}$ and so does γ_i. This contradicts the fact that Z_i escape compact sets in $\widetilde{M}$ and finishes the proof. □

We claim that the limit sets of V_i above shrink to a single point in S^2_∞. The limit sets form a weakly monotone decreasing sequence, because the L_i are nested and so are the V_i. If the limit set does not have diameter going to zero, then there are points in the limit set of L_i which are at least $2\delta_1$ apart for some fixed $\delta_1 > 0$. By the previous lemma the L_i cannot escape compact sets in $\widetilde{M}$, contradiction. Since the limit sets of V_i shrinks to a point in S^2_∞, let $\Psi(v)$ be this point. Clearly $\Psi(r)$ limits to this point and so does $\Psi(r')$ for any other geodesic ray r' in F with ideal point v.

<u>Case 2</u> $-$ Suppose that v is an ideal point of a leaf of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$.

Let l be a ray in say $\widetilde{\Lambda}^s_F$ which limits on v and r a geodesic ray on F with ideal point v. Then l is contained in L leaf of $\widetilde{\Lambda}^s$. Either $\Theta(l)$ escapes in $\Theta(L)$ or limits to a point x in $\Theta(L)$.

Consider the first case. Then in the intrinsic geometry of L, the ray l converges to the positive ideal point of L, hence in $\widetilde{M} \cup S^2_\infty$, the image $\Psi(l)$ converges to L_+. In the other option let $\beta = x \times \mathbf{R}$, an orbit of $\widetilde{\Phi}$. As l escapes in F then in L it either escapes up or down. If it escapes down then it converges to the negative ideal point of β in $L \cup \partial_\infty L$ and hence $\Psi(l)$ converges to β_-. Otherwise l escapes up in L as $\Theta(l)$ approaches x. In this case β is in the boundary of a lift annulus and l converges to the positive ideal point in $L \cup \partial_\infty L$ and so $\Psi(l)$ converges to L_+ again. Let $\Psi(v)$ be the limit point in any case.

Every point in r it is $2\delta_0$ close to a point in l in F, hence the limit of $\Psi(r)$ in $\widetilde{M} \cup S^2_\infty$ is the same as that of l. If l' is another ray of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$ converging to p, then it will have points boundedly close to r which escape in l' and therefore $\Psi(l')$ has the same ideal point in S^2_∞. Therefore $\Psi(v)$ is well defined.

This finishes the construction of the extension of Ψ to $\partial_\infty F$.

<u>Proof of continuity of the extension</u> $-$

<u>Case 1</u> $-$ v is not an ideal point of a ray in $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$.

Let r be a geodesic ray in F with ideal point v. Recall the extension construction. There are l_i in $\widetilde{\Lambda}^s_F$ shrinking to v in $F \cup \partial_\infty F$ and similarly b_i in $\widetilde{\Lambda}^u_F$, assumed to be nested with the l_i. Let $\{l^*_i\}$ define a neighborhood basis of v in $F \cup \partial_\infty F$. Let L_i in $\widetilde{\Lambda}^s$ with $l_i \subset L_i$, and $b_i \subset B_i \in \widetilde{\Lambda}^u$ as in the construction case 1. Then as seen in the construction, the L_i, B_i escape in $\widetilde{M}$. Let U_i be the component of $F - l_i$ containing a subray of r and V_i the component of $\widetilde{M} - L_i$ containing U_i.

Notice that $\Psi(U_i) \subset V_i$. Let now z in $\overline{U}_i$ with the closure taken in $F \cup \partial_\infty F$ and $\overline{V}_i$ the closure of V_i in $\widetilde{M} \cup S^2_\infty$. Then $\overline{U}_i$ is a neighborhood of v in $F \cup \partial_\infty F$. If z is in $\Psi(\overline{U}_i)$ then using either of the constructions in the extension part shows that z is a limit of points in $\Psi(U_i) \subset V_i$. As seen in the construction arguments the diameter of $\overline{V}_i$ in the visual distance is converging to 0. Hence we obtain continuity of Ψ at v. This finishes the proof in this case.

Case 2 $-$ v is an ideal point of a ray of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$.

This case is considerably more complicated, with several possibilities.

Case 2.1 $-$ v is an ideal point of $\widetilde{\Lambda}^s_F$ but not of $\widetilde{\Lambda}^u_F$ (or vice versa).

Suppose the first option occurs. There is l ray in $\widetilde{\Lambda}^s_F$ with ideal point v. We may assume that l is not in a leaf of $\widetilde{\Lambda}^s_F$ with same ideal points. Otherwise we can choose l to be one of the boundary leaves of the corresponding spike region. Since v is not an ideal point of $\widetilde{\Lambda}^s_F$, there are g_i line leaves in $\widetilde{\Lambda}^u_F$ defining a basis neighborhood system at v. Let g_i be contained in G_i leaves of $\widetilde{\Lambda}^u$. Let L in $\widetilde{\Lambda}^s$ containing l. If G_i escapes in $\widetilde{M}$ as $i \to \infty$, then as seen in case 1, we are done. Let then G_i converge to the finite set of leaves

$$\mathcal{V} \ = \ H_1 \cup H_2 .... \cup H_m \quad \text{leaves of} \ \ \widetilde{\Lambda}^u$$

We can assume that $G_i \cap l$ is not empty for all i.

Case 2.1.1 $-$ Suppose that L intersects $\mathcal{V}$, say $L \cap H_1 \neq \emptyset$.

Then l escapes down as $\Theta(l)$ approaches $\Theta(L \cap H_1)$. Otherwise $L \cap H_1$ is in the boundary of a lift annulus A and l has a subray contained in this lift annulus. But then A is also contained in the unstable leaf $\widetilde{W}^u(L \cap H_1)$ and so G_i cannot intersect l, contradiction. As l escapes down in L, then the ideal point of $\Psi(l)$ is $(L \cap H_1)_-$ which is equal to $(H_1)_-$, the negative ideal point of H_1.

Since the values of $\Psi(p)$ for p in $\partial_\infty F$ are obtained as limits of values in $\Psi(F)$, then we only need to show that if z_k is in F and z_k converges to p as $k \to \infty$, then $\Psi(z_k)$ converges to $\Psi(p)$. Suppose this is not the case.

By taking a subray if necessary, we may assume that l does not intersect a lift annulus and hence it is transverse to the unstable foliation $\widetilde{\Lambda}^u_F$ in F. Parametrize the leaves of $\widetilde{\Lambda}^u_F$ intersected by l as $\{g_t, \ t \in \mathbf{R}_+\}$, contained in $G_t \in \widetilde{\Lambda}^u$ (by an abuse of notation think of the G_i as a discrete subcollection of the $G_t, t \in \mathbf{R}_+$). Let

$$\mathcal{U} \ = \ \bigcup_{t > 0} G_t$$

No g_t (or leaf of $\widetilde{\Lambda}^u_F$) has ideal point v in $\partial_\infty F$. This implies that g_t escapes compact sets in F as $t \to \infty$ and the ideal points of g_t converge to v on either side of v. If ideal points do not converge to v then since ideal points of leaves of $\widetilde{\Lambda}^u_F$ are dense in $\partial_\infty F$, there will be leaf g in the limit of the g_t. Then since $\pi_1(M)$ is negatively curved there can only be finitely many leaves in the limit and consecutive leaves share an ideal point, because of the denseness again. It would then follow that some limit leaf has to have ideal point v, contradiction.

Up to subsequence assume that all of the elements of the sequence $\{z_k\}$ are either entirely contained in $\mathcal{U}$ or disjoint from $\mathcal{U}$.

Situation 1 $-$ Suppose that z_k is not in $\mathcal{U}$ for any k.

Since z_k is very close to p in $F \cup \partial_\infty F$ and g_t converges to v in $F \cup \partial_\infty F$ when $t \to \infty$, then there are t, s with z_k between g_t and g_s (in F). Notice z_k is not in any of them. Now there is a unique time t_k so that exactly at that time $\Psi(z_k)$ switches from being in one side of G_t in $\widetilde{M}$ to the other

Figure 13: *a. Line leaf separating points, b. Non separated leaf separating points.*

(equivalently compare the z and g_t in F). In particular, either there is a line leaf L_{t_k} of G_{t_k} which separates $\Psi(z_k)$ from all the other G_t, see fig. 13, a, or there is a leaf L_{t_k} non separated from G_{t_k} with $\Psi(z_k)$ either in L_{t_k} or L_{t_k} separates $\Psi(z_k)$ from all G_t, see fig. 13, b. This can be seen in the leaf space of $\widetilde{\Lambda}^u$, which is a non Hausdorff tree [Fe10, Ga-Ka, Ro-St].

<u>Claim</u> $-$ In the Gromov-Hausdorff topology of closed sets of $\widetilde{M} \cup S^2_\infty$, the sets $\overline{L}_{t_k}$ converge to $(H_1)_-$ as $k \to \infty$.

If L_{t_k} is a line leaf of G_{t_k}, then $(L_{t_k})_- = (G_{t_k})_-$. If L_{t_k} is not separated from G_{t_k} then also $(L_{t_k})_- = (G_{t_k})_-$. This is because there are E_i leaves of $\widetilde{\Lambda}^u$ with E_i converging to $L_{t_k} \cup G_{t_k}$. So there are $x_i,\ y_i$ in E_i with $x_i \to x,\ y_i \to y$ and $x \in L_{t_k},\ y \in G_{t_k}$. Then

$$\eta_-(x_i) \ \to \eta_-(x) \ = \ \eta_-(L_{t_k}), \quad \eta_-(y_i) \ \to \eta_-(y) \ = \ \eta_-(G_{t_k}) \quad \text{and} \ \ \eta_-(x_i) \ = \ \eta_-(y_i).$$

The last equality occurs because x_i, y_i are in the same unstable leaf E_i. Therefore $(L_{t_k})_-$ converges to $(H_1)_-$ when $k \to \infty$. Suppose that $\overline{L}_{t_k}$ does not converge to $(H_1)_-$ in $\widetilde{M} \cup S^2_\infty$. Since

$$(L_{t_k})_- \ \text{ converges to } \ (H_1)_-,$$

then lemma 6.5 shows that L_{t_k} does not escape compact sets in $\widetilde{M}$. Up to subsequence there are u_k in L_{t_k} with u_k converging to u in $\widetilde{M}$. The first possibility is that the L_{t_k} are subsets of the leaves G_{t_k}. This implies that $\widetilde{\Phi}_{\mathbf{R}}(u)$ is in the limit of the sequence of leaves G_{t_k} (in $\widetilde{M}$), so it is contained in $\mathcal{V}$. The second possibility is L_{t_k} non separated from G_{t_k} so L_{t_k} is between $G_{t_{k-1}}$ and $G_{t_{k+1}}$ hence u is again in the limit of the G_t so u is in $\mathcal{V}$. The leaves H_j in $\mathcal{V}$ are non singular in the side the G_t are limiting on, so there is a neighborhood of u on that side of H_j which has no singularities hence the u_k will be in $\mathcal{U}$ for k big enough. This contradicts the hypothesis in this case.

This shows that $\overline{L}_{t_k}$ converges to $(H_1)_-$ in $\widetilde{M} \cup S^2_\infty$. Also L_{t_k} either contains $\Psi(z_k)$ or separates it from a base point in $\widetilde{M}$. It follows that $\Psi(z_k)$ converges to $(H_1)_-$, which is what we wanted to prove. This finishes the analysis in situation 1.

<u>Situation 2</u> $-$ For all k assume that $\Psi(z_k)$ is in $\mathcal{U}$.

Let t_k with $\Psi(z_k)$ in G_{t_k}, hence z_k is in $G_{t_k} \cap F = g_{t_k}$. Then $(\Psi(z_k))_- = (G_{t_k})_-$ converges to $(H_1)_-$. Assume up to taking a subsequence that $\Psi(z_k)$ converges to q different from $(H_1)_-$. As above, up to subsequence assume $\widetilde{\Phi}_{\mathbf{R}}(\Psi(z_k))$ converges to $\widetilde{\Phi}_{\mathbf{R}}(z)$. Since $\Psi(z_k)$ is in G_{t_k} then z is in $\mathcal{V}$, say z is in H_j. Let $p = \Theta(z)$. At this point notice that F does not intersect any leaf H_i in $\mathcal{V}$. If it did, say in w then F intersects the nearby leaves G_t (for any t big enough) near w. This would imply $F \cap G_t = g_t$ does not escape compact sets in F, contradiction. Therefore $\Theta(p)$ is in $\partial\Theta(F)$. Let x_k in $g_{t_k} \cap l$. Then $\Theta(x_k)$ converges to a point in $\Theta(H_1 \cap L)$. There are segments b_k in $F \cap G_{t_k} = g_{t_k}$ from x_k to z_k. Then $\Theta(b_k)$ converges to a ray in $\Theta(H_1)$ and a ray in $\Theta(H_j) \subset \mathcal{O}^u(p)$ and possibly other unstable leaves. Then there is a ray in $\mathcal{O}^u(p)$ contained in $\partial\Theta(F)$. This implies that F escapes down as $\Theta(F)$ approaches this ray of $\Theta(H_j)$. Hence $\Psi(z_k)$ is getting closer to z_- which is $(H_j)_-$, which is also equal to $(H_1)_-$. This is what we wanted to prove anyway.

This finishes the proof of case 2.1.1, that is, when L intersects $\mathcal{V}$.

Lemma 6.6. *Let A in $\widetilde{\Lambda}^u$, B in $\widetilde{\Lambda}^s$ satisfying: there are R_i leaves of $\widetilde{\Lambda}^u$ intersecting B with R_i converging to A and $R_i \cap B$ escaping compact sets in B. Then A_- is equal to B_+.*

Proof. Since R_i converges to A then $(R_i)_-$ converges to A_-. Also R_i intersects B so $(R_i)_- = (R_i \cap B)_-$. As $R_i \cap B$ escapes compact sets in B then in the intrinsic geometry of B, the $R_i \cap B$ converges to the positive ideal point of B. This implies that $(R_i \cap B)_-$ converges to B_+. This implies the result. $\square$

Case 2.1.2 $-$ L does not intersect $\mathcal{V}$.

Then $\Theta(l)$ escapes in $\Theta(L)$ and so $\Psi(l)$ converges to L_+. By the previous lemma, this is also equal to $(H_1)_-$. From this point on, the proof is the same as in case 2.1.1. This finishes the proof of case 2.1.

Case 2.2 $-$ v is an ideal point of both $\widetilde{\Lambda}^s_F$ and $\widetilde{\Lambda}^u_F$.

Case 2.2.1 $-$ For any ray l of $\widetilde{\Lambda}^s_F$ and e of $\widetilde{\Lambda}^u_F$ with $l_\infty = e_\infty = v$, then l does not intersect e.

Let l', e' be rays as above. We may assume that l', e' do not have any singularities. Parametrize the leaves of $\widetilde{\Lambda}^s_F$ intersecting e' as $\{l_t, t \geq 0\}$ where $l_t \cap e'$ converges to v in $F \cup \partial_\infty F$ as t converges to infinity.

Since l' limits on v and is disjoint from e', then l' is on a side defined by e'. We will prove continuity of Ψ at v from the other side of e'. The point $p_t = l_t \cap e'$ disconnects l_t. For simplicity we only consider those l_t with $l_t \subset L_t \in \widetilde{\Lambda}^s$ and L_t non singular. Let l^1_t be the component of $(l_t - p_t)$ in the e' side union with p_t. Let l^2_t be the other component of $(l_t - p_t)$ union with p_t, see fig. 14, a.

The l^1_t are rays (here we use L_t non singular - but this is just a technicality) and $(l^1_t)_\infty$ are not equal v by hypothesis. They cannot escape compact sets of F since l' with ideal point v is on that side of e'. Hence as t converges to infinity l^1_t converges to a leaf l of $\widetilde{\Lambda}^s_F$ with a ray (also denoted by l) with ideal point v and maybe some other leaves as well. The leaf l either shares a subray with l' or separates l' from e. Let $e' \subset E$ leaf of $\widetilde{\Lambda}^u$ and $l \subset L$, leaf of $\widetilde{\Lambda}^s$.

Case 2.2.1.1 $-$ l^2_t escapes in F as $t \to \infty$.

Let b_t be the ideal point of l^2_t. Then $b_t \neq v$. Let L^2_t be the union of $\widetilde{\Phi}_{\mathbf{R}}(p_t)$ and the component of $L_t - \widetilde{\Phi}_{\mathbf{R}}(p_t)$ containing l^2_t. If L^2_t escapes in $\widetilde{M}$, then the arguments in case 1 show continuity of Ψ at v in the side of e' not containing l'.

Now assume that L^2_t converges to $R_1 \cup ... \cup R_m$ leaves of $\widetilde{\Lambda}^s$ with union $\mathcal{R}$. Notice F may intersect some of these leaves or not. If $\Theta(\Psi(p_t))$ does not escape in $\Theta(E')$, then one of the R_i, call it R_1, is a leaf intersecting E'. As seen in the arguments for case 2.1.1, F escapes up in this direction so $\Psi(p_t)$ converges to $(R_1)_+$. If $\Theta(\Psi(p_t))$ escapes in $\Theta(E')$, then lemma 6.6 shows that $\Psi(p_t)$ also converges to $(R_1)_+$. This is equal to $(R_j)_+$ for any j.

Suppose there are $t_k \to \infty$ and z_k in $l^2_{t_k}$ with $\Psi(z_k)$ not converging to $(R_1)_+$. Here there is no need to assume that L_{t_k} is non singular. Up to subsequence assume $\Psi(z_k)$ converges to another point q of $\widetilde{M} \cup S^2_\infty$. Then up to subsquence $\widetilde{\Phi}_{\mathbf{R}}(z_k)$ converges to $\widetilde{\Phi}_{\mathbf{R}}(z)$ and hence z is in $\mathcal{R}$, say in R_i. Then $\widetilde{\Phi}_{\mathbf{R}}(z_k)$ are near $\widetilde{\Phi}_{\mathbf{R}}(z)$ and since a ray of $\Theta(R_i)$ is in $\partial\Theta(F)$, then this is stable boundary. So F escapes up as $\Theta(F)$ approaches $\Theta(z)$ and hence $\Psi(z_k)$ converges to $(R_i)_+$. This is equal to $(R_1)_+$. The arguments of Case 2.1.1, situation 1 then show continuity of Ψ at v on this side of e'. This finishes the analysis of case 2.2.1.1.

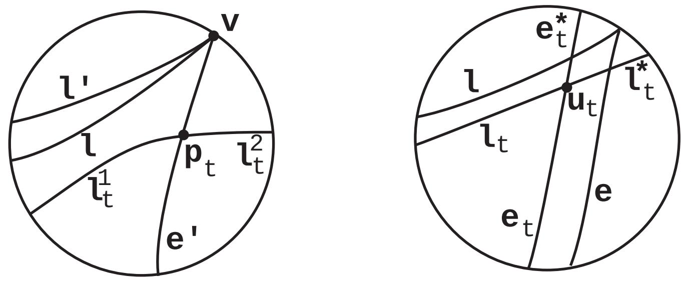

Figure 14: *a. Convergence on one side, b. Case 2.2.1.2 - intersection of leaves.*

<u>Case 2.2.1.2</u> $-$ The l_t^2 limit to r in F as $t \rightarrow \infty$.

Choose the leaf r with a ray which has ideal point v. Then the leaves r, l are not separated from each other in the leaf space of $\widetilde{\Lambda}^s_F$. Proposition 5.6 shows that the region bounded by these rays of r, l with ideal point v projects in M to a set asymptotic to a Reeb annulus. It follows that in F this region is a bounded distance from a geodesic ray with ideal point v. Now we restart the process with the ray r of $\widetilde{\Lambda}^s_F$ instead of e' of $\widetilde{\Lambda}^u_F$. Let $\{b_t, t \geq 0\}$ be a parametrization of the leaves of $\widetilde{\Lambda}^u_F$ through the corresponding points x_t of r. If the components of $(b_t - x_t)$ on the side opposite of e' escapes compact sets in F, then the analysis of case 2.2.1.1 shows continuity of Ψ at v in that side of r. Since r and e' are a bounded distance from each other in F, this shows continuity of Ψ at v on that side of e'.

Otherwise this process keeps being repeated. Let $A_0 = L$, A_1 be the leaf of $\widetilde{\Lambda}^s$ containing r. If the process above does not stop, we keep producing A_i in $\widetilde{\Lambda}^s$, so that they all disjoint and A_i is non separated from A_{i+1}. By theorem 2.6 up to covering translations there are only finitely many leaves of $\widetilde{\Lambda}^s$ which are not separated from some other leaf of $\widetilde{\Lambda}^s$. There is then $m > n$ and h covering translation with $h(A_n) = A_m$. Let f be the generator of the joint stabilizer of A_0, A_1. This is non trivial by theorem 2.6. Then f preserves all the prongs of A_1 and therefore leaves invariant all the A_i. Hence $h^{-1} f h(A_n) = A_n$ and so $h^{-1} f h = f^a$ for some integer a. This implies there is a $\mathbf{Z} \oplus \mathbf{Z}$ in $\pi_1(M)$, see detailed arguments in [Fe10]. This is a contradiction.

There is then a last leaf l_y (of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$) obtained from this process. The arguments of case 2.2.1.1 show continuity of Ψ at v on the other side of l_y. The region between e' and l_y is composed of a finite union of regions between non separated rays of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$. They are all a bounded distance from a geodesic ray with ideal point v, so the whole region also satisfies this property. It follows that this region can only limit in $\Psi(v)$ as well and this proves continuity of Ψ at v in that side of e'.

An entirely similar analysis shows continuity of Ψ at v from the side of l' not containing e'.

What remains to be analysed is the region of F <u>between</u> the rays l' and e'. Whenever there is non Hausdorfness involved, this region is a bounded distance (the bound is not uniform) from a geodesic rays with ideal point v. This is not the case a priori if there is no non Hausdorfness involved. In this case the region between l' and e' may not have bounded thickness in F and hence it is unclear whether its image under Ψ can only limit in $\Psi(v)$. We analyse this case now.

In this last case parametrize the leaves of $\widetilde{\Lambda}^u_F$ intersecting the ray l of $\widetilde{\Lambda}^s_F$ as $\{e_t \mid t \geq 0\}$. Since l_t converges to l, then for big enough t, the leaves l_t, e_t intersect $-$ let u_t be their intersection point, see fig. 14, b. Now define l_t^* to be the component of $l_t - u_t$ intersecting e and e_t^* the component of $e_t - u_t$ intersecting l. Since e' is on that side of l, the e_t cannot escape and converge to a leaf e of $\widetilde{\Lambda}^u_F$ with an ideal point v. Let $e \subset E$ leaf of $\widetilde{\Lambda}^u$.

Recall that L_t is the leaf of $\widetilde{\Lambda}^s$ containing l_t^* and similarly let E_t be the leaf of $\widetilde{\Lambda}^u$ containing e_t. Let L_t^* be the component of $L_t - \widetilde{\Phi}_{\mathbf{R}}(u_t)$ containing l_t^* and similarly define E_t^*. In this remaining case the l_t^* escape in F and so do the e_t^*. Hence $\mu_t = l_t^* \cup \{u_t\} \cup e_t^*$ defines a shrinking neighborhood system of v in $F \cup \partial_\infty F$. Consider the set

$$B_t = L_t^* \cup \widetilde{\Phi}_{\mathbf{R}}(u_t) \cup E_t^*$$

We want to show that $\overline{B}_t$ converges to L_+ in the topology of closed sets of $\widetilde{M} \cup S^2_\infty$.

First consider $L_t^* \cap E$ which intersects F in $(l_t^* \cap e)$. If $L_t^* \cap E$ does not escape compact sets in E then it limits to an orbit γ contained in a leaf H of $\widetilde{\Lambda}^s$. Then L, H are not separated from each other. But for t big enough then E_t is near enough E and will intersect H as well. This contradicts $E_t \cap L$ is not empty and L, H non separated. Hence $L_t^* \cap E$ escapes in E and similarly $E_t^* \cap L$ escapes in L. Hence L, E form a perfect fit. This implies that $L_+ = E_-$. Also $\Psi(e)$ limits to E_- and $\Psi(l)$ limits to $L_+ = E_-$.

The set $\overline{L}_t^*$ contains $(L_t^* \cap E)_+$ and this converges to E_- when $t \to \infty$. This is because $(L_t^* \cap E)$ escapes in E. If $\overline{L}_t^*$ does not converge to E_- in $\widetilde{M} \cup S^2_\infty$, then we find $t_k \to \infty$ and $x_k \in L^*_{t_k}$ with x_k converging to x not equal to E_-. Since $(x_k)_+ = (L_{t_k})_+$ converges to E_-, then up to subsequence assume $\widetilde{\Phi}_{\mathbf{R}}(x_k)$ converges to $\widetilde{\Phi}_{\mathbf{R}}(z)$ for some z in $\widetilde{M}$. Then z is in a leaf H of $\widetilde{\Lambda}^s$ which is non separated from L.

The leaf H does not intersect F, because l_t^* escapes in F by hypothesis in this final situation. It follows that $\Theta(H)$ has a ray contained in $\partial\Theta(F)$ and so this is stable boundary of $\Theta(F)$. Hence F escapes up as $\Theta(F)$ approaches $\Theta(H)$ and consequently $\Psi(x_k)$ limits to $H_+ = L_+ - = E_-$ – which is what we wanted anyway. This shows that $\overline{L}_t^*$ converges to E_- in $\widetilde{M} \cup S^2_\infty$.

Analysing the sets E_t^* in the same manner we obtain that $\overline{E}_t^*$ converges to L_+ as $t \to \infty$ as well. This implies that $\overline{B}_t$ converges to $L_+ = \Psi(v)$. Since $B_t \cap F = \mu_t$ and the μ_t define a neighborhood basis of v in $F \cup \partial_\infty F$, this shows continuity of Ψ at v. This finishes the proof of case 2.2.1.2 and hence of case 2.2.1.

<u>Case 2.2.2</u> – There are rays l of $\widetilde{\Lambda}^s_F$ and e of $\widetilde{\Lambda}^u_F$ starting at u_0 and having the ideal point v.

We will first prove continuity on the side of e not containing a subray of l. There will be an iteration of steps. Before we start the analysis we want to get rid of some problems as described now. Suppose that there are α_0, β_0 leaves of $\widetilde{\Lambda}^s_F$ (or leaves of $\widetilde{\Lambda}^u_F$) which have non separated rays converging to v in $\partial_\infty F$ and on that side of e. Suppose there are infinitely many of these on that side of e. Let them be α_i, β_i and G_i in $\widetilde{\Lambda}^s$ containing α_i. Each region B between α_0 and any α_i is a bounded distance from a geodesic ray in F with ideal point v. The image $\Psi(B)$ then can only limit in $\Psi(v)$. If the G_i do not escape in $\widetilde{M}$ then they converge to a leaf G of $\widetilde{\Lambda}^s$. Let A be an unstable leaf intersecting G tranversely. For i big enough then A intersects G_i transversely, which is impossible, as it would intersect α_i and β_i and these are not separated. Hence the the G_i escapes in $\widetilde{M}$. Then as seen in case 1, there is continuity of Ψ at v in that side of α_1.

Another situation is when there are leaves α_i in that side of e with two rays with ideal point v. Then they are in the interior of a spike region B with one boundary g with ideal point v. If there are infinitely many of these, where none of the α_i are nested with each other, then let G_i in $\widetilde{\Lambda}^s$ containing α_i. As in the previous paragraph, the G_i have to escape in $\widetilde{M}$ and we have continuity in that side of α_1.

Therefore we can assume there are only finitely many occurrences of spike regions or non separated leaves with ideal point on this side of e. If there is any of these let e_0 be the last ray in that side coming from such occurrences. Otherwise let e_0 be the ray given e by the hypothesis in this case. For simplicity assume that e_0 is a ray in $\widetilde{\Lambda}^u_F$, the other case being similar. Let $e_0 \subset E_0 \in \widetilde{\Lambda}^u$.

Parametrize the ray of e_0 as $\{p_t \mid t \geq 0\}$ with p_t converging to v as $t \to \infty$. Let l_t be the leaf of $\widetilde{\Lambda}^s_F$ through p_t and L_t in $\widetilde{\Lambda}^s$ with $l_t \subset L_t$. If L_t escapes $\widetilde{M}$ as $t \to \infty$ then as seen before we have

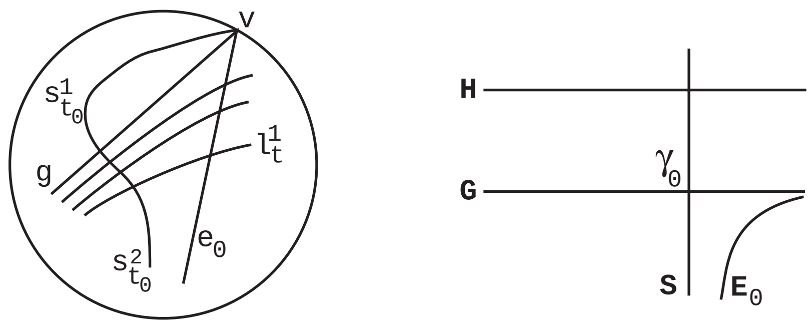

Figure 15: *Some limits in* F, *b. The picture in* $\widetilde{M}$.

continuity of Ψ at v in that side of e_0. So now suppose that L_t converges to $A_1 \cup ....A_m$, leaves of $\widetilde{\Lambda}^s$. This case is considerably more involved, with several possibilities.

<u>Claim</u> $-$ $\Psi(e_0)$ converges to $(A_i)_+$ (notice the $(A_i)_+, 1 \leq i \leq m$ are all equal).

If E_0 intersects some A_i, say A_1, then as seen in case 2.1.1, F escapes positively along $\Psi(e_0)$ as $\Theta(F)$ approaches A_1. This implies that $\Psi(e_0)$ converges to $(E_0 \cap A_1)_+ = (A_1)_+$. If E_0 does not intersect any A_i then $\Psi(e_0)$ converges to $(E_0)_- = (A_1)_+$. This proves the claim.

Let l^1_t be the component of $(l_t - p_t)$ in the side of e_0 we are considering. We are really interested in the behavior for $t \rightarrow \infty$, so we may assume p_t is not singular and there is only one such component.

Suppose first that no l^1_t has a ray with ideal point v and that l^1_t escapes in F as $t \rightarrow \infty$. In this case it is easy to show continuity of Ψ at v and in this side of e_0: Suppose there are x_i in $l^1_{t_i}$ with $t_i \rightarrow \infty$ and $\Psi(x_i) \not\rightarrow (A_i)_-$. Since $(x_i)_+$ converges to $(A_i)_+$ then up to subsequence assume that $(x_i)_- \rightarrow b \neq (A_i)_+$. Up to subsequence $\widetilde{\Phi}_{\mathbf{R}}(x_i) \rightarrow \widetilde{\Phi}_{\mathbf{R}}(x)$. Then x is in some A_i say $x \in A_2$. But F escapes positively as $\Theta(F)$ approaches $\Theta(A_2)$, so $\Psi(x_i) \rightarrow (A_i)_+$, as we wanted. Then as in case 2.1.1 this implies continuity.

There are 2 other options: 1) There is no t with l^1_t with an ideal point v and l^1_t does not escape in F; and 2) There is t with l^1_t having ideal point v. These two options interact and intercalate in appearance as explained below:

<u>Situation 1</u> $-$ There is no t with l^1_t with ideal point v and l^1_t does not escape in F.

There could be several leaves of $\widetilde{\Lambda}^s_F$ in the limit of l^1_t as $t \rightarrow \infty$ but there is a single leaf, call it g with ideal point v. If there is more than one such leaf with ideal point v, then there would have to be one with two rays with ideal point v. This leaf would be in a spike region and it is separated from any other leaf in $\widetilde{\Lambda}^s_F$, contradiction. Let g be contained in a leaf G of $\widetilde{\Lambda}^s$.

Parametrize the ray g as $\{q_t \mid t \geq 0\}$, with $q_t \rightarrow v$ as $t \rightarrow \infty$. Let s_t be the unstable leaf of $\widetilde{\Lambda}^u_F$ through q_t. Let s^1_t be the component of $(s_t - q_t)$ on the side of g opposite to e_0 and s^2_t the other component. Then s^2_t cannot have ideal point v: for t big enough it intersects l^1_t, see fig. 15, a. Then s^2_t converges to e_0. By hypothesis there are no more occurrences of non separated leaves of $\widetilde{\Lambda}^s_F$ with ideal point v on that side of e_0, which implies that s^1_t cannot limit to a leaf of $\widetilde{\Lambda}^u_F$ at $t \rightarrow \infty$ (it would distinct but non separated from e_0). Hence the s^1_t have to escape compact sets in F. If s^1_t does not have an ideal point at v for any t, then the previous analysis shows continuity of Ψ at v in that side of g. As in case 2.2.1.2 if B is the region between g and e_0 then $\Psi(B)$ can only limit in $\Psi(v)$.

Hence assume there is some t_0 so that $s^1_{t_0}$ has ideal point v, see fig. 15, a. Then for t bigger than t_0 all ideal points of s^1_t are v. Let $s^1_{t_0}$ be contained in a leaf S of $\widetilde{\Lambda}^u$ and s_t contained in S_t leaf of $\widetilde{\Lambda}^u$. Since

$$l^1_t \ \to \ g, \quad s^2_t \ \to \ e_0 \quad \text{when} \ \ t \to \infty,$$

$$\text{then} \ \ L_t \ \to \ G, \quad S_t \ \to \ E_0, \quad \text{when} \ \ t \to \infty.$$

It follows that E_0, G form a perfect fit, see fig. 15, b. Hence $(E_0)_- = G_+$. If $\Theta(s^1_{t_0})$ is a ray in $\Theta(S)$ then $\Psi(s^1_{t_0})$ converges to S_-. But $\Theta(s^1_{t_0})$ also converges to

$$\Psi(v) \ = \ (E_0)_- \ = \ G_+ \ = \ (G \cap S)_+.$$

Let $\gamma_0 = G \cap S$, an orbit of $\widetilde{\Phi}$ in G. The above equations imply that

$$(\gamma_0)_+ \ = \ (G \cap S)_+ \ = \ \Psi(v) \ = \ S_- \ = \ (\gamma_0)_-,$$

which is a contradiction. Hence $\Theta(s^1_t)$ is not a ray and has an endpoint x_1 in $\Theta(S)$. Let $\gamma_1 = x_1 \times \mathbf{R}$. Let $H = \widetilde{\Lambda}^s(\gamma_1)$. But F does not intersect H. If F escapes down as $\Theta(F)$ approaches x_1, then $\Psi(v) = (\gamma_1)_-$. But then

$$(\gamma_0)_- \ = \ (\gamma_1)_- \ = \ \Psi(v) \ = \ (\gamma_0)_+$$

contradiction. This implies that F escapes up as $\Theta(F)$ approaches x_1. Hence $\Theta(H)$ has a ray in $\partial\Theta(F)$. Therefore $\Psi(s^1_t)$ limits to $(\gamma_1)_+$. This implies that $(\gamma_0)_+ = (\gamma_1)_+$, where γ_0, γ_1 are distinct orbits of $\widetilde{\Phi}$ in the <u>same</u> unstable leaf S. This is dealt with by the following theorem proved in [Fe7]:

Theorem 6.7. *([Fe7]) Let Φ be a quasigeodesic almost pseudo-Anosov flow in M^3 with $\pi_1(M)$ negatively curved. Suppose there is an unstable leaf V of $\widetilde{\Lambda}^u$ and different orbits β_0, β_1 in V with $(\beta_0)_+ = (\beta_0)_+$. Then $C_0 = \widetilde{\Lambda}^s(\beta_0)$, $C_1 = \widetilde{\Lambda}^s(\beta_1)$ are both periodic, invariant under a nontrivial covering translation f, and the periodic orbits in C_0, C_1 are connected by an even chain of lozenges all intersecting V.*

Remark – This result is case 2 of theorem 5.7 of [Fe7]. In that article the proof is done for quasigeodesic Anosov flows in M^3 with $\pi_1(M)$ negatively curved. The proof goes verbatin to the case of pseudo-Anosov flows. The singularities make no difference. By the blow up operation, the same holds for almost pseudo-Anosov flows.

The theorem implies that G, H are in the boundary of a chain of adjacent lozenges all intersecting S. The first lozenge, call it $\mathcal{C}$ has one stable side contained in G and an unstable side D_1 which makes a perfect fit with G. Suppose first D_1 is in the side of S opposite to E_0, see fig. 16, a. The other unstable side of $\mathcal{C}$ is a leaf D_2 which intersects G on the other side of S. Hence G is some S_c with $c > t_0$. Then $S_c \cap F = s_c$ is a leaf of $\widetilde{\Lambda}^u_F$ and $\Psi(s_c)$ has ideal point $\Psi(v)$. Notice that $\Theta(s_c)$ (which is contained in $\Theta(F)$) escapes in $\Theta(F)$ – otherwise it would produce stable/unstable boundary in $\Theta(F)$ before it hits $\Theta(H)$ and $\Theta(F)$ could not limit on $\Theta(H)$, impossible. Hence $\Psi(s_c)$ limits to $(S_c)_-$ which is equal to $\Psi(v)$. Then

$$(S_c \cap G)_- \ = \ (S_c)_- \ = \ \Psi(v) \ = \ G_+$$

which contradicts the orbit $S_c \cap G$ being a quasigeodesic.

It follows that the perfect fits with G occurs in the E side of S, see fig. 16, b. Here $\Theta(H), \Theta(D_1)$ are contained in the boundary of $\Theta(F)$. We now look at the region B in F bounded by $s_{t_0} = S \cap F$ and $e_0 = E_0 \cap F$.

<u>Claim 1</u> – The image $\Psi(B)$ can only limit in $\Psi(v)$.

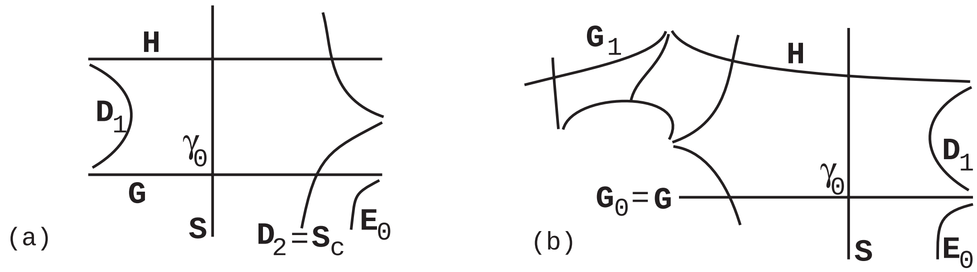

Figure 16: *Perfect fit with G in the side opposite to E_0, b. Perfect fit in the E side.*

The region $\Psi(B)$ is contained in the region $\mathcal{E}$ of $\widetilde{M}$ which is bounded by E, D_1 (maybe other unstable leaves non separated from D_1 as well), H and S, see fig. 16, b. Notice that F does not intersect D_1 or any leaf non separated from D_1 which is beyond D_1. Otherwise $b_0 = (D_1 \cap F)$ is contained in B and non separated from e_0, so it would have both ideal points v. Then it would be contained in the interior of a spike region and could not be non separated from another leaf $-$ impossible. On the other hand since $\Theta(H)$ has a line leaf in the stable boundary of $\Theta(F)$, then $\Theta(D_1)$ has a line leaf in the unstable boundary of $\Theta(F)$. Hence F escapes down as $\Theta(F)$ approaches $\Theta(D_1)$.

Let z_k in B escaping in F and hence converging to v in $\partial_\infty F$. Suppose that $\Psi(z_k)$ does not converge to $\Psi(v)$. Given that z_k escapes F and the structure of the region $\mathcal{E}$, it follows that up to subsequence either $\widetilde{W}^u(z_k)$ converges to D_1 or $\widetilde{W}^s(z_k)$ converges to H. Suppose that $\widetilde{W}^s(z_k)$ converges to H. In that case $(z_k)_+$ converges to $H_+ = \Psi(v)$. Then as seen before if $(z_k)_-$ does not converge to $\Psi(v)$ we can assume up to subsequence $\widetilde{\Phi}_{\mathbf{R}}(z_k)$ converges to $\widetilde{\Phi}_{\mathbf{R}}(z)$. Then z is in a leaf non separated from H and since $\Psi(z_k)$ has to be in $\mathcal{E}$ then z can only be in H. As F escapes up as $\Theta(F)$ approaches $\Theta(H)$ then $\Psi(z_k)$ converges to $H_+ = \Psi(v)$. The case $\widetilde{W}^u(z_k)$ converges to D_1 leads to $\widetilde{\Phi}_{\mathbf{R}}(z_k)$ converging to $\widetilde{\Phi}_{\mathbf{R}}(z)$ with z in unstable leaf non separated from D_1. As F escapes down as $\Theta(F)$ approaches these unstable leaves, then $\Psi(z_k)$ converges to $(D_1)_- = \Psi(v)$. Since this works for any subsequence of z_k, then $\Psi(z_k)$ has to converge to $\Psi(v)$ always. This proves claim 1.

Let $G_0 = G$. Notice that G is periodic and connected to H by an even chain of lozenges. We consider the ray $s_{t_0} = S \cap F$ which has ideal point v. Parametrize it as $\{z_t \mid t \geq 0\}$. Let y_t be the leaf of $\widetilde{\Lambda}^s_F$ through z_t and y^1_t the component of $(y_t - z_t)$ in the side opposite to e_0. The ray s_{t_0} has the same behavior as the original ray e_0. Hence we obtain continuity in that side of s_{t_0} unless y^1_t converges to a leaf μ of $\widetilde{\Lambda}^s_F$ with ideal point v. Let G_1 in $\widetilde{\Lambda}^s$ with $\mu \subset G_1$. Then G_1 is non separated from H, see fig. 16, b and therefore connected to it by a chain of lozenges. It follows that G_1 is connected to G_0 by a chain of lozenges. As in the proof of claim 1, the region B_1 of F between e_0 and $(F \cap G_1)$ has image $\Psi(B_1)$ which can limit only in $\Psi(v)$.

We restart the process with $g_1 = G_1 \cap F$ instead of g. The leaves of $\widetilde{\Lambda}^u_F$ through points of g_1 already converge to the unstable leaf $(D_3 \cap F)$ of $\widetilde{\Lambda}^u_F$ (D_3 is depicted in fig. 16, b). The leaf $(D_3 \cap F)$ cannot be non separated from any other leaf of $\widetilde{\Lambda}^u_F$ in that side of $(D_3 \cap F)$. It follows that the unstable leaves intersected by g_1 escape in F. The only case to be analysed is that some of these unstable leaves have ideal point v. This brings the process exactly to the situation of some s^1_t of $\widetilde{\Lambda}^u_F$ having ideal point v as described before (it was $s^1_{t_0}$). So this would produce H_1 of $\widetilde{\Lambda}^s$ with similar properties as H. This process can now be iterated. As in claim 1 the region of F between g_i and g_{i+1} maps to $\widetilde{M}$ to a region which can only limit in $\Psi(v)$.

We show that this process has to stop. Otherwise produce G_i leaves of $\widetilde{\Lambda}^s$ which are all connected to G_0 by a chain of lozenges. The G_i are all non separated from some other leaf of $\widetilde{\Lambda}^s$, Hence there are G_i, G_j which project to the same stable leaf in M. There is a covering translation h taking G_i to G_j. If f is a generator of the isotropy group of G_0 leaving all sectors invariant, then it leaves invariant all lozenges in any chain starting in G_0 so leaves invariant all the G_i. As before this leads to $h^{-1}fh = f^n$ for some n in $\mathbf{Z}$ and to a $\mathbf{Z} \oplus \mathbf{Z}$ in $\pi_1(M)$. This is disallowed. Therefore the process finishes after say j steps and we obtain continuity of Ψ at v in that side of $g_j = G_j \cap F$. As seen above the region between s_{t_0} and g_j maps by Ψ into a region that can only limit in $\Psi(v)$. This proves continuity of Ψ at v in that side of e_0. This finishes the analysis of situation 1.

Situation 2 $-$ There is $l^1_{t_0}$ with ideal point v.

Recall the setup before the analysis of situation 1. Let $\{u_t \mid t \geq 0\}$ be the collection of unstable leaves intersected by the ray $l^1_{t_0}$. The analysis is extremely similar to the analysis of situation 1, which shows all cases produce continuity in the first step except when u_t converges to a leaf u of $\widetilde{\Lambda}^u_F$ with ideal point v. Then consider the stable leaves intersecting u. The analysis of situation 1 shows continuity unless there is stable leaf with ideal point v. From now on the analysis is exactly the same as in situation 1, with unstable replaced by stable and vice versa.

So far we proved continuity of Ψ at v from the side of e_0 opposite to l. The same works for the other side of l, producing l_0 with similar properties as e_0. We now must consider the regions between e_0 and e, between e and l and between l and l_0.

First consider the region between e and e_0, which occurs only when they are distinct. This implies that the ray e_0 is a bounded distance from a geodesic ray in F with ideal point v. Let $\{\mu_t \mid t \geq 0\}$ be a parametrization of the stable leaves of $\widetilde{\Lambda}^s_F$ through e. Let μ^1_t be the component of $(\mu^1_t - e)$ in the side of e we are considering. If some μ^1_t has ideal point v then both ideal points of μ_t are v and μ_t is inside a spike region. The same is true for e and so e is a bounded distance from a geodesic ray in F with ideal point a. Hence the region between e and e_0 is a bounded distance from a geodesic ray and we are finished in this case.

The remaining case to be analysed here is that μ^1_t has no ideal point v. Then μ^1_t does not escape F as $t \to \infty$, because e_0 is in that side of e. So μ^1_t converges to a leaf μ which has ideal point v. Now consider a parametrization $\{\nu_t \mid t \geq 0\}$ of the unstable leaves intersected by μ. Then ν_t converges to the leaf e. If it converges to some other leaf, then e is a bounded distance from a geodesic ray in F and we are done. Otherwise it must be that some ν_t has ideal point v. Therefore we exactly in the setup analysed in situation 1 above.

This shows continuity of Ψ for the region between e and e_0 and similarly for the region between l and l_0.

Finally we analyse the region B between e and l. First notice there is no singularity in the interior of B. Otherwise there would be a line leaf in B and hence a leaf with both endpoints v. It would have to be part of a spike region and the spike region does not have any singularities in its interior.

Parametrize the leaves of $\widetilde{\Lambda}^u_F$ through l as $\{e_s \mid s \geq 0\}$ and similarly those of $\widetilde{\Lambda}^s_F$ through e as $\{l_t \mid t \geq 0\}$. Let L, L_t leaves of $\widetilde{\Lambda}^s$ with $l \subset L$, $l_t \subset L_t$ and similarly define E, E_t. There are 2 possibilities:

1) Product case $-$ Any l_t intersects every e_s and vice versa.

Equivalently $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$ define a product structure in the region B bounded by l_0 and e_0. If the L_t escapes in $\widetilde{M}$ as $t \to \infty$, then there is a stable product region defined by a segment in L_0. But then theorem 2.7 implies that Φ is topologically conjugate to a suspension, contradiction. It follows that the L_t converge to $H_1 \cup ... H_m$ as $t \to \infty$. Since the l_t are stable leaves, it follows that F escapes up as $\Theta(F)$ appraches $\Theta(H_i)$. This implies that $\Psi(e)$ limits to $(H_i)_+$ which is then equal to $\Psi(v)$. Similarly E_s converges to $V_1 \cup ... V_n$ and F escapes down as $\Theta(F)$ approaches $\Theta(V_j)$. Hence $\Psi(l)$

limits to $(V_j)_- = \Psi(v)$. If some H_i intersects some V_j, then

$$(V_j \cap H_i)_+ \;=\; (H_i)_+ \;=\; \Psi(v) \;=\; (V_j)_- \;=\; (V_j \cap H_i)_-,$$

contradiction. Let now $\{z_k\}$ be a sequence in B converging to v. The product structure implies that up to subsequence we may assume that either $\widetilde{W}^s(z_k)$ converges to H_i or $\widetilde{W}^u(z_k)$ converges to V_j. This is analysed carefully in Claim 1 above, which shows that $\Psi(z_k)$ must converge to $\Psi(v)$. This shows continuity of Ψ when restricted to the region B.

2) Non product case.

There are $t, u > 0$ with $l_t \cap e_u = \emptyset$. Consider one such u. Let a be the infimum of t with $l_t \cap e_u = \emptyset$. Now let b be the infimum of u with $l_a \cap e_u = \emptyset$. Then $l_a \cap e_b = \emptyset$, but for any $0 \leq t < a$ and $0 \leq u < b$ one has $l_t \cap e_u \neq \emptyset$. Since $l_a \cap e_b = \emptyset$, then $L_a \cap E_b = \emptyset$. It follows that L_a, E_b form a perfect fit, see fig. 17, a. If $\Theta(l_a)$ does not escape in $\Theta(L_a)$, then there would be unstable boundary of $\Theta(F)$ in the limit and that would keep F from intersecting E_b, contradiction. Hence $\Theta(l_a)$ escapes in $\Theta(L_a)$ and $\Theta(e_b)$ escapes in $\Theta(E_b)$. Hence $\Psi(l_a)$ limits to $(L_a)_+$ and $\Psi(e_b)$ limits to $(E_b)_-$. Also l_a, e_b limit to v in $\partial_\infty F$.

Let $p_t = l_t \cap e$. If $\Theta(p_t)$ escapes in $\Theta(E)$, then $\Psi(e)$ converges to E_-. Notice that $\Psi(e)$ converges to $\Psi(v)$ so:

$$E_- \;=\; \Psi(v) \;=\; (L_a)_+ \;=\; (L_a \cap E)_+$$

contradiction. It follows that $\widetilde{\Phi}_{\mathbf{R}}(p_t)$ converges to $\widetilde{\Phi}_{\mathbf{R}}(x)$ with x in E. Also F has to escape up as $\Theta(F)$ approaches $\Theta(x)$ – same as in Situation 1 above. Hence $\Psi(e)$ limits to x_+. So

$$x_+ \;=\; \Psi(v) \;=\; (L_a)_+ \;=\; (p_a)_+$$

Let $X = \widetilde{W}^s(x)$. Then x, p_a are in 2 distinct orbits of E with the same positive ideal point. Therefore theorem 6.7 implies that L_a, X are connected by an even chain of lozenges all intersecting E. Let $\mathcal{C}$ be the first lozenge. It has a stable side in L_a and one unstable side, call it D_1 which makes a perfect fit with L_a. Suppose first that D_1 is in the component of $\widetilde{M} - E$ opposite to E_b. Then the other unstable side of $\mathcal{C}$, call it D_2 has to intersect L_a in the other side of E. Then D_2 must be some E_t, let it be $E_{b'}$, see fig. 17, a. Then $\Theta(e_{b'})$ has to escape in $\Theta(E_{b'})$ or else one produces stable boundary to $\Theta(F)$ and $\Theta(F)$ cannot limit to $\Theta(x)$ contradiction. Hence $\Psi(e_{b'})$ converges to $\Psi(v)$ and also to $(E_{b'})_-$. But then

$$(E_{b'} \cap L_a)_- \;=\; (E_{b'})_- \;=\; \Psi(v) \;=\; (E_b)_- \;=\; (L_a)_+$$

again a contradiction.

This implies that D_1 is on the side of E containing E_b, see fig. 17, b.

If there are only 2 lozenges in the chain from L_1 to X, then D_1 also makes a perfect fit with X. Otherwise there are $D_2, ..., D_i$ all non separated from D_1 and so that D_i makes a perfect fit with X and the D_j are all in the boundary of the chain of lozenges. As seen in claim 1 above, F cannot intersect any D_j $(1 \leq j \leq i)$, but all $\Theta(D_j)$ are contained in the unstable boundary of $\Theta(F)$. Also F escapes down as $\Theta(F)$ limits to $\Theta(D_j)$. The set $\Theta(X)$ also has a line leaf which is a stable boundary of $\Theta(F)$ and F escapes up when $\Theta(F)$ approaches $\Theta(X)$.

The same discussion applies to L, so there is y in L, $Y = \widetilde{W}^u(y)$ with $\Theta(Y)$ having a line leaf in the unstable boundary of $\Theta(F)$ and F escapes down accordingly. There are $C_1, ..., C_n$ leaves in $\widetilde{\Lambda}^u$, all non separated from each other and in the boundary of the lozenges in the chain from E_b to Y so

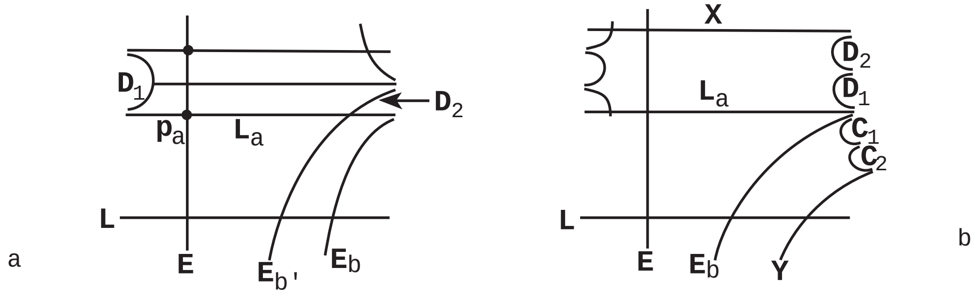

Figure 17: *a. Reaching before, b. Reaching at the exact time.*

that C_1 makes a perfect fit with E_b and C_n makes a perfect fit with Y, see fig. 17, b. Finally $\Theta(C_j)$ has a line leaf in the stable boundary of $\Theta(F)$ and F escapes up accordingly.

Let $\mathcal{E}$ be the region in $\widetilde{M}$ bounded by

$$E,\ L,\ X,\ Y,\ C_1, ..., C_n,\ D_1, ..., D_i$$

Then $\mathcal{E} \cap F$ is exactly the region B bounded by the rays e and l. Let z_k in B escaping to v. Then the region $\mathcal{E}$ shows that up to subsequence one of the following must occur:

1) $\widetilde{W}^s(z_k)$ converges to either X or C_1. The analysis of claim 1 above shows that $\Psi(z_k)$ converges to either X_+ or $(C_1)_+$ both of which are equal to $\Psi(v)$.

2) $\widetilde{W}^u(z_k)$ converges to either Y or D_1. Here $\Psi(z_k)$ converges to either Y_- or $(D_1)_-$ both of which are equal to $\Psi(v)$.

In any case this shows continuity of Ψ in the region B. This finishes the non product case.

This finishes the proof of theorem 6.3, the continuous extension theorem. $\square$

7 Ideal boundaries of pseudo-Anosov flows

Let Φ be a pseudo-Anosov flow. The orbit space $\mathcal{O}$ of $\widetilde{\Phi}$ (the lifted flow to $\widetilde{M}$) is homeomorphic to $\mathbf{R}^2$ [Fe-Mo]. We want to show there is a natural compactification of $\mathcal{O}$ with an ideal circle $\partial\mathcal{O}$ called the ideal boundary of the pseudo-Anosov flow. We will also show that $\mathcal{D} = \mathcal{O} \cup \partial\mathcal{O}$ is homeomorphic to a closed disk. In addition covering translations of $\widetilde{M}$, acting in $\mathcal{O}$ extend to an action in $\mathcal{O} \cup \partial\mathcal{O}$ by homeomorphisms, which are group equivariant. This holds for general pseudo-Anosov flows in 3-manifolds $-$ no metric (or atoroidal) assumptions on M or on the flow Φ. In the next section we specialize to $\pi_1(M)$ being negatively curved and Φ being a quasigeodesic flow. We will show that any section of $\mathcal{O}$ in $\widetilde{M}$ of the orbit map of $\widetilde{\Phi}$ extends to a continuous map from $\mathcal{D}$ to $\widetilde{M} \cup S^2_\infty$. The analysis here will apply equally well to almost pseudo-Anosov flows.

Before we give the formal definition of the ideal points of $\mathcal{O}$ we will give an idea of what they should be. This is done by analysing some examples. The first examples are the $\mathbf{R}$-covered Anosov flows and they have to be treated differently.

For every ray l of a leaf of $\mathcal{O}^s$ or $\mathcal{O}^u$ let l_∞ denote its ideal point. This will be a point in $\mathcal{D}$. Given g in $\pi_1(M)$ it acts in $\widetilde{M}$ and sends flow lines of $\widetilde{\Phi}$ to flow lines and hence acts in $\mathcal{O}$. It also preserves the foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u, \mathcal{O}^s, \mathcal{O}^u$.

1) Ideal boundary for $\mathbf{R}$-covered Anosov flows. The product case.

Recall the picture of product Anosov flow [Fe3, Ba1]: every leaf of $\mathcal{O}^s$ intersects every leaf of $\mathcal{O}^u$ and vice versa. Every ideal point of a ray in $\mathcal{O}^s$ or $\mathcal{O}^u$ should be a point of $\partial\mathcal{O}$ and they are all

(a) (b)

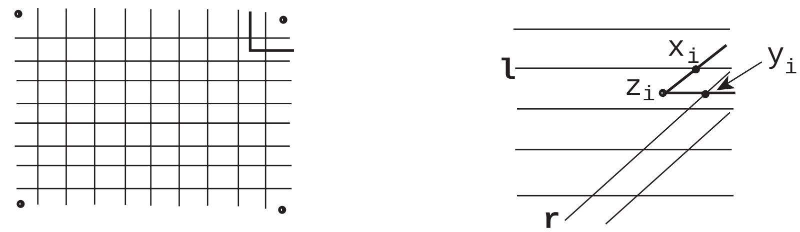

Figure 18: *Ideal points for product* **R***-covered Anosov flow, the dots represent the 4 special points, b. The picture in skewed case.*

distinct. Furthermore there are 4 additional ideal points corresponding to escaping quadrants in $\mathcal{O}$, see fig. 18, a. The quadrants are bounded by a ray in $\mathcal{O}^u$ and a ray in $\mathcal{O}^s$ which intersect only in their common starting point (or finite endpoints). In this case it is straightforward to put a topology in $\mathcal{D}$ so that it is a closed disk and covering transformations act in the extended object. If Λ^s, Λ^u are both transversely orientable, then: any covering translation g fixes the 4 distinguished points. It is associated to a periodic orbit if and only if fixes 4 additional ideal points: if x in $\mathcal{O}$ with $g(x) = x$, then g fixes the ideal points of rays of $\mathcal{O}^s(x), \mathcal{O}^u(x)$. When Λ^s, Λ^u are not transversely orientable, there are other restricted possibilities.

Since we want to define a topology in $\mathcal{D}$ using only the structure of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in $\widetilde{M}$, then a neighborhood basis of the ideal points has to be determined by sets in $\mathcal{O}$. A distinguished ideal point p has a neighborhood basis determined by (say nested) pairs of rays in $\mathcal{O}^s, \mathcal{O}^u$ intersecting at their common finite endpoint and so that the corresponding quadrants "shrink" to p. For an ordinary ideal point p, say a stable ideal point of a ray in $\mathcal{O}^s(x)$, we can use shrinking strips: the strips are bounded by 2 rays in $\mathcal{O}^s$ and a segment in $\mathcal{O}^u$ connecting the endpoints of the rays. The unstable segment intersects the original stable ray of $\mathcal{O}^s(x)$ and the intersections escape in that ray and also shrink in the transversal direction. Already in this case this leads to an important concept:

Definition 7.1. *A chain in $\mathcal{O}$ is either a leaf of $\mathcal{O}^s$ or $\mathcal{O}^u$ or a finite collection $l_1, ... l_n$ of arcs and two rays of $\mathcal{O}^s$ or $\mathcal{O}^u$ so that l_1 and l_n are rays in $\mathcal{O}^s$ or $\mathcal{O}^u$, the other l_i are finite arcs, for $i \neq j$, l_i intersects l_j if and only if $|i - j| = 1$ and only in their finite endpoint and l_i are alternatively in $\mathcal{O}^s$ and $\mathcal{O}^u$. The number n is the length of the chain.*

In the product **R**-covered case, the exceptional ideal points need neighborhoods basis formed by chains of length 2 and all the others need chains of length 3. Notice that the definition of chains implies that it cannot close back up and bound a compact region in $\mathcal{O}$.

2) **R**-covered Anosov flows $-$ skewed case.

Topologically $\mathcal{O}$ is homeomorphic to $(0, 1) \times \mathbf{R}$, a subset of the plane, so that stable leaves are horizontal segments and unstable leaves are segments making a constant angle $\neq \pi/2$ with the horizontal, see fig. 18, b. A leaf of $\mathcal{O}^s$ does not intersect every leaf of $\mathcal{O}^u$ and vice versa [Fe3, Ba1]. Here again the ideal points of rays define ideal points in $\mathcal{O}$. However as is intuitive from the picture, rays of $\mathcal{O}^s, \mathcal{O}^u$ which correspond to leaves of $\widetilde{\Lambda}^s$ and $\widetilde{\Lambda}^u$ which form a perfect fit should define the same ideal point of $\mathcal{O}$. In addition to these ideal points of rays in $\mathcal{O}^s$ or $\mathcal{O}^u$, there should be 2 distinguished ideal points $-$ one from the "positive" direction and one from the "negative" direction. Again it is very easy to put a topology in $\mathcal{D} = \mathcal{O} \cup \partial\mathcal{O}$ making it homeomorphic to a closed disk. In addition covering translations act as homeomorphisms of this disk $-$ a transformation without fixed points in $\mathcal{O}$ fixes only the 2 distinguished ideal points in $\partial\mathcal{O}$, one attracting and another repelling. If a transformation has a fixed point in $\mathcal{O}$, then it leaves invariant a leaf of $\mathcal{O}^s$ or $\mathcal{O}^u$. If this corresponds to an orientation reversing element then there are only 4 fixed points in $\partial\mathcal{O}$. Otherwise there are infinitely many fixed points, see [Fe3, Ba1].

A neighborhood basis of the distinguished ideal points can be obtained from leaves of $\mathcal{O}^s$ or $\mathcal{O}^u$ which escape in that direction (say positive or negative). In the case of non distinguished ideal points we get sequences of chains of length 2 escaping every compact set and "converging" to this ideal point, see fig. 18, b. More precisely if rays l, r of $\mathcal{O}^s, \mathcal{O}^u$ respectively form a perfect fit defining the ideal point p, then choose x_i in l converging to p, y_i in r converging to p and the chain of length two containing rays in the stable leaf through y_i and the unstable leaf through x_i (intersecting in z_i, see fig. 18, b).

3) Suspension pseudo-Anosov flows.

Here we assume it is really a singular flow (that is not an Anosov flow). The fiber is then an hyperbolic surface and every orbit intersects the fiber. The orbit space $\mathcal{O}$ is then identified with the universal cover of the fiber which is metrically like the hyperbolic plane $\mathbf{H}^2$. In this case there is a natural ideal boundary S^1_∞ – the circle at infinity of $\mathbf{H}^2$. But his uses the metric structure on the surface – in general we will not have a metric structure in $\mathcal{O}$, so again here we want to define $\partial\mathcal{O}$ only using the structure of $\mathcal{O}^s, \mathcal{O}^u$. Intuitively $\partial\mathcal{O}$ and S^1_∞ should be the same. From a geometric point of view, there are some points of S^2_∞ which are ideal points of rays of leaves of $\mathcal{O}^s$ or $\mathcal{O}^u$. Notice that in this case no distinct rays have the same ideal point [Bl-Ca, Th4]. This is not true in general [Fe7]. But there are many other points. The foliations $\mathcal{O}^s, \mathcal{O}^u$ can be split into geodesic laminations (of $\mathbf{H}^2$) which have only complementary regions which are finite sided ideal polygons. This implies that there always is a sequence of leaves l_i (in $\mathcal{O}^s$ or $\mathcal{O}^u$) which is nested, escapes to infinity and "shrinks" to the ideal point p. So we can obtain all ideal points in this way.

At this point we are getting close to being able to define an ideal point of $\mathcal{O}$. First we analyse the following situation: let l be a ray (say) in $\mathcal{O}^s$ and assumed without singularity. Then take x_i in l, nested and escaping to the ideal point p in l. This p should be a point in $\partial\mathcal{O}$. For simplicity assume that the leaves g_i of $\mathcal{O}^u$ through x_i are non singular. We would like to say that these leaves g_i "define" the ideal point p. If they escape in $\mathcal{O}$, then that will be the case. However it is not always true that they escape in $\mathcal{O}$. If they do not escape in $\mathcal{O}$, then they limit on a collection of unstable leaves $\{h_j \mid j \in J\}$ – maybe infinitely many. But there is one of them, call it h which makes a perfect fit with l on that side of l. The perfect fit is the obstruction to leaves g_i escaping in $\mathcal{O}$. Consider now e_i stable (non singular) leaves intersecting h and escaping in the direction of the perfect fit with l. Since l and e form a perfect fit, then for big enough i, the e_i and g_i intersect and form a chain of length 2. If the e_i now escape in $\mathcal{O}$, this should define the ideal point p, otherwise we iterate this process. If this stops after a while we can always take chains of a fixed length. Otherwise as the chains escape in $\mathcal{O}$, then we use chains of bigger and bigger lengths, in order to cross over more and more of the collection of perfect fits emanating from l. Such a sequence of chains will be called a *standard sequence* for the ray l of $\mathcal{O}^s$ or $\mathcal{O}^u$.

These sequences are enough to define "neighborhood basis" of ideal points of $\mathcal{O}$. We are now ready to define ideal points of $\mathcal{O}$.

Definition 7.2. *(convex chains) A chain $\mathcal{C}$ in $\mathcal{O}$ is convex if there is a complementary region V of $\mathcal{C}$ in $\mathcal{O}$ so that at any given corner z of $\mathcal{C}$ the local region of V determined by $\mathcal{O}^s(z), \mathcal{O}^u(z)$ is not just a single sector. (notice that if z is non singular there are exactly 4 sectors. If z is a p-prong point there are $2p$ sectors). Let $U = \mathcal{O} - (\mathcal{C} \cup V)$. This region is the convex region of $\mathcal{O}$ associated to the convex chain $\mathcal{C}$. The definition implies that if U contains 2 endpoints of a segment in a leaf of $\mathcal{O}^s$ or $\mathcal{O}^u$, then it contains the entire segment. This is why $\mathcal{C}$ is called convex.*

Definition 7.3. *(equivalent rays) Two rays l, r of $\mathcal{O}^s, \mathcal{O}^u$ are* equivalent *if there is a finite collection* *of rays $l_i, 1 \leq i \leq n$, alternatively in $\mathcal{O}^s, \mathcal{O}^u$ so that $l = l_0, r = l_n$ and l_i forms a perfect fit with l_{i+1} for all $1 \leq i < n$. If $n \geq 3$ then for all $1 \leq i \leq n-2$ the leaves l_i and l_{i+2} are non separated.*

Definition 7.4. *(admissible sequences of chains) An admissible sequence of chains in $\mathcal{O}$ is a sequence of convex chains $\mathcal{C}_i$ so that the associated convex regions U_i are nested and escape to infinity in $\mathcal{O}$. In addition we require that for any i, the 2 rays at the ends of $\mathcal{C}_i$ are not equivalent.*

Intuitively an ideal point of $\mathcal{O}$ is determined by an admissible sequence of chains. Two differenct admissible sequences may define the same ideal point, so we first need to determine when 2 such sequences are equivalent, that is, they define the same ideal point and then we need to define a topology in $\mathcal{O} \cup \partial\mathcal{O}$. At first one might think that any 2 sequences associated to (what we intuitively think is) the same ideal point of $\mathcal{O}$ would have to be eventually nested with each other. This would make it simple and nice. However it is easy to see that such is not the case. For example there are sequences of chains so that one end ray is always in the same leaf and they just shrink on the other side. One could then construct chains on the other side as well, making two disjoint sequences which cannot be nested. We need a couple more ideas.

Given an admissible sequence $\{\mathcal{C}_i \mid i \in \mathbf{N}\}$, since it escapes compact sets, there is a unique component U_i of $\mathcal{O} - \mathcal{C}_i$ denoted by $\widetilde{c}_i$ which does not contain a fixed basepoint in $\mathcal{O}$ (if necessary eliminate a few elements of the sequence or move the basepoint).

Definition 7.5. *Given two admissible sequences of chains $C = \{c_i\}$, $D = \{d_i\}$, we say that C is smaller than D, denoted by $C < D$, if: for any i there is $k_i > i$ so that $\widetilde{c}_{k_i} \subset \widetilde{d}_i$. Two admissible sequences of chains $C = \{c_i\}$, $D = \{d_i\}$ are equivalent and denoted by $C \cong D$ if there is a third admissible sequence $E = \{e_i\}$ so that $C < E$ and $D < E$.*

The ideal points of $\mathcal{O}$ will be the equivalence classes of admissible sequences of chains. We have to prove that $\cong$ is an equivalence class and several other properties. We should stress here that the requirement that the chains are <u>convex</u> is fundamental for the whole discussion. It is easy to see in the skewed $\mathbf{R}$-covered Anosov case, then given any two distinct ideal points p, q on the "same side" of the distinguished ideal points then the following happens: Let l, r be stable rays defining p, q respectively. Then there is a sequence of chains that escape and contain subrays of both l and r. The chains can be chosen to satisfy all the properties, except that they are convex. On the other hand convexity does imply important properties as shown in the next lemma:

Lemma 7.6. *(fundamental lemma) Suppose that Φ is not topologically conjugate to a suspension Anosov flow. Let l, r be rays of $\mathcal{O}^s$ or $\mathcal{O}^u$, which are not equivalent. Then there is no pair of admissible sequence of chains $E = \{e_i\}$, $F = \{f_i\}$ so that: $\widetilde{e}_i \cap \widetilde{f}_i \neq \emptyset$ (for all i) and $\widetilde{e}_i \cap r \neq \emptyset$, $\widetilde{f}_i \cap l \neq \emptyset$, for all i.*

Proof. We assume that both l and r are rays of $\mathcal{O}^s$, other cases are treated similarly. By taking subrays if necessary we may assume that l, r are disjoint, have no singularities and miss a big compact set around a base point in $\mathcal{O}$. Join the initial points of l, r by an arc α' missing a big compact set to produce a properly embedded biinfinite curve $\alpha = l \cup \alpha' \cup r$. Let V be the component of $\mathcal{O} - \alpha$ which misses the basepoint.

We first show that there is no admissible sequence of chains $E = \{e_i\}$ such that $\widetilde{e}_i$ always intersects l and r. This implies that the phenomenon described above (in the skewed Anosov flow case) for non convex chains cannot happen for convex chains.

Suppose this is not true. Let $B_i = \widetilde{e}_i \cup e_i$. Notice first that $B_i \cap r$ is connected. Otherwise there is a compact subarc r_0 of r with ∂r_0 in e_i and the rest of r_0 contained in $\mathcal{O} - B_i$, see fig. 19, a. Starting from an endpoint of r_0 and moving along e_i towards the other endpoint, the only way to

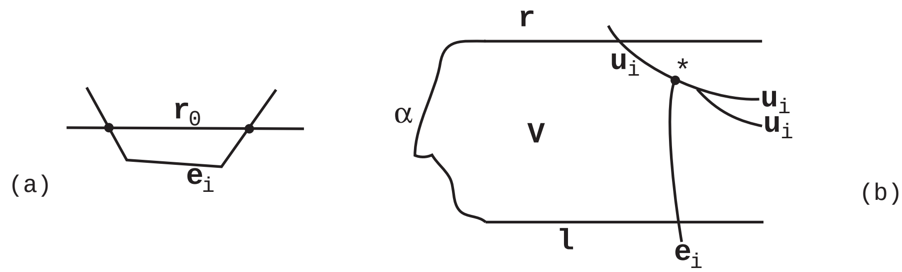

Figure 19: *a. Convexity implies connected intersection of r and B_i. b. All rays of u_i stay in V forever. There is a non convex switch at $*$.*

intersect r again is if there is a non convex switch in e_i, contradiction. This shows that $B_i \cap r$ is connected. This also shows that $\widetilde{e}_i$ is convex as stated in definition 7.2.

Notice that $B_j \cap r$ forms a nested family of sets in r. Since B_j escapes compact sets as $j \to \infty$ and $B_j \cap r$ is connected it follows that $B_j \cap r$ is a subray of r for any j. If $e_i \cap r$ contains a non trivial segment, then by convexity again it follows that $\widetilde{e}_i \cap r = \emptyset$ contradiction. It follows that e_i intersects r in a single point. Let $u'_i = \mathcal{O}^u(e_i \cap r)$, the unstable leaf through the intersection. Up to subsequence we may assume no two u'_i are the same.

Since r has no singularities there are two components of $u'_i - (u'_i \cap r)$. There is only one of them called u_i which locally enters V, see fig. 19, b. There are two possibilities: the first is that some rays of u_i stay in V for all time. By taking another bigger i we may assume that all rays of u_i stay in V forever. In that case, in order for e_i to reach l there must be a non convex switch in e_i, impossible see fig. 19, b. (the non convex switch is at $*$).

The other option is that all rays of u_i exit V. One possibility here is that all u_i intersect l. In that case let z_i be the part of u_i between l and r. If the z_i escapes compact sets, then the region between l and r is an unstable product region as in definition 2.4. Theorem 2.7 then implies that Φ is topologically conjugate to a product **R**-covered Anosov flow. This is disallowed by hypothesis. The lemma fails for product **R**-covered Anosov flows. Hence the u_i converges to a collection of leaves, analysed below. Another possibility is that the u_i does not intersect l. Since l obstructs the u_i to escape compact sets, it also follows that u_i converges to a collection of leaves. Let u be one of the limit leaves.

Consider the set of stable leaves non separated from u and in that side of r. By theorem 2.6 there are only finitely many stable leaves between any given u and r, so we may assume that u is the closest one. If u does not make a perfect fit with r, then when considering the stable leaves intersected by u, they do not converge to r, but rather to some other stable leaf r'. The region between r, r' would then be a product region (since there are no leaves non separated from u in this region). As seen above, this would imply Φ is topologically conjugate to a suspension Anosov flow, contradiction. It follows that r, u form a perfect fit and so the rays r, u are equivalent.

Notice now u is a ray of $\mathcal{O}^u$ and we restart the proof. There is a convex polygonal curve from r to u and so now we need to go from u to l. Notice there is a ray of u in V. Suppose first that the other ray of u is also contained in V. This occurs for instance when all u_i intersect l. The chains $\{e_i\}$ escape compact sets in $\mathcal{O}$ and eventually have to cross over to the other side of u. Since they have to intersect l later on, they will have to cross back to the l side of u again. This would force a non convex switch, contradiction.

It follows that there is a ray of u exiting V. The same argument as above produces v_1 a ray of $\mathcal{O}^s$ making a perfect fit with u and consequently non separated from r, see fig. 20, a. Now iterate to obtain $v_2, ...$ etc.. They are obviously nested and the sequence cannot limit on any leaf of $\mathcal{O}^s, \mathcal{O}^u$,

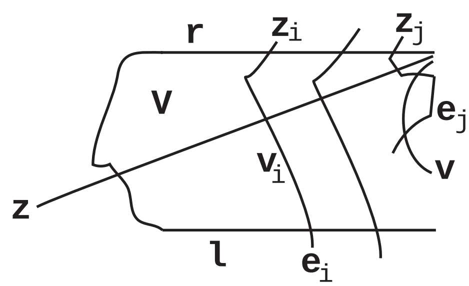

Figure 20: *Producing a perfect fits.*

since v_i, v_{i+2} are non separated from each other in the corresponding leaf space. Since they escape $\mathcal{O}$, eventually all rays of u_i are in V and there is no convex chain to l, forcing a non convex switch. This proves that no escaping sequence of convex chains can always intersect both l and r.

We now prove the more general result. For the general result assume r, l are as above and $E = \{e_i\}, F = \{f_i\}$ are admissible sequences with $\widetilde{e}_i \cap \widetilde{f}_i \neq \emptyset$, $r \cap \widetilde{e}_i \neq \emptyset, l \cap \widetilde{f}_i \neq \emptyset$. As before consider the region V bounded by l, r and an arc connecting them. We go from r to l along e_i and then f_i. The switches are all convex, except for a single one when it moves from e_i to f_i. Once a non convex switch is used, then all subsequent switches have to be convex.

Consider the unstable leaf z_i through $e_i \cap r$. If some z_i has a ray which is entirely in V, then as seen above for $j > i$ all rays of z_j which enter V must be entirely in V. This implies that there has to be a non convex switch from e_i to f_i right in z_i. On the other hand notice that if two rays of $\mathcal{O}^u$ are equivalent, then they are connected by a chain of consecutively non separated leaves. Therefore the rays never intersect the same unstable leaf. Since z_i, z_j intersect r it follows that they are not equivalent. For $k > j$, then one has to spend a non convex switch at z_k and then everything else must be convex. In particular there must be one escaping sequence of convex chains intersecting z_k and z_j. Since z_k, z_j are not equivalent, this is ruled out by the first part of the proof.

We conclude that all rays of z_i which enter V actually exit V. We now mimic the proof of the first part. The z_i converge to a leaf z of $\mathcal{O}^u$ so that z forms a perfect fit with r. We can have a convex chain from r to z and we restart the proof with z, l.

By the above arguments we can assume that we have not spent the non convex switch to go from r to z and consequently e_i intersects z. Let v_i be the unstable leaf through $e_i \cap z$, see fig. 20. Then z_i, v_i form a chain of length 2 connecting r, z. Suppose first that every v_i (for i sufficiently big) intersects l see fig. 20 b. If the region of v_i between z and l escapes in $\mathcal{O}$ then we have a product region of $\mathcal{O}^s$ contradiction to Φ not being a product Anosov flow, by theorem 2.7.

Therefore the v_i limit to a leaf v which we can choose to make a perfect fit with z, see fig. 20. Since E, F are sequences which escape in $\mathcal{O}$, then some e_j crosses to the other side of v. Parts of e_j and f_j form a chain that has to cross v back again, since it has to reach l. But that requires at least 2 non convex switches see fig. 20, b or fig. 19, a. However only one non convex switch is allowed by the hypothesis.

We conclude that there is a v_i not intersecting l. If there is some v_i with a ray entirely in V then as seen above we again produce a contradiction. The remaining option is that every ray of every v_i which enters V has to escape V. As before this produces a leaf v which makes a perfect fit with z. Iterate this process, producing leaves equivalent to u. As seen in the first part of the proof, these leaves have to escape $\mathcal{O}$, producing a contradiction. This finishes the proof of the lemma. $\square$

We now prove that $\cong$ is an equivalence relation.

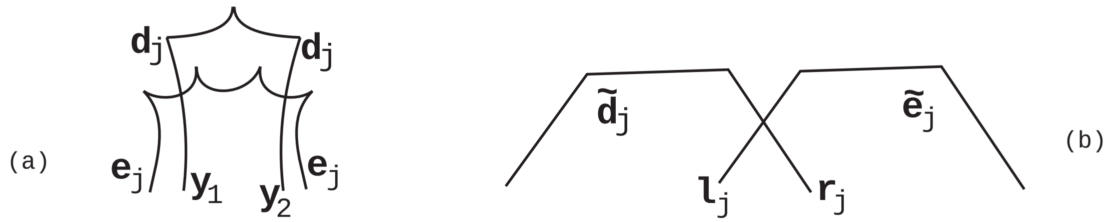

Figure 21: *a. The intersection of convex neighborhoods, b. Intersecting master sequences.*

Lemma 7.7. $\cong$ *is an equivalence relation for admissible sequences of chains.*

Proof. Clearly $\cong$ is reflexive and symmetric. Suppose now that $A = \{a_i\}, B = \{b_i\}, C = \{c_i\}$ are admissible sequences of chains and $A \cong B,\ B \cong C$. Then there is $D = \{d_i\}$ with $A < D,\ B < D$ and $E = \{e_i\}$ with $B < E,\ C < E$. If for some i, j the $\widetilde{d}_i$ and $\widetilde{e}_j$ do not intersect this contradicts $B < D$, $B < E$.

Claim $-$ Fix j. Then either there is i with $\widetilde{a}_i \subset \widetilde{e}_j$ or there is i with $\widetilde{c}_i \subset \widetilde{d}_j$.

Suppose not. Then for each i, then $\widetilde{a}_i \not\subset \widetilde{e}_j$ and $\widetilde{c}_i \not\subset \widetilde{d}_j$. Clearly this implies that none of $\widetilde{d}_i, \widetilde{e}_i$ is contained in the other.

Let y_1, y_2 be the rays of d_j and z_1, z_2 be the rays of e_j. Since there is i with

$$\widetilde{b}_i \ \subset \ \widetilde{e}_j \cap \widetilde{d}_j \ = \ Z$$

then this last intersection is non compact and its boundary ∂Z has two rays which are contained in $y_1 \cup y_2 \cup z_1 \cup z_2$. If there are subrays of both rays in this boundary ∂Z which are contained in $y_1 \cup y_2$, then it follows that $\widetilde{d}_j \cup d_j - (\widetilde{e}_j \cup e_j)$ is contained in a compact set in $\mathcal{O}$, see fig. 21, a. Since the sequence $\{\widetilde{d}_i\}, i \in \mathbf{N}$ escapes compact sets in $\mathcal{O}$, then there would be k with $\widetilde{d}_k \subset \widetilde{e}_j$. But then there is i with $\widetilde{a}_i \ \subset \ \widetilde{d}_k \ \subset \ \widetilde{e}_j$ and this would contradict the assumption in the proof.

It follows that one and only one boundary ray of $\widetilde{e}_j \cap \widetilde{d}_j$ must be contained in $y_1 \cup y_2$ and one and only one boundary ray is in $z_1 \cup z_2$. This last fact also implies that if a boundary ray is contained in $y_1 \cup y_2$ then it cannot have a subray in $z_1 \cup z_2$. Let l_j be the boundary ray contained in $z_1 \cup z_2$. Then this ray is in $\widetilde{e}_j \cup e_j$ and since it cannot have a subray contained in e_j it follows that it has a subray contained in $\widetilde{e}_j$. It also follows that the other ray of d_j has to be eventually disjoint from $\widetilde{e}_j \cup e_j$. Similarly there is a ray r_j of d_j contained in $\widetilde{e}_j$, see fig. 21, b.

Now consider $i \geq j$. Then

$$\widetilde{d}_i \cap \widetilde{e}_j \ \neq \ \emptyset \quad \text{and} \quad \widetilde{d}_i \ \not\subset \ \widetilde{e}_j$$

so the same analysis as above produces a ray of e_j contained in $\widetilde{d}_i$. It can only be l_j since the other ray of e_j is disjoint from $d_j \cup \widetilde{d}_j$, so certainly disjoint from $d_i \cup \widetilde{d}_i$. It now follows that for any $i \geq j$ there is a subray of the fixed ray r_j which is contained in $\widetilde{e}_i$. Similarly for any $i \geq j$ there is a subray of l_j contained in $\widetilde{d}_i$.

The set $\widetilde{d}_j \cap \widetilde{e}_j$ has boundary which contains subrays of the rays r_j, l_j. If r_j, l_j are equivalent then because there is i with $\widetilde{b}_i \subset \widetilde{e}_j \cap \widetilde{d}_j$, the two rays of b_i would be equivalent, contradiction. Therefore r_j, l_j are not equivalent. But for any $i \geq j$, then $\widetilde{d}_i \cup \widetilde{e}_i$ is a union of two convex regions containing subrays of l_j and r_j (j is fixed!). This is now disallowed by the fundamental lemma 7.6. This proves the claim.

Suppose then there are infinitely many j's so that for each one of them, there is $i > j$ with $\widetilde{a}_i \subset \widetilde{e}_j$. Then for any k there is one such j with $j > k$ and so there is $i > j$ with $\widetilde{a}_i \subset \widetilde{e}_j \subset \widetilde{e}_k$. This means that $A < E$ and so $A \cong C$. The other option is there are infinitely many j and for each such j there is $i \geq j$ and $\widetilde{c}_i \subset \widetilde{d}_j$. This now implies that $C < D$ and again $C \cong A$. This finishes the proof that $\cong$ is an equivalence relation. □

Given these results we now define the ideal points of $\mathcal{O}$.

Definition 7.8. *A point in $\partial\mathcal{O}$ or an ideal point of $\mathcal{O}$ is an equivalence class of admissible sequences of chains. Let $\mathcal{D} = \mathcal{O} \cup \partial\mathcal{O}$.*

Given R, an admissible sequence of chains, let $\overline{R}$ be its equivalence class under $\cong$.

Lemma 7.9. *For any equivalence class $\overline{R}$ there are master sequences: An admissible sequence C defining $\overline{R}$ is a master sequence for $\overline{R}$, if for any $B \cong R$, then $B < C$.*

Proof. Intuitively the elements of this sequence approach the ideal points from "both sides".

Case 1 – Suppose that for any $A = \{a_i\},\ B = \{b_i\}$ in $\overline{R}$ and for any i, j then $\widetilde{a}_i \cap \widetilde{b}_j \neq \emptyset$.

We claim that in this case any $A \cong R$ will serve as a master sequence. Let $A \cong R$ and let $B \cong R$. We want to show that $B < A$. Otherwise there is a fixed i so that for any j, $\widetilde{b}_j \not\subset \widetilde{a}_i$. In case 1, $\widetilde{b}_j \cap \widetilde{a}_i$ is not empty for any j. Since $\widetilde{b}_j$ escapes in $\mathcal{O}$, so does $\widetilde{b}_j \cap \widetilde{a}_i$. The arguments of the previous lemma show that $\widetilde{a}_i$ cannot contain subrays of both rays in b_j and in fact for j big enough, then $\widetilde{b}_j$ contains a subray of a single ray of a_i and no singular point. This implies that a_i cuts $\widetilde{b}_j$ into 2 convex regions: one contained in $\widetilde{a}_i$ and the other (call it $\widetilde{b}'_j$) disjoint from $\widetilde{a}_i$. It is essential here that the intersection of $\widetilde{b}_j$ and $\widetilde{a}_i$ does not contain a switch of a_i, otherwise one of the cut up regions in $\widetilde{a}_i$ may well have a non convex corner. Let b'_j be the boundary of $\widetilde{b}'_j$ defining a sequence of chains $B' = \{b'_j\}$. Clearly the sequence $\{b'_j\}$ is nested and escapes $\mathcal{O}$ when $j \to \infty$. Because of the conditions above, they are also convex and therefore B' is an admissible sequence of chains. Clearly $B' < B$, hence B and B' are equivalent and so in $\overline{R}$. But $\widetilde{b}'_j \cap \widetilde{a}_i$ (i fixed) is empty for j big enough. This violates the hypothesis in case 1. This finishes the proof in this case.

Case 2 – There are A, B in $\overline{R}$ and i so that $\widetilde{a}_i,\ \widetilde{b}_i$ are disjoint.

Let C be an admissible sequence with $A < C,\ B < C$. Notice that C is in $\overline{R}$ as well – since $A < C$ implies that $A \cong C$. We claim that C is a master sequence for $\overline{R}$. Let $D \cong A$. Suppose that $D \not< C$. Hence there is i so that for no j we have $\widetilde{d}_j \not\subset \widetilde{c}_i$. As in case 1, we can chop $\widetilde{d}_j$ and produce $E = \{e_j\}$ so that $\widetilde{e}_j$ is disjoint from $\widetilde{c}_i$. Up to deleting a few initial terms in the sequences we have $A \cong B \cong E$ and $\widetilde{a}_1, \widetilde{b}_1, \widetilde{e}_1$ disjoint. Suppose without loss of generality that $\widetilde{b}_1$ is between $\widetilde{a}_1$ and $\widetilde{e}_1$ – that is $\widetilde{e}_1$ is on the $\widetilde{b}_1$ side of $\widetilde{c}_1$. Notice there is $F = \{f_i\}$ with $A < F,\ E < F$. Since $\widetilde{a}_i, \widetilde{b}_i, \widetilde{e}_i$ are all disjoint and $\widetilde{b}_1$ is between $\widetilde{a}_i$ and $\widetilde{c}_i$ then: $\widetilde{f}_i$ contains subrays of the rays of b_1. But f_i escapes compact sets in $\mathcal{O}$. The fundamental lemma 7.6 implies this is impossible. We conclude that C is a master sequence for the equivalence class. This finishes the proof of lemma 7.9. □

Notice that by definition, for any 2 master sequences A, B for an equivalence class $\overline{R}$, then both $A < B$ and $B < A$.

Lemma 7.10. *Let p, q in $\partial\mathcal{O}$. Then p, q are distinct if and only if there are master sequences $A = \{a_i\}, B = \{b_i\}$ associated to p, q respectively with $\widetilde{a}_i \cap \widetilde{b}_j = \emptyset$ for some i, j. Equivalently for some other master sequences this is true for all i, j.*

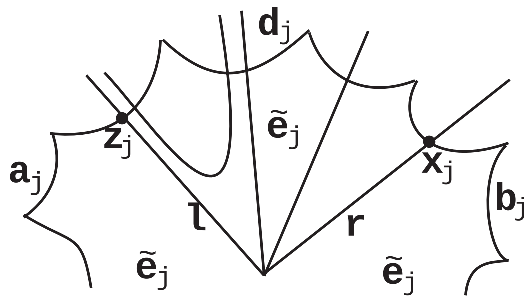

Figure 22: *Interpolating chains that intersect to produce a new convex chain.*

Proof. ($\Leftarrow$) Suppose that $p = q$. Let A, B be any master sequences associated to $p = q$. Then since A, B are master sequences associated to the same equivalence class then $A < B$ and $B < A$. Therefore we can never have $\widetilde{a}_i \cap \widetilde{b}_j = \emptyset$. This is the easy implication.

($\Rightarrow$) Suppose that for any master sequences $A = \{a_i\}$ and $B = \{b_i\}$ associated to p, q respectively and any i, j then $\widetilde{a}_i \cap \widetilde{b}_j \neq \emptyset$. Let A, B be such a pair and define $\widetilde{c}_i = \widetilde{a}_i \cap \widetilde{b}_i$ and let $c_i = \partial \widetilde{c}_i$. Let $C = \{c_i\}$. First of all $\widetilde{b}_i \cap \widetilde{a}_j$ can never be compact or else for some $i' > i$ then $\widetilde{a}_j \cap \widetilde{b}_{i'} = \emptyset$. Also the rays in c_i are subrays of rays of a_i or b_i.

Suppose first that there is i with the rays of c_i equivalent. The rays are u, v and there is a collection $\mathcal{Y} = \{u_0 = u, u_1, ..., u_n = v\}$ so that u_k, u_{k+1} make a perfect fit for every k. Since $\{\widetilde{c}_j\}$ is nested, the rays of c_j for $j > i$ have to be in the collection $\mathcal{Y}$. Up to subsequence we can assume they are all subrays of fixed rays r, l. Notice that $r \neq l$, or else $\widetilde{b}_j \cap \widetilde{a}_j = \emptyset$ for some j. Since r, l are equivalent they cannot both be in a_j (or in b_j either). Hence up to renaming objects, a_j has a subray in r and b_j has a subray in l, for all $j > i$, see fig. 22.

Let $z_j = a_j \cap l$, $x_j = r \cap b_j$. Then z_j escapes in l and x_j escapes in r. Therefore we can connect z_j, x_j by a finite convex chain d_j which extends a_j, b_j and their union is a convex chain e_j. This chain d_j interpolates between a_j, b_j, see fig. 22. Notice that a_j has a subray of r so it goes to r, but a_j may reach r in a point different than x_j. If we just connect this to x_j and then follow b_j this will produce a non convex switch in r. That is why we use the interpolating chain d_j. Then the chains e_j are convex and one can construct the interpolating chain d_j so that e_j escapes compact sets as $j \to \infty$. Then $E = \{e_j\}$ defines an admissible sequence of chains. It is easy to see that $A < E$ and $B < E$ so that $A \cong B$ and $p = q$.

The remaining situation is that the rays of c_i are not equivalent for any i. Then c_i is a convex chain, non empty and C is an admissible sequence. Also $C < A$, $C < B$, which implies that $A \cong C \cong B$ and $p = q$. This proves the lemma. ☐

We now define the topology in $\mathcal{D} = \mathcal{O} \cup \partial \mathcal{O}$.

Definition 7.11. *(topology in $\mathcal{D} = \mathcal{O} \cup \partial \mathcal{O}$) Let $\mathcal{T}$ be the collection of subsets U of $\mathcal{D} = \mathcal{O} \cup \partial \mathcal{O}$ satisfying the following two conditions:*

(a) If x is in $U \cap \mathcal{O}$, then there is an open set O in $\mathcal{O}$ with $x \in O \subset U$.

(b) If p is in $U \cap \partial \mathcal{O}$ and $A = \{a_i\}$ is any master sequence associated to p, then there is i_0 satisfying two conditions: (1) $\widetilde{a}_{i_0} \subset U \cap \mathcal{O}$ and (2) For any z in $\partial \mathcal{O}$, if it admits a master sequence $B = \{b_i\}$ so that for some j_0, one has $\widetilde{b}_{j_0} \subset \widetilde{a}_{i_0}$ then z is in U.

First notice that if the second requirement works for a master sequence $A = \{a_i\}$ with index i_0, then for any other master sequence $C = \{c_k\}$ defining p, we can choose k_0 with $\widetilde{c}_{k_0} \subset \widetilde{a}_{i_0}$. Then $\widetilde{c}_{k_0} \subset U$. If q point of $\partial \mathcal{O}$ has a master sequence $B = \{b_j\}$ with

$$\widetilde{b}_{j_0} \subset \widetilde{c}_{j_0} \quad \text{then} \quad \widetilde{b}_{j_0} \subset \widetilde{a}_{i_0}$$

so q is in U. Therefore (b) works for C instead of A with k_0 instead of i_0. So we only need to check the requirements for a single master sequence.

Lemma 7.12. *The collection of sets in $\mathcal{T}$ forms a topology in $\mathcal{D} = \mathcal{O} \cup \partial\mathcal{O}$.*

Proof. Clearly $\mathcal{D}, \emptyset$ are in $\mathcal{T}$.

Consider unions. If $\{U_\alpha | \alpha \in \mathcal{A}\}$ is a collection of sets in $\mathcal{T}$, then let U be their union. If x is in U and x is in $\mathcal{O}$, there is open set O in $\mathcal{O}$ with $x \in O \subset U_\alpha$ for some index α, hence satisfying condition (a). Let now p in $U \cap \partial\mathcal{O}$. There is $\beta \in \mathcal{A}$ with $p \in U_\beta$. Let $A = \{a_i\}$ be a master sequence associated to p. There is i_0 with

$$\widetilde{a}_{i_0} \subset U_\beta \cap \mathcal{O} \subset U \cap \mathcal{O} \subset \mathcal{O}.$$

In addition if $q \in \partial\mathcal{O}$ and q has a master sequence $B = \{b_j\}$ and j_0 with $\widetilde{b}_{j_0} \subset \widetilde{a}_{i_0}$ then q is in $U_\beta \subset U$. Hence this i_0 works for U as well. This proves that U is in $\mathcal{T}$.

Now consider intersections. Let U_1, U_2 be in $\mathcal{T}$ and $U = U_1 \cap U_2$. Clearly $U_1 \cap U_2 \cap \mathcal{O}$ is open in $\mathcal{O}$. Let $u \in U_1 \cap U_2 \cap \partial\mathcal{O}$. Given a master sequence $A = \{a_i\}$ associated to u there is i_1 with $\widetilde{a}_{i_1} \subset U_1$ and if q has master sequence $B = \{b_j\}$ with $\widetilde{b}_{j_0} \subset \widetilde{a}_{i_1}$ then q is in U_1. Similarly considering $u \in U_2$, there is index i_2 satisfying the conditions for U_2. Let $i_0 = max(i_1, i_2)$. Then $\widetilde{a}_{i_0}$ is contained in U_1 and U_2 (since $\widetilde{a}_i$ are nested). If now q is in $\partial\mathcal{O}$ has a master sequence $B = \{b_j\}$ with $\widetilde{b}_{j_0} \subset \widetilde{a}_{i_0}$ for some j_0 then q is in U_1 and is in U_2 by choice of i_1, i_2. Therefore q is in U. Hence U is in $\mathcal{T}$.

We conclude that $\mathcal{T}$ is a topology in $\mathcal{O} \cup \partial\mathcal{O}$. ◻

Lemma 7.13. *The set $\partial\mathcal{O}$ has a natural cyclic order.*

Proof. Let p, q, r in $\partial\mathcal{O}$ distinct points. As shown in lemma 7.10, there are master sequences $A = \{a_i\}, B = \{b_i\}, C = \{c_i\}$ associated to p, q, r respectively with $\widetilde{a}_1, \widetilde{b}_1, \widetilde{c}_1$ disjoint. From a basepoint x in $\mathcal{O}$ we can draw embedded arcs α, β, γ from x to $\widetilde{a}_1, \widetilde{b}_1, \widetilde{c}_1$ satisfying: the arcs are disjoint except for the endpoint x. This defines a cyclic order on p, q, r. It is easy to see that this is independent of the choice of master sequences (since they are all equivalent) and also choice of arcs. It is also invariant under the action of $\pi_1(M)$ in $\mathcal{O}$. This defines a natural cyclic order in $\partial\mathcal{O}$. ◻

Definition 7.14. *For any convex chain c there is an associated open set U_c of $\mathcal{D}$: let $\widetilde{c}$ be the corresponding convex set of $\mathcal{O}$ (if c has length 1 there are two possibilities). Let*

$$U(c) \;=\; \widetilde{c} \,\cup\, \{x \in \partial\mathcal{O} \mid \text{ there is a master sequence } \; A = \{a_i\} \; \text{ with } \; \widetilde{a}_1 \subset \widetilde{c}\,\}$$

It is easy to verify that $U(c)$ is always an open set in $\mathcal{D}$. In particular it is an open neighborhood of any point in $U(c) \cap \partial\mathcal{O}$. The rays of c are equivalent if and only if $U(c)$ is contained in $\mathcal{O}$. The notation $U(C)$ will be used frequently from now on.

Lemma 7.15. *$\mathcal{D}$ is Hausdorff.*

Proof. Any two points in $\mathcal{O}$ are separated. If p, q are distinct in $\partial\mathcal{O}$ choose disjoint master sequences $A = \{a_i\}$ and $B = \{b_i\}$. Let $U(a_1)$ be the open set of $\mathcal{D}$ associated to a_1 and $U(b_1)$ associated to b_1. By definition $U(a_1)$ is an open neighborhood of p and likewise $U(b_1)$ for q. They are disjoint open sets.

Finally if p is in $\mathcal{O}$ and q is in $\partial\mathcal{O}$, choose U a neighoborhood of q as above so that $U \cap \mathcal{O}$ does not have p in its closure - always possible because master sequences are escaping sets. Hence there are disjoint neighborhoods of p, q. This finishes the proof. ◻

Our goal is to show that $\partial\mathcal{O}$ is homeomorphic to $\mathbf{S}^1$ and that $\mathcal{D}$ is homeomorphic to a closed disk. We need a couple simple facts:

Lemma 7.16. *For any ray l of $\mathcal{O}^s$ or $\mathcal{O}^u$, there is an associated point in $\partial\mathcal{O}$. Two rays generate the same point of $\mathcal{O}$ if and only if the rays are equivalent (as rays!).*

Proof. Given a ray l consider a standard sequence associated to it. The description of standard sequences was done just before the definition of of convex chains (definition 7.2). These sequences define an ideal point of $\mathcal{O}$. It is clearly associated to the ray since every element in the convex chain contains a subray of the original ray.

Two such points in $\partial\mathcal{O}$ are equivalent if there is a chain than contains subrays of both l, r. By the fundamental lemma, this occurs if and only if the rays are equivalent. □

The following result is very useful.

Lemma 7.17. *Suppose that $A = \{a_i\}$ is an admissible sequence of chains and that all a_i have a ray in a fixed leaf l of $\mathcal{O}^s$ or $\mathcal{O}^u$. Then A is associated to the ideal point of l and A is not a master sequence for this point.*

Proof. Consider $B = \{b_i\}$, a standard sequence associated to l. Then for each i we claim that for big enough j, then $\widetilde{a}_j \subset \widetilde{b}_i$. Suppose not and fix such i. Notice l contains a subray of a_j. If $\widetilde{a}_j$ is not contained in $\widetilde{b}_i$ then the other ray u of b_i has a subray contained in $\widetilde{a}_j$. The ray u is the ray of b_i in the $\widetilde{a}_j$ side. By construction of standard sequences, it follows that the rays u and l are not equivalent. Notice then that $\widetilde{a}_j$ always contains a subray of u (as j varies), $\widetilde{b}_k$ always contains a subray of l (as k varies) and $\widetilde{a}_j \cap \widetilde{b}_k$ is never empty. Since u, l are not equivalent, this is disallowed by the fundamental lemma. This is impossible.

We conclude that there is j with $\widetilde{a}_j \subset \widetilde{b}_i$ and so $A < B$. Therefore $A \cong B$ and A is associated to the ideal point of l. In addition A is not a master sequence because using $B = \{b_j\}$ we can cut $\widetilde{b}_j$ along l and produce another admissible sequence in the other side of l, which is equivalent from A and disjoint from A. Hence A cannot be a master sequence. □

Lemma 7.18. *Let $A = \{a_i\}$ be an admissible sequence defining a point p in $\partial\mathcal{O}$. Then one of the following mutually exclusive possibilities occurs:*

(i) There are infinitely many i and side rays l_i of a_i which are equivalent to each other. Then p is the ideal point of any of the l_i and A is not a master sequence for p.

(ii) There are only finitely many sides which are equivalent to any given side and A is a master sequence for p.

Proof. Suppose first that there are infinitely many l_i contained in a ray l. As seen in the previous lemma, this implies that p is associated to the ideal point of l and A is not a master sequence.

Next suppose there are infinitely many l_i which are distinct but equivalent. Up to discarding a few initial terms we can assume all l_i are equivalent. There is a ray l of a_1 so that all l_i are equivalent to l. Without loss of generality we can assume all l_i are in stable leaves v_i of $\mathcal{O}^s$. Then as the v_i are distinct, they escape compact sets of $\mathcal{O}$ as $i \to \infty$. Also v_i separates $\widetilde{a}_i$ from a basepoint. Let $B = \{b_j\}$ be a standard sequence associated to the ideal point of l. Then for any j, then chain b_j intersects only finitely many of the $\{v_i, i \in \mathbf{N}\}$, so $b_j \cap v_i = \emptyset$ for i big enough which implies that $\widetilde{a}_i \subset \widetilde{b}_j$. It follows that $A < B$ and A is associated to the ideal point of l, or of l_k for any k. As in the previous lemma the sequence $\{a_i\}$ is in only one side of l and hence it cannot be a master sequence for the ideal point of l.

The other case is that there are only finitely many sides equivalent to any given side. If A is not a master sequence for p, then there is B an admissible sequence for the same point and $B \not< A$. As in the proof of lemma 7.7 we can cut along rays of the chains to produce $D = \{d_i\}$ so that $\widetilde{d_i} \cap \widetilde{a}_i = \emptyset$ for i big enough. By hypothesis there is $k > i$ so that a_k does not have rays equivalent with rays in a_i. There is C admissible sequence for p with $A < C$, $D < C$ and C a master sequence. But $\widetilde{c}_k$ will always have to contain subrays of the inequivalent rays a_i and a_k. The fundamental lemma 7.6 shows that $\widetilde{c}_k$ cannot escape compact sets in $\mathcal{O}$. This is a contradiction and shows that A is a master sequence for p. This finishes the proof of the lemma. □

Lemma 7.19. *The space $\mathcal{D}$ is first countable.*

Proof. Let p be a point in $\mathcal{D}$. The result is clear if p is in $\mathcal{O}$ so suppose that p is in $\partial\mathcal{O}$. Let $A = \{a_i\}$ be a master sequence associated to p. We claim that $U(a_i)$ is a neighborhood basis at p. Let U be an open set containing p. By definition 7.11 there is i_0 with $\widetilde{a}_{i_0} \subset U$ and if z in $\partial\mathcal{O}$ admits a master sequence $B = \{b_i\}$ so that for some j_0 then $\widetilde{b}_{j_0} \subset \widetilde{a}_{i_0}$ then z is in U. By the definition of $U(a_{i_0})$, it follows that $U(a_{i_0}) \subset U$.

This shows that the collection $U(a_i), i \in \mathbf{N}$ forms a neighborhood basis at p. □

More importantly we have the following:

Lemma 7.20. *The space $\mathcal{D} = \mathcal{O} \cup \partial\mathcal{O}$ is second countable.*

Proof. We first construct a candidate for a countable basis. Since $\mathcal{O}$ is homeomorphic to $\mathbf{R}^2$ it has a countable basis. For points in $\partial\mathcal{O}$ we proceed as follows: we claim that it suffices to consider admissible sequences where for every i the sides of b_i are in periodic leaves. Let $A = \{a_i\}$ be a master sequence associated to p. Consider a given i. If all sides of a_i are in periodic leaves we are done. Otherwise push them in slightly, that is, make the $\widetilde{a}_i$ smaller. First we do this for the finite sides. The obstruction to pushing it in slightly, still intersecting the same adjacent sides is that there is a singularity in this segment. But then this segment is already in a periodic leaf and we leave it as is. We push a segment slightly to produce a chain b_i. Do this for all i. Given i, then since a_j escapes in $\mathcal{O}$ with increasing j, then it is eventually in the region $\widetilde{b}_i$, that is $a_j \subset \widetilde{b}_i$. We take a subsequence of the original sequence b_i to make it nested.

Given i, consider one ray l of b_i and $\{l_t \mid t \geq 0\}$ leaves of the same foliation as l, with l_t converging to l as $t \to 0$. We want l_t intersecting the side of b_i adjacent to l. Note that this intersection of l and the adjacent side is not a singular point, otherwise we would be done. If the l_t converge to another leaf (in $\widetilde{b}_i$ or not), this implies that l is in a periodic leaf and we are done. There is $j > i$ so that l is not a side of a_j – otherwise $A = \{a_j\}$ would not be a master sequence, by lemma 7.18. Choose j big enough so that this is true. Then there is t sufficiently small so that l_t separates l from a_j. This is true because l_t does not converge to any other leaf besides l. Choose also one t for which l_t is a periodic leaf and change b_i to have this ray in l_t. With the new b_i, there is a_j with j big enough with $a_j \subset \widetilde{b}_i$. Proceeding in this way we produce a master sequence $B = \{b_i\}$ which is nested with A and therefore defines the same ideal point of $\mathcal{O}$. In addition all sides of b_i are periodic, for every i.

Since each sequence has finitely many elements and there are countably many periodic leaves, there are countably many sets of the form b_i as above. Hence there are countably many $U(b)$ with b some b_i as above. The previous lemma shows that this collection forms a countable basis for the points in $\partial\mathcal{O}$. This finishes the proof of the lemma. □

Next we show that $\mathcal{D}$ is a regular space.

Lemma 7.21. *The space $\mathcal{D}$ is a regular space.*

Proof. Let p be a point in $\mathcal{D}$ and V be a closed set not containing p. Suppose first that p is in $\mathcal{O}$. Here V^c is an open set with x in V^c, so there are open disks D_1, D_2 in $\mathcal{O}$, so that $p \in D_1 \subset \overline{D}_1 \subset D_2 \subset V^c$, producing disjoint neighborhoods D_1 of p and $(D_2)^c$ of V.

Suppose now that p is in $\partial\mathcal{O}$. Since p is not in the closed set V, there is an open set O containing p and disjoint from V. Let $A = \{a_i\}$ be a master sequence associated to p. Then there is i_0 so that $U(a_{i_0})$ defined above is contained in O. Notice that the closure of $\widetilde{a}_{i_0}$ in $\mathcal{D}$ is $\widetilde{a}_{i_0}$ with a_{i_0} plus the two ideal points of the rays in a_{i_0}. Clearly the closure in $\mathcal{O}$ is just adjoining a_{i_0}. The 2 ideal points of the rays of a_i are clearly in the closure as any neighborhood contains a subray. Any other point in $\partial\mathcal{O}$, if in $U(a_{i_0})$ are are done, otherwise we find a master sequence disjoint from master sequences of both ideal points of a_{i_0}. Therefore they are not in the closure of $\widetilde{a}_{i_0}$.

Choose j big enough so that a_j does not share any leaf with a_{i_0} and that the rays of a_j are not equivalent to any ray of a_{i_0}, again possible by lemma 7.18. If follows that the closure of $\widetilde{a}_j$ is contained in $U(a_{i_0})$, hence

$$p \in U(a_j) \ \subset \ \ \text{closure}(\widetilde{a}_j) \ \subset \ U(a_{i_0}) \ \subset \ V^c$$

This proves that $\mathcal{D}$ is regular. $\square$

This implies that $\mathcal{D}$ is metrizable:

Corollary 7.22. *The space $\mathcal{D}$ is metrizable.*

Proof. Since $\mathcal{D}$ is second countable and regular, a classical theorem of general topology $-$ the Urysohn metrization theorem (see [Mu] pg. 215) implies that $\mathcal{D}$ is metrizable. $\square$

Therefore in order to prove that $\mathcal{D}$ is compact it suffices to show that any sequence in $\mathcal{D}$ has a convergent subsequence. Still it is a bit tricky to get a handle on an arbitrary sequence of points in $\mathcal{O}$ or in $\partial\mathcal{O}$. We will analyse one case which seems very special, but which in fact implies the general case without much additional work.

Lemma 7.23. *Let $\{ \ l_i \mid i \in \mathbf{N}\}$ be a sequence of slices in leaves of $\mathcal{O}^s$ (or $\mathcal{O}^u$). Suppose that for each i the set $\mathcal{O} - l_i$ has a component C_i so that all C_i are disjoint and the C_i are linearly ordered as seen from a basepoint in $\mathcal{O}$. Then in $\mathcal{D}$, the sequence $C_i \cup l_i$ converges to a point p in $\partial\mathcal{O}$.*

Proof. We assume that l_i is always in $\mathcal{O}^s$, other cases are treated similarly. If the l_i does not escape compact sets in $\mathcal{O}$ when $i \to \infty$ then there are i_k and z_{i_k} in l_{i_k} with z_{i_k} converging to a point z. Clearly the C_{i_k} cannot all be disjoint, contradiction. Therefore the $\{l_i\}$ escape in $\mathcal{O}$ and the C_i are uniquely defined.

The order is defined as follows: Let x in $\mathcal{O}$ be a basepoint. For each i choose an embedded path γ_i from x to l_i. We can do this inductively so that the γ_i are all disjoint except for x. The collection $\{\gamma_i\}$ now induces an order in $\{l_i\}$.

The first situation is that there is an infinite subsequence, which we assume is the original sequence so that l_i are all non separated from each other. Here we state a useful lemma also for future use:

Lemma 7.24. *If p is a point of $\partial\mathcal{O}$ associated to an infinite collection of non separated leaves in $\mathcal{O}^s$ or $\mathcal{O}^u$, then a master sequence is obtained with 2 elements in each chain as described below.*

Proof. See fig. 23, a. Recall the structure of the set of infinitely many non separated leaves of a pseudo-Anosov flow, theorem 2.6. Let $\{z_j, j \in \mathbf{Z}\}$ be the set of leaves non separated from each other and ordered as in theorem 2.6. We can find pairs of rays a_j, b_j, a_j in $\mathcal{O}^u$, b_j in $\mathcal{O}^s$, with $a_j \cup b_j$ intersecting only in their finite boundary point and satisfying: $a_j \cup b_j$ escapes in $\mathcal{O}$ with j and a_j, b_j

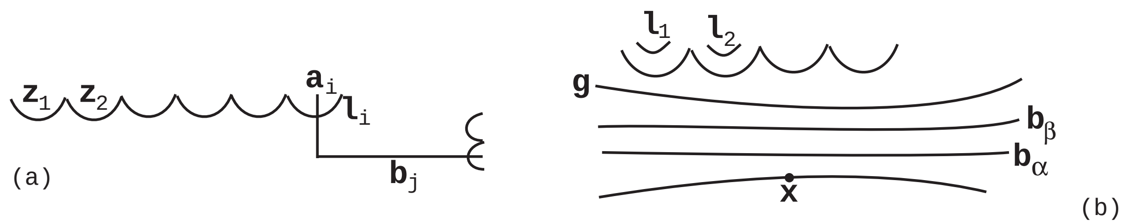

Figure 23: *a. Infinitely many non separated leaves converge to a single ideal point, b. A more interesting situation.*

are not equivalent. Also the a_j intersects z_j, see fig. 23, a. Let $d_j = a_j \cup b_j$, a convex chain and $D = \{d_j\}$. We can choose the a_j, b_j so that the d_j form a nested sequence. Then D is an admissible sequence of chains and defines an ideal point p in $\partial\mathcal{O}$. The collection D is also a master sequence associated to p. $\square$

The $\{l_i, i \in \mathbf{N}\}$ forms a subcollection of a set of leaves non separated from each other. It follows that we can find a_j, b_j as in the lemma and also for any i, l_i intersects a_j where j goes to infinity with i. As in the lemma let $d_j = a_j \cup b_j$ and $D = \{d_j\}$. Then D is a master sequence converging to a point p in $\partial\mathcal{O}$. In addition given any j then for i big enough l_i is contained in $\widetilde{d}_j$. Hence $l_i \cup C_i$ converges to p in $\mathcal{D}$.

The second possible situation is that up to subsequence, then for any distinct i, j, the l_i is separated from l_j. The procedure will be to inductively choose a leaf g_i so that the collection $\{g_i\}, i \in \mathbf{N}$ is nested and escapes compact sets in $\mathcal{O}$. Hence it defines an ideal point of $\mathcal{O}$ and the $l_i \cup C_i$ will converge to it.

Fix x in $\mathcal{O}$. Consider the slices b of $\mathcal{O}^s$ which separate x from ALL of the l_i. Choose x so there is at least one. The collection of slices is clearly ordered by separation properties so we can index then as $\{b_\alpha \mid \alpha \in I\}$ where I is an index set. Put an order in I so that $\alpha < \beta$ if and only if b_α separates b_β from x. Since the b_α cannot escape $\mathcal{O}$ as α increases (they are bounded by all the l_i) then it limits to a collection of leaves and there is a single leaf g which is either equal to some l_{i_0} or separates every l_i from all other elements b_α, see fig. 23, b. In the second case let g_1 be the leaf g thus constructed.

The choice of g_1 in the first case is more complicated. In the first case since l_{i_0} is separated from any other l_i then no other leaf which is non separated from g is equal to some l_i. Hence for each i there is a leaf h_i non separated from g which separates l_i from l_0.

Suppose there are infinitely many distinct h_i. Then as seen in situation one above the h_i converge to a point p in $\partial\mathcal{O}$ and by construction so do the l_i. This finishes the proof in this case. The remaining option is that there are only finitely many distinct h_i. In particular there is some h so that $h = h_i$ for infinitely many i. Let g_1 be this h. Now we are going to restart the process, but we keep only the l_i's separated from l_{i_0} by this h $-$ which must be all l_i for i sufficiently big since the collection $\{l_i, i \in \mathbf{N}\}$ is ordered. Now throw out the first leaf l_i still in the sequence and redo the process. This iterative process produces $\{g_j, j \in \mathbf{N}\}$ which is a weakly monotone sequence of leaves, that is, it is weakly nested. We explain the weak behavior. For instance in the first case, after throwing out l_1 (or whatever first leaf was still present), it may be that only g_1 is a slice which separates x from all other l_i. In that case $g_2 = g_1$. So the g_j may be equal, but they are weakly monotone with j.

If the $\{g_j, j \in \mathbf{N}\}$ escapes in $\mathcal{O}$ with j, then since each g_j separates infinitely many l_i from x we quickly obtain that the l_i converge to a point in $\partial\mathcal{O}$. Suppose then that the $\{g_j, j \in \mathbf{N}\}$ does not escape $\mathcal{O}$. The first option is that there are infinitely many distinct g_j. Up to taking a subsequence assume all g_j are distinct and let g_j converge to $H = \cup h_k$, a collection of stable leaves in $\mathcal{O}^s$. Then

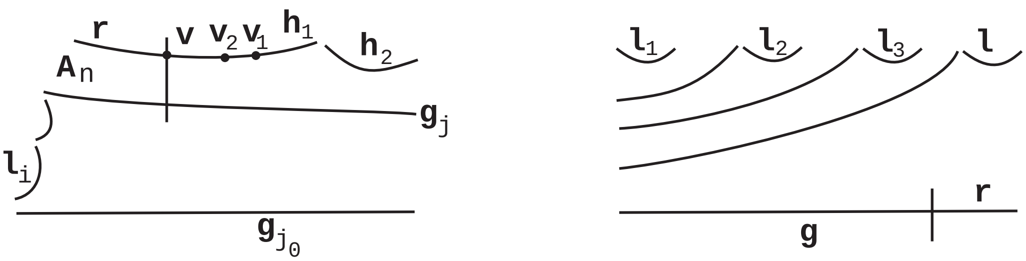

Figure 24: *a. Forcing convergence on one side, b. The case that all g_j are equal.*

by construction, for each j_0, the g_{j_0} separates some l_i from x but for a bigger j, the g_j does not separate l_i from x, see fig. 24, a. Also, for each i there is some j so that g_j separates l_i from x.

We analyse the case there are finitely many line leaves of $\mathcal{O}^s$ in H, the other case being similar. As seen in theorem 2.6 the set of leaves in H is ordered and we choose h_1 to be the leaf closest to the l_i. Also there is a ray r of l which points in the direction of the l_i, see fig. 24, a. Let p be the ideal point of r in $\partial\mathcal{O}$. We want to show that $l_i \cup C_i$ converges to p.

Choose points v_n in r converging to p. For each n then $\mathcal{O}^u(v_n)$ intersects g_j for j big enough $-$ since the sequence g_j converges to H. Choose one such g_j with j converging to infinity with n. We consider a convex set A_n of $\mathcal{O}$ bounded by a subray of r starting at v_n, a segment in $\mathcal{O}^u(v_n)$ between h_1 and g_j for a suitable big g_n and a ray in g_n starting in $g_n \cap \mathcal{O}^u(v_n)$ and going in the direction of the l_i, see fig. 24, a. We can choose j so that the $\{A_n, n \in \mathbf{N}\}$ forms a nested sequence. Let $a_n = \partial A_n$. Since h_1 is the first element of H it follows that $\{a_n\}$ escapes compact sets in $\mathcal{O}$ and clearly it converges to p. For each n and associated j, there is i_0 so that for $i > i_0$ then g_j separates l_i from x. If follows that $l_i \cup C_i$ is contained in A_n and therefore $l_i \cup C_i$ converges to p in $\mathcal{D}$. This finishes the proof in this case.

If H is infinite let $H = \{h_k, k \in \mathbf{Z}\}$ with k increasing as h_k moves in the direction of the l_i. Then h_i converges to a point $p \in \mathcal{O}$. A similar analysis as in the case that H is finite shows that $l_i \cup C_i$ converges to p in $\mathcal{D}$. Use the convex chains $a_j \cup b_j$ as described in lemma 7.24.

The final case to be considered is that up to subsequence all g_i are equal and let g be this leaf. In particular no l_i is equal to g. This can certainly occur as shown in fig. 24, b. If we remove finitely many of the l_i, then g is still the farthest leaf separating x from all the remaining l_i.

Consider the collection of leaves $\mathcal{C}$ of $\mathcal{O}^s$ non separated from g in the side containing the l_i. Put an order in $\mathcal{C}$ with increasing corresponding with increasing i in l_i. Suppose there is h in $\mathcal{C}$ with $h > g$ in $\mathcal{C}$ and either h is equal to some l_k or h separates some l_k from g, see fig. 25, a. There are only finitely many leaves in $\mathcal{C}$ between g, h and since no l_i can go beyond g, it follows that there is some h' in $\mathcal{C}$ so that there are infinitely many i with h' separating g from l_i, see fig. 25, a. In particular there is i_0 so that h' separates g from l_i for all $i > i_0$. Hence the g_j cannot be equal to g for j big enough, contradiction.

It follows the all l_i are in the same component of $\mathcal{O} - \mathcal{C}$. For simplicity assume that $\mathcal{C}$ is finite. (The case where there are infinitely many leaves non separated from g on that side is very similar with proof left to the reader). Let h be the biggest element of $\mathcal{C}$. Let r the ray of h associated to the increasing direction of the the l_i and let p in $\partial\mathcal{O}$ be the ideal point of r. We want to show that l_i converges to p.

Suppose that this is not true. Then there is some convex neighborhood A of p bounded by a convex chain a, so that l_i is disjoint from A for all i, see fig. 25, b. This follows because l_i is ordered. Consider the ray r_1 of a in the side containing l_i. Suppose first this ray is in an unstable

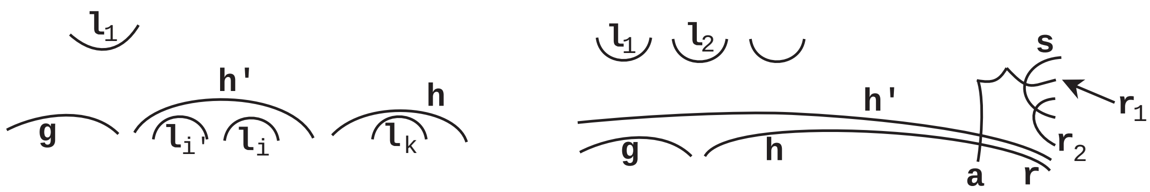

Figure 25: *a. The l_i flip to the other side of a a leaf non separated from g, b. Convex neighborhood disjoint from all.*

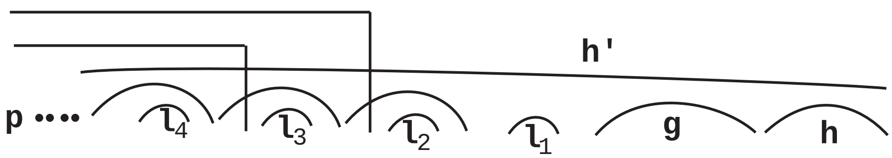

Figure 26: *The final possibility.*

leaf. Consider all the stable leaves through r_1. They cannot all intersect h because the ray of h and r_1 are not equivalent. In addition by going further intersecting r_1 we find a stable leaf s intersecting r_1 so that s and h are not equivalent, see fig. 25.

We do the same analysis and find r_2 stable leaf of $\mathcal{O}^s$ intersecting s and does not have a ray equivalent to r, see fig. 25, b. Now take h' stable leaf very near h and in the l_i side. Because r_2 and r are not equivalent we can assume that h' separates h from r_2, see fig. 25, b. Suppose first that some such h' separates some l_i from h, see fig. 25, b. Then for all i big enough h' does that and so we obtain that g_j would have eventually to move past h' contradiction. The contradiction shows that in this case the l_i are eventually in A and $l_i \cup C_i$ converges to p in $\mathcal{D}$.

The remaining possibility is that for any such h' then it does not separate h from any l_i. As h' gets closer and closer to h, it limits to a leaf f in $\mathcal{C}$ which is either equal to some l_{i_0} or separates all l_i from g. In the second case the g_j would have to eventually move beyond f and could not be constant. In the first case either there is some f' separating infinitely many l_i from g – again g_j would move beyond f' generating a contradiction or any f in $\mathcal{C}$ can only separate finitely many l_i from g. In this case it follows that $\mathcal{C}$ is infinite and the sequence of leaves of $\mathcal{C}$ in this side of g converges to an ideal point p of $\partial \mathcal{O}$, see fig. 26. It follows that the $l_i \cup C_i$ converges to p, as seen before.

This finishes the proof of lemma 7.23. $\square$

The proof of lemma 7.23 was very involved because there are so many places the sequence $\{l_i\}$ can slip through.

Proposition 7.25. *The space $\mathcal{D}$ is compact.*

Proof. Since $\mathcal{D}$ is metrizable, it suffices to consider the behavior of sequences z_i in $\mathcal{D}$. Up to taking subsequences there are 2 cases:

1) Assume the z_i are all in $\mathcal{O}$. If there is a subsequence of z_i in a compact set of $\mathcal{O}$ we are done. So assume that z_i escapes compact sets in $\mathcal{O}$. Let l_i be slice leaves of $\mathcal{O}^s$ with z_i in l_i. Suppose there is a subsequence l_{i_k} converging to l and assume that all l_{i_k} are in one sector of l or in l itself. A small transversal to l intersects l_{i_k} for k big enough and up to subsequence assume all z_{i_k} are in one side of that transversal. Suppose for simplicity there are only finitely many leaves non separated from l in that side. Let l' be the last one in the side the z_{i_k} are in and let p be the ideal point of l' in

that direction. The argument is similar to a previous one: let v_n in l' converging to p in $\mathcal{D}$. Choose a convex chain a_n made up of the ray in l' starting in v_n and converging to p, then the segment in $\mathcal{O}^u(v_n)$ from v_n to $\mathcal{O}^u(v_n) \cap l_{i_k}$ for apropriately big k and then a ray in l_{i_k} starting in this point. As before we can choose the a_n nested and converging to p in $\mathcal{D}$. It follows that z_{i_k} converges to p and we are done in this case. The case of infinitely many leaves non separated from l is treated similarly as seen in the proof of lemma 7.23.

Suppose now that the sequence $\{l_i, i \in \mathbf{N}\}$ escapes compact sets in $\mathcal{O}$. Fix a base point x in $\mathcal{O}$ and assume that x is not in any l_i and define $\widetilde{l_i}$ to be the component of $\mathcal{O} - l_i$ not containing x. Then $\widetilde{l_i}$ escapes compact sets in $\mathcal{O}$. If there is a subsequence l_{i_k} so that l_{i_k} is nested then this defines an admissible sequence of convex chains (of length one) converging to an ideal point p.

Otherwise the has to be i_1 so that it only has finitely many i with $\widetilde{l_i} \subset \widetilde{l}_{i_1}$. Choose $i_2 > i_1$ with $\widetilde{l}_{i_2} \not\subset \widetilde{l}_{i_1}$ and hence $\widetilde{l}_{i_2} \cap \widetilde{l}_{i_1} = \emptyset$ and also so that there are finitely many i with $\widetilde{l_i} \subset \widetilde{l}_{i_2}$. In this way we construct a subsequence $i_k, k \in \mathbf{N}$ with $\widetilde{l}_{i_k}$ disjoint from each other. The collection of leaves

$$\{ \ \widetilde{l}_{i_k} \ \mid \ k \in \mathbf{N}\}$$

is obviously circularly ordered and if we remove one element of the sequence (say the first one) then it is ordered. As such it can be mapped into $\mathbf{Q}$ in an order preserving way. Therefore there is another subsequence (call it still l_{i_k}) for which $\widetilde{l}_{i_k}$ is monotone. Now apply the previous lemma and obtain that l_{i_k} converges to a point p in $\partial\mathcal{O}$ and hence so does z_{i_k}.

This finishes the analysis of case 1.

<u>Case 2</u> $-$ Suppose the z_i are in $\partial\mathcal{O}$.

Here we will eventually use the analysis of case 1. We may assume that the points z_i are distinct. To start we can find a convex neighborhood bounded by a convex chain a_1 so that $\overline{U(a_1)}$ is a neighborhood of z_1 in $\mathcal{D}$ and also it does not contain any other z_i. Otherwise there is a subsequence of $\{z_i\}$ which converges to z_1. Then find a_2 convex chain with $\overline{U(a_2)}$ neighborhood of z_2 in $\mathcal{D}$ disjoint from $\overline{U(a_1)}$ and not containing any other z_i either. Inductively construct a_i convex chains with $\overline{U(a_i)}$ neighborhood of a_i in $\mathcal{D}$ and all $\overline{U(a_i)}$ disjoint from each other. By taking smaller convex neighborhoods we can assume that the $U(a_i)$ escapes compact sets in $\mathcal{O}$ as $i \to \infty$. Up to subsequence we may assume that the $U(a_i)$ forms an ordered set as seen with respect to a collection of disjoint arcs (except for initial points) from x to a_i. Let w_i be a point in a_i. Since a_i escapes compact sets in $\mathcal{O}$ we obtain by case 1 that there is a subsequence w_{i_k} converging to a point p in $\partial\mathcal{O}$. Consider a master sequence $B = \{b_j\}$ associated to p. Let j be an integer. If for any i we have that $\widetilde{a}_i \not\subset \widetilde{b}_j$, then $\widetilde{a}_{i_k}$ has a point w_{i_k} converging to p and also has points outside $\widetilde{b}_j$. This contradicts the $\widetilde{a}_{i_k}$ being all disjoint. Therefore $\widetilde{a}_i \subset \widetilde{b}_j$ for i big enough $-$ this follows because the sequence $\widetilde{a}_i$ is ordered. It follows that w_i converges to p. Therefore there is always a subsequence of the original sequence which converges to a point in $\mathcal{D}$.

This finishes the proof of proposition 7.25, compactness of $\mathcal{D}$. $\square$

We now prove a couple of additional properties of $\mathcal{D}$.

Proposition 7.26. *The space $\partial\mathcal{O}$ is homeomorphic to a circle.*

Proof. The space $\partial\mathcal{O}$ is metrizable and circularly ordered. It is compact as a closed subset of a compact space. We now show that $\partial\mathcal{O}$ is connected, no points disconnect the space and any two points disconnect the space.

Let p, q be distinct points in $\partial\mathcal{O}$. Choose disjoint convex neighborhoods $\overline{U(a)}$, $\overline{U(b)}$ of p, q defined by convex chains a, b. There are ideal points in $\overline{U(a)}$ distinct from p, hence there is a point in $\partial\mathcal{O}$ between p, q. Hence any "interval" in $\mathcal{O}$ is a linear continuun, being compact and satisfying the property that between any two points there is another point. This shows that $\partial\mathcal{O}$ is connected and also that no point in $\partial\mathcal{O}$ disconnects it. In addition as $\partial\mathcal{O}$ is circularly ordered, then any two points disconnect $\partial\mathcal{O}$. By theorem I.11.21, page 32 of Wilder [Wi], the space $\partial\mathcal{O}$ is homeomorphic to a circle. $\square$

We are now ready to prove that $\mathcal{D}$ is homeomorphic to a disk.

Theorem 7.27. *The space $\mathcal{D} = \mathcal{O} \cup \partial\mathcal{O}$ is homeomorphic to a closed disk D^2.*

Proof. This proof will use classical results of general topology, namely a theorem of Zippin characterizing the closed disk $\mathbf{D}^2$, see theorem III.5.1, page 92 of Wilder [Wi].

First we need to show that $\mathcal{D}$ is a Peano continuum, see page 76 of Wilder [Wi]. A Hausdorff topological space C is a *Peano space* if it is not a single point, it is second countable, normal, locally compact, connected and locally connected. Notice that Wilder uses the term perfectly separable (definition in page 70 of [Wi]) instead of second countable. If in addition C is compact then C is a *Peano continuum.*

By proposition 7.25 our space $\mathcal{D}$ is compact, hence locally compact. It is also Hausdorff – lemma 7.15 – hence normal. By lemma 7.20 it is second countable and it is clearly not a single point. What is left to show is that $\mathcal{D}$ is connected and locally connected.

We first show that $\mathcal{D}$ is connected. Suppose not and let A, B be a separation of $\mathcal{D}$. Since $\partial\mathcal{O}$ is connected (this is done in the proof of proposition 7.26), then $\partial\mathcal{O}$ is contained in either A or B, say it is contained in A. Then B is contained in $\mathcal{O}$. If B is not contained in a compact set of $\mathcal{O}$ then there is a sequence of points in B escaping compact sets in $\mathcal{O}$. As $\mathcal{D}$ is compact there is a subsequence converging in $\mathcal{D}$, which must converge to a point in $\partial\mathcal{O}$, contradiction to A, B forming a separation. It follows that there is a closed disk D in $\mathcal{O}$ with B contained in the interior of D. Then $D \cap A, D \cap B$ form a separation of D, contradiction.

Next we show that $\mathcal{D}$ is locally connected. Since $\mathcal{O} \cong \mathbf{R}^2$, then $\mathcal{D}$ is locally connected at every point of $\mathcal{O}$. Let p in $\partial\mathcal{O}$ and let W be a neighborhood of p in $\mathcal{D}$. If $A = \{a_i\}$ is a master sequence associated to p, there is i with $\overline{U(a_i)}$ contained in W and $U(a_i)$ is a neighborhood of p in $\mathcal{D}$. Now $U(a_i) \cap \mathcal{O} = \widetilde{a}_i$ is homeomorphic to $\mathbf{R}^2$ also and hence connected. The closure of $\widetilde{a}_i$ in $\mathcal{D}$ is $\overline{U(a_i)}$. Since

$$\widetilde{a}_i \quad \subset \quad U(a_i) \quad \subset \quad \overline{U(a_i)}$$

then $U(a_i)$ is connected. This shows that $\mathcal{D}$ is locally connected and that $\mathcal{D}$ is a Peano continuum.

To use theorem III.5.1 of [Wi] we need the idea of spanning arcs. An arc in a topological space X is a subspace homeomorphic to a closed interval in $\mathbf{R}$. Let ab denote an arc with endpoints a, b. If K is a point set, we say that *ab spans K* if $K \cap ab = \{a, b\}$. We now state theorem III.5.1 of [Wi].

Theorem 7.28. *(Zippin) A Peano continuum C containing a 1-sphere J and satisfying the following conditions below is a closed 2-disk with boundary J:*

(i) C contains an arc that spans J,

(ii) Every arc that spans J separates C,

(iii) No closed proper subset of an arc spanning J separates C.

Here E separates C mean that $C - E$ is not connected.

In our case J is $\partial\mathcal{O}$. For condition (i) let l be a non singular leaf in $\mathcal{O}^s$ or $\mathcal{O}^u$. Then l has 2 ideal points in $\mathcal{O}$ which are distinct. The closure $\overline{l}$ is an arc that spans $\partial\mathcal{O}$. This proves (i).

We prove (ii). Let ζ be an arc in $\mathcal{D}$ spanning $\partial\mathcal{O}$. Then $\zeta \cap \mathcal{O}$ is a properly embedded copy of $\mathbf{R}$ in $\mathcal{O}$. Hence $\mathcal{O} - (\zeta \cap \mathcal{O})$ has exactly two components A_1, B_1. In addition $\partial\mathcal{O} - (\zeta \cap \partial\mathcal{O})$ has exactly two components A_2, B_2 and they are connected, since $\partial \mathcal{O}$ is homeomorphic to a circle by proposition 7.26. If p is in A_2 there is a connected neighborhood U of p in $\mathcal{D}$ which is disjoint from ζ – since ζ is closed in $\mathcal{D}$. Hence $U \cap \mathcal{O}$ is contained in either A_2 or B_2. This also shows that a neighborhood of p in $\partial \mathcal{O}$ will also satisfy the same property. By connectedness of A_2, B_2, then after switching A_1 with B_1 if necessary it follows that: for any $p \in A_2$ there is a neighborhood U of p in $\mathcal{D}$ with $U \cap \zeta = \emptyset$ and $U \cap \mathcal{O} \subset A_1$. Similarly B_2 is paired with B_1. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. The arguments above show that A, B are open in $\mathcal{D}$ and therefore they form a separation of $\mathcal{D} - \zeta$. This proves (ii).

In addition since $\mathcal{O} - (\zeta \cap \mathcal{O})$ has exactly two components A_1, B_1 then $\zeta \cap \mathcal{O}$ is contained in $\overline{A}_1 \cap \overline{B}_1$ and so $\zeta \subset \overline{A} \cap \overline{B}$. It follows that no proper subset of ζ separates $\mathcal{D}$. This proves property (iii).

Now Zippin's theorem implies that $\mathcal{D}$ is homeomorphic to a closed disk. This finishes the proof of theorem 7.27. ☐

Notice that $\pi_1(M)$ acts on $\mathcal{O}$ by homeomorphisms. The action preserves the foliations $\mathcal{O}^s, \mathcal{O}^u$ and also preserves convex chains, admissible sequences, master sequences and so on. Hence $\pi_1(M)$ also acts by homeomorphisms of $\mathcal{D}$. The action has some nice properties: g in $\pi_1(M)$ has a fixed point in $\mathcal{O}$ if and only if it is associated to a periodic orbit of the flow Φ. The action in $\partial \mathcal{O}$ also has good properties.

The same holds for almost pseudo-Anosov flows:

Corollary 7.29. *Let Φ be an almost pseudo-Anosov flow. Then $\mathcal{O}$ has a natural compactification to a closed disk.*

Proof. Let Φ_1 be the pseudo-Anosov flow obtained from Φ. The orbit space $\mathcal{O}_1$ of $\widetilde{\Phi}_1$ is obtained as a blow down of the orbit space of $\widetilde{\Phi}$. Going backwards from Φ_1 to Φ produces blown up standard sequences, master sequences, etc.. All the definitions concerning ideal boundary of $\mathcal{O}_1$ also work for $\mathcal{O}$ producing a compactification $\mathcal{D}$. There is a natural blow down map from $\mathcal{D}$ to $\mathcal{D}_1$ which is a homeomorphism in the boundary. Clearly all the constructions are group equivariant. ☐

8 Quasigeodesic pseudo Anosov flows in M^3 with $\pi_1(M)$ negatively curved

We will apply the results of the last section to study metric properties of flows. First we derive a further property of ideal points of general pseudo-Anosov flows.

Proposition 8.1. *Let Φ be a pseudo-Anosov flow in M^3 closed. Let p be an ideal point of $\mathcal{O}$. Then one of the 3 mutually exclusive options occurs:*

1) There is a master sequence $L = \{l_i\}$ where l_i are slices in leaves of $\mathcal{O}^s$ or $\mathcal{O}^u$.

2) p is an ideal point of a ray l of $\mathcal{O}^s$ or $\mathcal{O}^u$ so that l makes a perfect fit with another ray of $\mathcal{O}^s$ or $\mathcal{O}^u$. There are master sequences which are standard sequences associated to a ray in $\mathcal{O}^s$ or $\mathcal{O}^u$ as described before definition 7.2.

3) p is an ideal point associated to infinitely many leaves non separated from each other. Then a master sequence for p is obtained as shown in lemma 7.24.

Proof. Fix a basepoint x in $\mathcal{O}$. Let $A = \{a_i\}$ be a master sequence defining p. Assume that x is not in the closure of any $\widetilde{a}_i$. Each a_i is a convex chain, $a_i = b_1 \cup ... b_n$ where b_i is either a segment or a ray in $\mathcal{O}^s$ or $\mathcal{O}^u$. For simplicity we omit the dependence of the b_j's on the index i.

Claim – For each i there is some b_j as above, with b_j is contained in a slice z of $\mathcal{O}^s$ or $\mathcal{O}^u$, so that z separates x from $\widetilde{a}_i$.

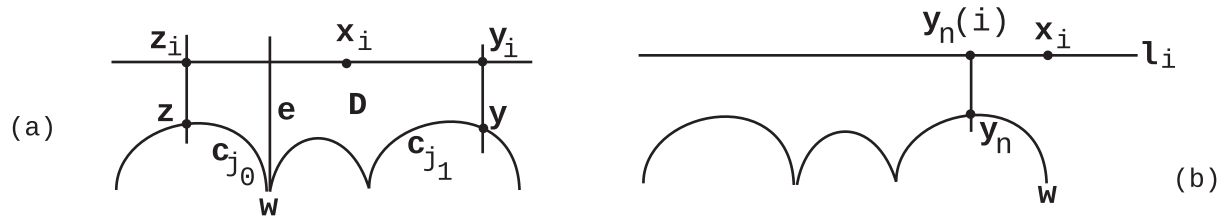

Figure 27: *a. The case that x_i is between some unstable leaves, b. The case x_i escapes to one side.*

For each j consider an endpoint y of b_j. Suppose wlog that b_j is in a leaf of $\mathcal{O}^s$. Since $\widetilde{a}_i$ is a convex chain, we can extend b_j along $\mathcal{O}^s(y)$ beyond y and entirely outside $\widetilde{a}_i$. Here the hypothesis that $\widetilde{a}_i$ is convex is necessary, for otherwise at a non convex switch any continuation of b_j along $\mathcal{O}^s(y)$ beyond y would have to enter $\widetilde{a}_i$. Suppose that y is also in b_{j+1}. If one encounters a singular point in $\mathcal{O}^s(y)$ (which could be y itself), then continue along the prong closest to b_{j+1}. In this way we construct a slice c_j of $\mathcal{O}^s(y)$ with $b_j \subset c_j$. Notice that there is a component V_j of $\mathcal{O} - c_j$ containing $\widetilde{a}_i$. By construction

$$\bigcap_{j=1}^{n} V_j \quad = \quad \widetilde{a}_i$$

Since x is not in $\widetilde{a}_i$, then there is j with x not in V_j and so c_j separates x from $\widetilde{a}_i$. Let z be this slice c_j. This proves the claim.

Using the claim then for each i produce such a slice and denote it by l_i. Let $\widetilde{l_i}$ be the component of $\mathcal{O} - l_i$ containing $\widetilde{a}_i$. Up to subsequence assume all the l_i are in (say) $\mathcal{O}^s$. Since A is a master sequence for p, we may also assume, by lemma 7.18, that all the l_i are disjoint from each other.

We now analyse what happens to the l_i. The first possibility is that the sequence $\{l_i\}$ escapes compact sets in $\mathcal{O}$. Then this sequence defines an ideal point of $\mathcal{O}$. As $\widetilde{a}_i \subset \widetilde{l_i}$, it follows that $L = \{l_i\}$ is an admissible sequence for p. Since $A = \{a_i\}$ is a master sequence for p, then given $\widetilde{a}_i$, there is $j > i$ with $l_j \subset \widetilde{a}_i$. It follows that $L = \{l_i\}$ is also a master sequence for p. This is case 1) of the possibilities.

Suppose from now on that for <u>any</u> master sequence $A = \{a_i\}$ for p and any l_i as constructed above, then l_i does not escape compact sets. Then l_i converges to a set of leaves $\mathcal{C} = \{c_k\}$, $k \in J \subset \mathbf{Z}$. This is a family of leaves of $\mathcal{O}^s$ non separated from each other. We assume $\mathcal{C}$ to be ordered as described in theorem 2.6. Here J is either $\{1, ...k_0\}$ or is $\mathbf{Z}$. Choose $x_i \in b_i = a_i \cap l_i$. Hence x_i converges to p in $\mathcal{D}$ as $i \to \infty$. For any y in $\mathcal{C}$, then $y \in c_k$ for some k and $\mathcal{O}^u(y)$ intersects l_i for i big enough in a point denoted by $y(i)$. Similarly for z in $\mathcal{C}$ define $z(i)$. This notation will be used for the remainder of the proof.

<u>Situation 1</u> $-$ Suppose there are $y, z \in \mathcal{C}$ so that for big enough i, x_i is between $y(i)$ and $z(i)$ in l_i, see fig. 27, a. Let z in c_{j_0}, y in c_{j_1}, with $j_0 \leq j_1$. If $j_0 = j_1$ then the segment u_i of l_i between $z(i), y(i)$ converges to the segment in $\mathcal{O}^s(z)$ between z and y. Then x_i does not escape compact sets, contradiction.

For any k the leaves c_k, c_{k+1} are non separated from each other and there is a leaf e of $\mathcal{O}^u$ making perfect fits with both c_k and c_{k+1}. This defines an ideal point w of $\partial\mathcal{O}$ which is an ideal point of equivalent rays of c_k, c_{k+1} and e, see fig. 27, a. Consider the region D of $\mathcal{O}$ bounded by the ray of c_{j_0} defined by z and going in the y direction, the segment in $\mathcal{O}^u(z)$ from z to $z(i)$, the segment u_i in l_i from $z(i)$ to $y(i)$, the segment in $\mathcal{O}^u(y)$ from $y(i)$ to y, the ray in $\mathcal{O}^s(y)$ defined by y and going towards the z direction and the leaves c_k with $j_0 < k < j_1$ (this last set may be empty). By the remark above, the only ideal points of D in $\partial\mathcal{O}$, that is the set $\overline{D} \cap \partial\mathcal{O}$ (closure in $\mathcal{D}$), are those associated to rays of c_k with $j_0 \leq k \leq j_1$. One such point is the w defined above. It follows that x_i converges to one of these and p is one of these points. So p is an ideal point of a ray of $\mathcal{O}^s$ or $\mathcal{O}^u$ which makes a perfect fit with another leaf. There is a master sequence which is a standard sequence associated to p. This is case 2 of the proposition. Notice it is disjoint from case 1).

Situation 2 – For any y, z in $\mathcal{C}$ the x_i is eventually not between the corresponding $y(i), z(i)$.

Suppose first that $\mathcal{C}$ is an infinite collection of non separated leaves. Let $y \in \mathcal{C}$. Then up to subsequence the x_i are in one side of $y(i)$ in l_i, say in the side corresponding to increasing k in the order of $\mathcal{C}$ (this is in fact true for the original sequence as x_i converges in $\mathcal{D}$). Let w be the ideal point associated to infinitely many non separated leaves as in lemma 7.24. Let $g_m = e_m \cup f_m$ and let $G_m = \{g_m\}$ be a master sequence associated to w as in lemma 7.24. Fix m. Then x_i is eventually in $\widetilde{g}_m$. Therefore x_i converges to w and $w = p$. Here we are in case 3). Notice all 3 cases are mutually exclusive.

Finally suppose that $\mathcal{C}$ is finite. We may assume that x_i escapes in the positive direction, that is y is in c_{k_0}. Let w be the ideal point of that ray of c_{k_0}. Let y_n in c_{k_0} converging to w. Let

$$y_n(i) \quad = \quad \mathcal{O}^u(y_n) \cap l_i$$

Fix n. Then eventually in i, the x_i is in the component of $\mathcal{O}^s(y_n(i)) - y_n(i)$ corresponding to the ideal point w. Consider a master sequence defining w which is a standard sequence defining w so that: it is arbitrary in the side of $\mathcal{O} - c_{k_0}$ not containing x_i and in the other side we have an arc in $\mathcal{O}^u(y_n)$ from y_n to $y_n(i)$ and then a ray in l_i. Since c_{k_0} is the biggest element in $\mathcal{C}$, there is no leaf of $\mathcal{O}^s$ non separated from c_{k_0} in that side. Hence the l_i cannot converge to anything on that side and those parts of l_i escape in $\mathcal{O}$. As the x_i are in these subarcs of l_i then $x_i \to w$ in $\mathcal{D}$ and so $p = w$.

Let $r_n = \mathcal{O}^u(y_n)$. If r_n escapes compact sets in $\mathcal{O}$ as $n \to \infty$, then it defines a master sequence for p and we are in case 1). Otherwise r_n converges to some r making a perfect fit with c_{k_0} and we are in case 2).

This finishes the proof of the proposition. ☐

We now study metric properties of flows. Suppose that M^3, closed has $\pi_1(M)$ negatively curved and that Φ is a quasigeodesic pseudo-Anosov flow in M. Since $\widetilde{M}$ is an $\mathbf{R}$-bundle over $\mathcal{O}$ it is a trivial bundle. Choose any continuous section $\sigma : \mathcal{O} \to \widetilde{M}$. We now prove that this section extends continuously to a map also $\sigma : \mathcal{D} \to \widetilde{M} \cup S^2_\infty$ (by an abuse of notation, also denoted by σ) and the map restricted to $\partial\mathcal{O}$ is uniquely defined and group equivariant, providing a group invariant Peano curve.

Theorem 8.2. *Let Φ be a quasigeodesic pseudo-Anosov flow in M^3 closed, with $\pi_1(M)$ negatively curved. For any continuous section $\sigma : \mathcal{O} \to \widetilde{M}$, it extends to a continuous map $\sigma : \mathcal{D} \to \widetilde{M} \cup S^2_\infty$. The ideal map $\sigma : \partial\mathcal{O} \to S^2_\infty$ is group equivariant and is uniquely determined by Φ.*

Proof. Let σ be a section of the bundle and let p in $\partial\mathcal{O}$. We use the 3 cases from the previous proposition.

Case 1 – There is $L = \{l_i\}$ master sequence for p with l_i slices in leaves of $\mathcal{O}^s$ or $\mathcal{O}^u$ and l_i escaping in $\mathcal{O}$.

Suppose up to subsequence that l_i are all slices in $\mathcal{O}^s$. Then there are slices L_i of $\widetilde{\Lambda}^s$ with L_i escaping in $\widetilde{M}$. The basic lemma (lemma 6.5) shows that $\overline{L}_i$ shrinks in visual diameter. The closure $\overline{L}_i$ is taken in $\widetilde{M} \cup S^2_\infty$. In addition L_i also separates a set B_i in $\widetilde{M}$ with $\{B_i\}$ nested in i and $\overline{B}_i$

shrinking in diameter. Therefore in $\widetilde{M} \cup S^2_\infty$, the sets $\overline{B}_i$ converge to a single point in S^2_∞. This is $\sigma(p)$. Any other master sequence is nested with this master sequence, so it will produce the same limit $\sigma(p)$.

<u>Case 2</u> $-$ p is an ideal point of a ray l in $\mathcal{O}^s$ or $\mathcal{O}^u$.

Without loss of generality suppose that $l \subset \Theta(L)$ with $L \in \widetilde{\Lambda}^s$. Then since p is an ideal point of l, it follows that $\sigma(l)$ converges to L_+ $-$ same arguments as in the proof of continuous extension property for leaves of foliations $-$ see the begining of case 2 of the extension construction. Let $\sigma(p) = L_+$. Recall that L_+ is the common positive ideal point of any flow line in L.

<u>Case 3</u> $-$ Suppose p is associated to infinite branching.

Use a master sequence $\mathcal{G} = \{g_i\}$, where $g_i = e_i \cup f_i$ as in situation 2 of the previous proposition. Let $E_i = e_i \times \mathbf{R}$, $F_i = f_i \times \mathbf{R}$ and $G_i = g_i \times \mathbf{R}$, all subsets of $\widetilde{M}$. Let $B_i = \widetilde{g}_i \times \mathbf{R}$. Then the B_i are nested in $\widetilde{M}$. By the basic lemma, since f_i escapes in $\mathcal{O}$, then F_i shrinks in visual diameter in $\widetilde{M} \cup S^2_\infty$. The same same is true for E_i and hence for G_i and finally B_i. Let $\sigma(p)$ be the limit of the shrinking sequence $\overline{B}_i$.

This defines an extension map $\sigma : \mathcal{D} \to \widetilde{M} \cup S^2_\infty$. In cases 1) and 3) the definition of $\sigma(p)$ is obtained using master sequences $A = \{a_i\}$ so that $\sigma(\widetilde{a}_i)$ is nested, has diameter converging to 0 and converges to $\sigma(p)$. Recall the sets $U(a_i)$ which form a neighborhood basis of p in $\mathcal{D}$ $-$ see definition 7.14 and lemma 7.19. The $U(a_i)$ are contained in the closure of $\widetilde{a}_i$ in $\mathcal{D}$. Therefore the arguments above show that $\sigma(U(a_i))$ also shrinks in visual measure to the point $\sigma(p)$. This proves continuity of σ at p in cases 1) and 3).

We now prove continuity at p in case 2). We use the same notation as above with p ideal point of l ray of $\mathcal{O}^s$.

The basic lemma refers to a collection of sets C_i each of which in a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. The proof of continuity of σ at p in cases 1) and 3) uses the fact that the chains in the master sequences always have length one or two and so the basic lemma can be applied twice directly yielding the result. In case 2) chains can have arbitrary length. It is true that pieces of the chain intersecting l will have to escape in $\mathcal{O}$ and so the corresponding sets in $\widetilde{M}$ shrink to $\sigma(p)$, but since there can be more and more links in the chains, it is unclear whether the complete chains converge to $\sigma(p)$. An additional argument is needed as follows:

Fix a metric in $\widetilde{M} \cup S^2_\infty$ making it homeomorphic to a closed ball and so that covering translations act as homeomorphisms. Let $\epsilon > 0$. Let $A = \{a_i\}$ be a master sequence which is a standard sequence associated to p. Suppose there are infinitely many leaves L_j in $\widetilde{\Lambda}^s$ which are equivalent to L. Suppose they form a nested collection, $L_0 = L$. Then L_j escapes compact sets in $\widetilde{M}$ as $j \to \infty$. The basic lemma (lemma 6.5) implies that $diam(\overline{L}_j)$ converges to 0 and so suppose j is chosen so that this diameter is less than ϵ. Since L_j is equivalent to L, then

$$(L_j)_+ \quad = \quad L_+ \quad = \quad \sigma(p)$$

Then $\overline{L}_j$ is very close to $\sigma(p)$ and so is the part of $\sigma(\widetilde{a}_i)$ beyond L_j. This shows that we only have to worry about continuity of σ at p for the regions between l and $l_j = \Theta(L_j)$. If there are only finitely many leaves equivalent to L, then this first part is not needed.

The number j is fixed. Hence it suffices to consider the region between l and a leaf u of $\mathcal{O}^u$ making a perfect fit with l. Let $U \in \widetilde{\Lambda}^u$ with $u \subset \Theta(U)$. Let $\widetilde{b}_i$ be the subregion of $\widetilde{a}_i$ bounded by a ray in l, a ray in u and 2 segments c_i, d_i in a_i: c_i starts in $l \cap a_i$, d_i starts in $u \cap a_i$ and c_i, d_i share the other endpoint. Let

$$b_i \;=\; \partial \widetilde{b_i} \;=\; c_i \cup d_i \cup e_i \cup f_i$$

where e_i is a ray in l, f_i is a ray in u. Let

$$C_i = c_i \times \mathbf{R},\ D_i = d_i \times \mathbf{R},\ E_i = e_i \times \mathbf{R},\ F_i = f_i \times \mathbf{R}$$

Then E_i, F_i converge to $\sigma(p)$ in $\widetilde{M} \cup S^2_\infty$, because $U_- = L_+$. The D_i have flow lines $D_i \cap L$ which converge to $\sigma(p)$ in $\widetilde{M} \cup S^2_\infty$. If D_i does not converge to $\sigma(p)$, then again lemma 6.5 implies that D_i cannot escape compact sets in $\widetilde{M}$ as $i \rightarrow \infty$. This contradicts the fact that $\Theta(D_i) \subset a_i$ and a_i escapes compact sets in $\mathcal{O}$. It now follows that $C_i \cup D_i \cup E_i \cup F_i$ converges to $\sigma(p)$. This proves continuity of σ at p in case 2). This finishes the proof of continuity of σ.

The map $\sigma : \partial \mathcal{O} \rightarrow S^2_\infty$ so defined is clearly group equivariant as it was defined using group invariant objects such as master sequences and such. As $\sigma : \partial \mathcal{O} \rightarrow S^2_\infty$ is continuous, then $\sigma(\partial \mathcal{O})$ is a closed subset of S^2_∞. It is group invariant hence it is S^2_∞. This provides several examples of group invariant Peano curves whenever there is a quasigeodesic pseudo-Anosov flow in $\widetilde{M}$. Notice that σ restricted to $\partial \mathcal{O}$ is independent of the section σ chosen. This is because for any sequence x_i in $\mathcal{O}$ converging to p in $\mathcal{D}$, then the <u>entire</u> orbits $x_i \times \mathbf{R}$ converge to $\sigma(p)$ in $\widetilde{M} \cup S^2_\infty$.

This finishes the proof of the theorem. $\square$

In the same way as in theorem 8.2 we prove:

Corollary 8.3. *Let Φ be a quasigeodesic almost pseudo-Anosov flow in M^3 closed, with $\pi_1(M)$ negatively curved. Let $\sigma : \mathcal{O} \rightarrow \widetilde{M}$ be any section. Then it extends continuously to a map $\mathcal{O} \cup \partial \mathcal{O} \rightarrow \widetilde{M} \cup S^2_\infty$. The restriction to the boundary is a map $\sigma : \partial \mathcal{O} \rightarrow S^2_\infty$, which is a naturally defined group invariant Peano curve.*

9 Global circle maps and encoding of limit sets

We now relate the results of the last 2 sections. Let $\mathcal{F}$ be a foliation almost transverse to a pseudo-Anosov flow Φ_1 and transverse to an almost pseudo-Anosov flow Φ. Let $\mathcal{O}$ be the leaf space of $\widetilde{\Phi}$.

Definition 9.1. *Let $F \in \widetilde{\mathcal{F}}$. The limit set of F in $\mathcal{D}$ is the closure of $\Theta(F)$ in $\mathcal{D}$ intersected with $\partial \mathcal{O}$. This is denoted by B_F.*

Lemma 9.2. *There is a continuous, circularly monotone map $c_F : \partial \mathcal{O} \rightarrow \partial_\infty F$. This is an encoding of the boundary of F. The map $\zeta_F : \Theta(F) \cup B_F \rightarrow F \cup \partial_\infty F$ given by $\zeta_F(x) = (x \times \mathbf{R}) \cap F$ if x is in $\Theta(F)$ and $\zeta_F(x) = c_F(x)$ if x is in B_F is surjective and continuous.*

Proof. Let $z \in B_F$. Then there are z_i in $\Theta(F)$ with $z_i \rightarrow z$ in $\mathcal{D}$. Let u_i in F with $\Theta(u_i) = z_i$. Up to subsequence assume u_i converges to u in $\partial_\infty F$. If w_i is another sequence in $\Theta(F)$ with $w_i \rightarrow z$ and v_i in F with $\Theta(v_i) = w_i$ assume v_i converges to v in $\partial_\infty F$. If v, u are distinct, let τ be a component of $\partial_\infty F - \{u, v\}$. There is a stable leaf l of $\widetilde{\Lambda}^s_F$ with both endpoints in τ. Let $l_1 = \Theta(l \times \mathbf{R})$. If l_1 has both endpoints in $\partial \mathcal{O}$ then they separate the limits of z_i and w_i, contradiction, because they have the same limit in $\mathcal{D}$. If both endpoints are in slice leaves r_1, r_1 of $\mathcal{O}^u$ then r_1, r_2 do not intersect $\Theta(F)$ and the other side of r_1, r_2 is disjoint from $\Theta(F)$. Then $l \cap r_1 \cap r_2$ also shows that the limits of z_i and w_i cannot be the same. We conclude that v, u distinct is impossible.

So there is a well defined map c_F from B_F to $\partial_\infty F$ and this is weakly circularly monotone – it can collapse points but not reverse the order: Let p_1, p_2, p_3 be an ordered triple of points in B_F.

Find α_i properly embedded half infinite arcs in $\Theta(F)$ converging to p_i and disjoint except for their common starting points. Let β_i be their lifts to F, that is $\Theta(\beta_i) = \alpha_i$. Then in F the β_i carry the same ordering as the α_i. Since β_i has points which limit to $c_F(p_i)$, the the order of p_1, p_2, p_3 cannot be reversed under the images by c_F. However it is quite possible that some or all of the p_i get mapped to the same point in $\partial_\infty F$.

In addition every point of $\partial_\infty F$ is obtained this way. Let u_i in F converging to an arbitrary point u of $\partial_\infty F$. Let $w_i = \Theta(u_i)$. Up to subsequence, assume that $w_i \to w$ in $\mathcal{D}$. If w is in $\partial\mathcal{O}$ we are done. Otherwise suppose that w is in a stable boundary leaf l of $\partial\Theta(F)$. Let p be an ideal point of l in $\partial\mathcal{O}$. Since l is contained in $\partial\Theta(F)$ there are z_i in $\Theta(F)$ converging to p and v_i in F with $\Theta(v_i) = z_i$, with v_i also assumed to converge in $F \cup \partial_\infty F$ to a point v. If $u \neq v$, then construct as above a stable leaf in between them and as above arrive at a contradiction.

The same type of argument also shows that the map $c_F : B_F \to \partial_\infty F$ thus defined is continuous. The boundary $\partial_\infty F$ is essentially obtained by collapsing the endpoints of complementary intervals of B_F in $\partial\mathcal{O}$. The arguments above show that endpoints of any complementary interval of B_F in $\partial\mathcal{O}$ map under c_F to the same point of $\partial_\infty F$. Extend the map c_F to $\partial\mathcal{O}$, by sending a complementary interval to the image of the endpoints.

Finally we analyse continuity of ζ_F: If z_i in $\Theta(F)$ and $z_i \to z$ in B_F then by construction $\zeta_F(z_i)$ converges to $\zeta_F(z)$. Since ζ_F restricted to B_F or $\Theta(F)$ are already continuous, this implies that ζ_F is continuous. □

Remarks 1) Notice that the proof shows that if l is a boundary leaf of $\Theta(F)$ and $\bar{l}$ is the closure of l in $\mathcal{D}$, then for any sequence w_i in $\Theta(F)$ converging to a point w in $\bar{l}$, and any $z_i \in F$ with $\Theta(z_i) = w_i$, then the limit of the z_i in $F \cup \partial_\infty F$ is uniquely defined. In fact there is always a limit otherwise find a subsequence converging to something else.

2) Also this lemma in particular shows that if z is in $\partial_\infty F$, then there is a in $\partial\mathcal{O}$ with $c_F(a) = z$. This occurs even though for an arbitrary z_i in F converging to z and $w_i = \Theta(z_i)$, then w_i may not converge to a and in fact w_i may converge to a point in $\mathcal{O}$. Still there is always some sequence y_i in F converging to z, with $\Theta(y_i)$ converging to a.

Now we have the main result of this section:

Theorem 9.3. *Let Φ_1 be a quasigeodesic pseudo-Anosov flow almost transverse to a Reebless foliation $\mathcal{F}$ in M^3 closed, with $\pi_1(M)$ Gromov hyperbolic. As shown before $\mathcal{F}$ has the continuous extension property. Given F in $\widetilde{\mathcal{F}}$, let $\varphi_F : \partial_\infty F \to S^2_\infty$ be the induced continuous map. Let Φ be the almost pseudo-Anosov flow transverse to $\mathcal{F}$. Let $\sigma : \partial\mathcal{O} \to S^2_\infty$ be the ideal map defined in theorem 8.2 associated to $\widetilde{\Phi}$. Then σ encodes all ideal maps $\varphi_F : \partial_\infty F \to S^2_\infty$ in the following way: if F is in $\widetilde{\mathcal{F}}$, then $\sigma|_{B_F} = \varphi_F \circ c_F|_{B_F}$.*

Proof. In other words for any z in $\partial_\infty F$ and w in $(c_F)^{-1}(z) \cap B_F$ then $\sigma(w)$ is equal to $\varphi_F(z)$.

Let w in $(c_F)^{-1}(z) \cap B_F$. There are z_i in F with $\Theta(z_i)$ converging w.

By the previous lemma, the continuity of ζ_F implies that z_i converges to z in $F \cup \partial_\infty F$. Consider the picture in $\widetilde{M} \cup S^2_\infty$: The continuous extension property for $\widetilde{\mathcal{F}}$ applied to F shows that z_i converges to $\varphi_F(z)$. On the other hand, theorem 8.2 shows that $\sigma(\Theta(z_i))$ converges to $\sigma(w)$. This shows that $\sigma(w) = \varphi_F(z)$, which is the desired equation. This finishes the proof of the theorem. □

Remark: $-$ Let w in $\partial\Theta(F)$ which is a subset of $\mathcal{O}$. Let z_i in F with $\Theta(z_i)$ converging to w. Assume that z_i converges to z in $F \cup \partial_\infty F$. Let x in $\widetilde{M}$ with with $\Theta(x) = w$. Assume without loss of generality that w is in a stable boundary leaf l of $\partial\Theta(F)$. Then F escapes up as $\Theta(F)$ nears w. Therefore z_i converges to x_+ which is then the value of $\varphi_F(z)$. Let L slice of $\widetilde{\Lambda}^s$ with $l = \Theta(L)$. Let

a in $\partial\mathcal{O}$ be an ideal point of l. The proof of theorem 8.2, case 2 shows that $\sigma(a) = L_+$ for any ideal point a of l. By the previous lemma $c_F(a) = z$. Hence

$$\sigma(a) \;=\; L_+ \;=\; x_+ \;=\; \varphi_F(z) \;=\; \varphi_F(c_F(a))$$

10 Limit sets and identification of ideal points

Let $\mathcal{F}$ be a foliation in M^3 with $\pi_1(M)$ negatively curved and $F \in \widetilde{\mathcal{F}}$. The limit set Λ_F is the set of accumulation points of F in S^2_∞. Usually the limit set Λ_F of F is not a Jordan curve and if $\mathcal{F}$ has the continuous extension property, then the map $\varphi_F : \partial_\infty F \rightarrow S^2_\infty$ is not injective. The map φ_F is injective if $\pi(F)$ is quasi-isometrically embedded in M [Th1, Gr, Gh-Ha] $-$ for example if $\pi(F)$ is a compact leaf which is not a fiber or a virtual fiber [Th1, Th2, Bon]. Here we analyse the identifications about ideal points of quasigeodesic pseudo-Anosov flows. First we mention a result of identified ideal points:

Theorem 10.1. *([Fe7]) Let Φ be a quasigeodesic almost pseudo-Anosov flow in M^3 closed, with $\pi_1(M)$ negatively curved. Let γ, β be two flow lines with the same positive ideal points in S^2_∞. Then there is a sequence of leaves $S_i, 0 \leq i \leq m$ alternatively in $\widetilde{\Lambda}^s$ and $\widetilde{\Lambda}^u$ so that: $S_0 = \widetilde{W}^s(\gamma)$, $S_m = \widetilde{W}^s(\beta)$ and S_i, S_{i+1} form a perfect fit. In general if any S_i is periodic then all of them are periodic and left invariant by a common non trivial covering translation. In that case the leaves S_0, S_m are in fact connected by a chain of lozenges. In particular this happens whenever there is i so that S_i and S_{i+2} are non separated from each other in the leaf space of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. If on the other hand $\gamma_+ = \beta_-$, then there is a sequence of perfect fits from $\widetilde{W}^s(\gamma)$ to $\widetilde{W}^u(\beta)$. If any leaf in the sequence is periodic, then there is a sequence of lozenges.*

Remark $-$ This is contained in parts (2) and (3) of Theorem 5.7 of [Fe7] and the last two statements are corollary 5.9 of [Fe7]. In [Fe7] these results are proved for quasigeodesic Anosov flows in M^3 with negatively curved fundamental group. The proof goes verbatin to the case of pseudo-Anosov flows. The singularities make no difference. By the blow up operation, the same holds for almost pseudo-Anosov flows.

Theorem 10.2. *Let Φ be a quasigeodesic pseudo-Anosov flow in M^3 closed, with $\pi_1(M)$ negatively curved. Let $\sigma : \partial\mathcal{O} \rightarrow S^2_\infty$ be the ideal map associated to this flow. Suppose p, q in $\partial\mathcal{O}$ with $\sigma(p) = \sigma(q)$. Then p, q are ideal points or rays of leaves l, r of $\mathcal{O}^s$ or $\mathcal{O}^u$ so that l, r are connected by a finite chain of perfect fits between leaves of $\mathcal{O}^s, \mathcal{O}^u$. If there is any element in the chain that is periodic then the perfect fits in the chain are parts of lozenges producing a chain of lozenges. The leaves l, r may be the same one and p, q the ideal points of different rays of l.*

Proof. First of all notice that if there are infinitely many leaves of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ which are non separated from each other, then theorem 2.6 implies that there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$, contradiction to $\pi_1(M)$ being negatively curved. Hence only options 1) and 2) of proposition 8.1 can occur.

Let $\sigma(p) = \sigma(q)$ and suppose first that one of p, q, say p, is not an ideal point of $\mathcal{O}^s$ or $\mathcal{O}^u$. By proposition 8.1 we can choose a master sequence $\mathcal{L} = \{l_i\}$ for p with l_i slices in (say) $\mathcal{O}^s$. Let $l_i \subset \Theta(L_i)$ where L_i are slices in leaves of $\widetilde{\Lambda}^s$.

Eventually the $\overline{l}_i$ separates p from q in $\mathcal{D}$. This is because no ray of $\mathcal{O}^s$ or $\mathcal{O}^u$ has ideal point p and q is different than p. It follows that $\sigma(p), \sigma(q)$ are obtained using points in $\widetilde{M}$ which are in opposite sides of L_i. Since $\sigma(p) = \sigma(q)$ then $\sigma(p)$ is in the limit set of L_i which is denoted by Λ_{L_i}. Notice Λ_{L_i} consists of a single point which is the forward limit of all flow lines in L_i and all other points in Λ_{L_i} are negative ideal points of flowlines [Fe3, Fe7].

Case 1 – Suppose all L_i have the same positive ideal point.

Then theorem 10.1 implies that for any i, j then L_i and L_j are connected by a chain of perfect fits. Notice that $\{L_i\}$ is a nested collection, hence the chain of perfect fits from R_1 from R_j has to contain R_i for any $1 < i < j$. It follows that there are at least $2j - 2$ perfect fits in the chain from R_1 to R_j.

If some L_j is periodic then the perfect fits show that all L_i are periodic and left invariant by a common non trivial covering translation g. Let δ_i be the periodic orbit in L_i, that is $g(\delta_i) = \delta_i$ for all i. Since Φ is quasigeodesic, there is a global $a_1 > 0$ so that δ_i is in the a_1 neighborhood of a minimal geodesic with the same ideal points [Gr, Gh-Ha]. Therefore δ_i has points in a fixed compact set of $\widetilde{M}$ for any i. This contradicts the fact that L_i escapes $\widetilde{M}$.

So now assume no L_i is periodic. Let γ_i be orbits in L_i. Since they all share the same positive ideal points and flow lines are uniform quasigeodesics, then given γ_1 and γ_i, they have subrays in the positive direction which are a uniform distance apart [Gr, Gh-Ha]. This is dependent on the index i. Look at $\pi(\gamma_1), \pi(\gamma_i)$. They are non compact and so accumulate in points in M. If the only limit is a singular orbit then the $\pi(\gamma_1)$ is in the stable manifold of this singular so L_1 is periodic, contrary to assumption. Under those conditions the orbit $\pi(\gamma_1)$ gets arbitrarily close to itself and one can apply the closing lemma for pseudo-Anosov flows [Man] to obtain the following: orbits α_i, β_i of $\widetilde{\Phi}$ with α_i having a point very close to γ_1 and β_i having a point very close to γ_i. In addition $\pi(\alpha_i)$ is freely homotopic to $\pi(\beta_i)$. See the detailed construction in proposition 4.2 of [Fe4] for the case of quasigeodesic Anosov flows – the same arguments work for pseudo-Anosov flows. Theorem 2.5 then implies that α_i, β_i are connected by a chain of lozenges. The number of lozenges from α_i to β_i is exactly the same number of perfect fits in the chain from from γ_1 to γ_i. Therefore it is at least $2i - 2$. This goes to infinity as i grows without bound.

But then there is a uniform bound on the distance from α_i to β_i, because they are uniform quasigeodesics with same ideal points. Hence given any point in α_i, say x_i there are points $y_0 = x_i, y_1, ..., y_k$ $(k \geq 2i - 2)$ with y_j in the corners of the lozenges from α_i to β_i and $d(y_0, y_j) \leq a_1$. So we can assume that there are 2 indices j_1, j_2 with $\widetilde{W}^s(y_{j_1})$ intersecting $\widetilde{W}^u(y_{j_2})$ transversely in a non singular point. But this contradicts the fact that $\widetilde{W}^s(y_{j_1}), \widetilde{W}^u(y_{j_2})$ are both periodic and left invariant by the same covering translation.

This shows that case 1) cannot happen. Clearly the same holds if infinitely many of the L_i have the same positive ideal point.

Case 2 – Up to subsequence in i, assume there are x_i in L_i with $(x_i)_- = \sigma(p)$, so L_i share some negative ideal point.

Let $H_i = \widetilde{W}^u(x_i)$. Now they all have the same negative ideal points and therefore by theorem 10.1, H_i and H_j are connected by a chain of perfect fits for any i, j. Only finitely many of the H_i can be the same or else up to subsequence they are all the same and equal to H. But then as $\Theta(x_i)$ converges to p, it follows that $\Theta(H)$ has one ideal point p, contrary to assumption. So we may assume that all H_i are distinct.

It follows that the $\Theta(H_i)$ escape compact sets in $\mathcal{O}$. Otherwise they accumulate in some $\Theta(H)$ and any stable leaf intersecting $\Theta(H)$ transversely will intersect infinitely many $\Theta(H_i)$ contradiction, because they are connected by perfect fits. In addition if there is any non Hausdorffness involved in the chain of perfect fits from H_i to H_j for any i, j or if any H_i or H_j is singular or periodic, then all H_i are periodic and left invariant by a common non trivial covering translation g. The proof of case 1) shows that only finitely many H_i can occur, contradiction.

Therefore up to subsequence the H_i have to be nested with each other. The proof is then exactly as in the last arguments of case 1) with unstable objects switched with stable ones.

This proves that if p is not an ideal point of a leaf of $\mathcal{O}^s$ or $\mathcal{O}^u$, then $\sigma(p)$ is not equal to any

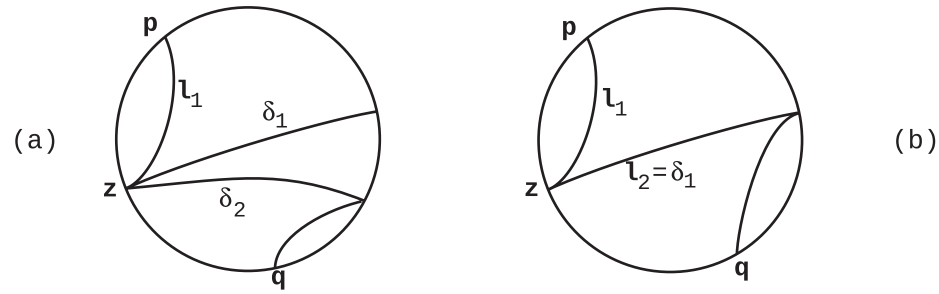

Figure 28: *a. The sequence of perfect fits, b. Jumping in the sequence.*

$\sigma(q)$ for an arbitrary q in $\partial\mathcal{O}$ distinct from p.

Suppose then that p, q are ideal points of leaves l, r of $\mathcal{O}^s, \mathcal{O}^u$. Suppose first that l, r are both in leaves of $\mathcal{O}^s$ or both in leaves of $\mathcal{O}^u$, say the first option. Let $l \subset \Theta(L)$, $r \subset \Theta(R)$ with L, R slices of $\mathcal{O}^s$. Then $\sigma(p) = L_+$ and $\sigma(q) = R_+$. We can now apply theorem 10.1 directly to show that L, R are connected by a chain of lozenges. The only case remaining is that up to renaming objects then $L \in \mathcal{O}^s$ and $R \in \mathcal{O}^u$. Then $\sigma(p) = L_+$ and $\sigma(q) = R_-$. There are orbits γ, β of $\widetilde{\Phi}$ with $\gamma \subset L$, $\beta \subset R$ and $\gamma_+ = \beta_-$. Then again theorem 10.1 shows that $\widetilde{W}^s(\gamma)$ and $\widetilde{W}^u(\beta)$ are connected by a finite chain of perfect fits.

Finally the arguments in cases 1) and 2) show that if there are infinitely many p_i in $\partial\mathcal{O}$ with $\sigma(p_i)$ the same point, then we produce a contradiction.

This shows that the map $\sigma : \partial\mathcal{O} \to S^2_\infty$ is a finite to one map and completely characterizes the identifications. This finishes the proof of the theorem. $\square$

Using the blow up and blow down operations we obtain the following:

Corollary 10.3. *Exactly the same results as in theorem 10.2 hold for quasigeodesic almost pseudo-Anosov flows in M^3 with negatively curved fundamental group.*

We can now study idenfication of ideal points of leaves of foliations:

Theorem 10.4. *Let $\mathcal{F}$ be a Reebless foliation in M^3 closed, with negatively curved fundamental group. Let Φ_1 be a quasigeodesic pseudo-Anosov flow almost transverse to $\mathcal{F}$ and Φ be a corresponding almost pseudo-Anosov flow transverse to $\mathcal{F}$. Let $\mathcal{O}$ be the orbit space of $\widetilde{\Phi}$ with compactification $\mathcal{D}$ and ideal map $\sigma : \partial\mathcal{O} \to S^2_\infty$. Given F a leaf of $\widetilde{\mathcal{F}}$ and φ_F the extension map from $\partial_\infty F$ to S^2_∞, then suppose that e_0, e_1 distinct in $\partial_\infty F$ with $\varphi_F(e_0) = \varphi_F(e_1)$. Then e_0, e_1 are ideal points of leaves of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$ and correspond to ideal points of leaves l, m of $\mathcal{O}^s$ or $\mathcal{O}^u$. The leaves l, m are connected by a chain of perfect fits. The same holds true for the leaves of $\widetilde{\Lambda}^s_F$ or $\widetilde{\Lambda}^u_F$ defining e_1, e_2.*

Proof. Let e_1, e_2 in $\partial_\infty F$ distinct with $\varphi_F(z) = \varphi_F(y)$. By theorem 9.3, there are p, q are in B_F with $c_F(p) = e_1, c_F(q) = e_2$, hence $p \neq q$. It follows that

$$\sigma(p) \;=\; \varphi_F(e_1) \;=\; \varphi_F(e_2) \;=\; \sigma(q).$$

By theorem 10.2 there are rays l, r of $\mathcal{O}^s$ or $\mathcal{O}^u$ with p ideal point of l, q ideal point of r and l, r connected by a chain of perfect fits.

What remains to be proved is that e_1, e_2 are connected by a chain of perfect fits in F. The problem is that we do not know how the sides of the perfect fits in $\mathcal{O}$ from p to q relate to F:

whether they are they contained in $\Theta(F)$ (in which case we can lift the perfect fits in $\mathcal{O}$ to perfect fits in F) and so on. The analysis for the rest of the proof is to understand this situation.

We start with the sequence of the perfect fits in $\mathcal{O}$ from p to q.. For simplicity we consider the sides of the perfect fits to be not just rays, but rather slices in $\mathcal{O}^s$ or $\mathcal{O}^u$ containing these rays. We first construct a subsequence of the slices which will be used later. Let l_1 be the first slice with one ideal point p and the other ideal point z, see fig. 28, a.

If q is the other ideal point of l_1 we finish the selection of the l_i. Otherwise l_1 makes a perfect fit with another slice in the chain, called δ_1. If q is the other ideal point of δ_1 let l_2 be δ_1. Otherwise there is another perfect fit in the sequence with one side in δ_1. Let δ_2 be the other side of this second perfect fit. If δ_1, δ_2 share the ideal point z, see fig. 28, a, then throw out δ_1 and restart with δ_2. Otherwise let $l_2 = \delta_1$ and restart with l_2 and the subsequent perfect fits, see fig. 28, b.

In this way we inductively define $l_1, l_2, ..., l_n$ so that l_i, l_{i+1} have equivalent rays and l_i, l_{i+2} do not have equivalent rays, for any i for which this makes sense. This means that the rays of l_i, l_{i+1} define the same ideal point in $\partial\mathcal{O}$, but the rays of l_{i+2} do not. We now prove a collection of properties of these leaves l_i.

<u>Claim 1</u> $-$ If $\Theta(F)$ intersects l_i, then l_i is contained in $\Theta(F)$.

Suppose that x_0 is in $l_i \cap \Theta(F)$. If l_i is not contained in $\Theta(F)$, then starting from x_0 go along l_i until hitting x in $l_i \cap \partial\Theta(F)$. Then there is a slice η of either $\mathcal{O}^s$ or $\mathcal{O}^u$, so that x is in η and $\eta \subset \partial\Theta(F)$. Then $\overline{\eta}$ separates the endpoints of l_i in $\mathcal{D}$. It follows that it separates p from q in $\mathcal{D}$. As $\eta \subset \partial\Theta(F)$, then η separates $\Theta(F)$ from one of p or q. This is impossible, since p, q are in B_F so $\Theta(F)$ limits to p, q in $\mathcal{D}$. This proves claim 1.

<u>Claim 2</u> $-$ If $\Theta(F)$ does not intersect l_i, then l_i is contained in $\partial\Theta(F)$.

It suffices to show this for l_1, because then the other ideal point z of l_1 is in $\partial\Theta(F)$ and in addition $\sigma(p) = \sigma(z)$. Also $c_F(p) = c_F(z)$. Hence we can then restrict the process to l_2 and the other leaves.

So we show this fact for l_1. Suppose for simplicity that l_1 is a slice of $\mathcal{O}^s$. Let ν be a ray of l_1 defining p so that ν has no singularity, see fig. 29, a. Let τ be a segment in a leaf of $\mathcal{O}^u$ transversal to ν going into the side containing $\Theta(F)$ and let η be an arbitrary non singular leaf of $\mathcal{O}^s$ intersecting τ close to ν.

We first show that the rays of η defined by $\tau \cap \eta$ and in the direction of p have to intersect $\Theta(F)$ if they are close enough to l_1. To prove this first notice that since l_1 and η intersect a common unstable segment τ, then ν and that ray of η cannot be equivalent rays. If they are equivalent, they would have to be at least non separated in the leaf space of $\mathcal{O}^s$ which is impossible. This means that these rays do not define the same ideal point of $\mathcal{O}$. Suppose that $\eta \cap \Theta(F)$ is empty for all η near v. As $\Theta(F)$ limits to p then $\Theta(F)$ is always contained on the same side of η that l_1 is. It follows that as η gets closer and closer to l_1 then the corresponding rays of η cannot converge only to ν: there is a slice γ of $\mathcal{O}^s$ with γ non separated from l_1 and so that γ separates $\Theta(F)$ from l_1, see fig. 29, a. But then $\Theta(F)$ cannot limit on q, contradiction.

We conclude that η has points of $\Theta(F)$. If η stops intersecting $\Theta(F)$ before getting to $\eta \cap \tau$ then one has an unstable boundary leaf τ_1 of $\partial\Theta(F)$ separating τ from $\Theta(F)$. Again τ_1 separates $\Theta(F)$ from q, contradiction. This shows that τ is contained in $\Theta(F)$ and the ray ν is contained in $\partial\Theta(F)$. There is a slice of $\mathcal{O}^s(l_1)$ contained in $\partial\Theta(F)$. If this slice is not l_1 ,then there is a prong γ' in $\mathcal{O}^s(l_1)$ with γ' contained in $\partial\Theta(F)$ and again we obtain $\Theta(F)$ cannot limit on q, see fig. 29, b. This proves claim 2.

We now use this information to analyse the situation in F. We want to show that e_1, e_2 are connected by a chain of perfect fits between rays of $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$. Recall that z is the other ideal point of l_1. If l_1 does not intersect $\Theta(F)$, then $l_1 \subset \partial\Theta(F)$ and so z is also in B_F. As seen before

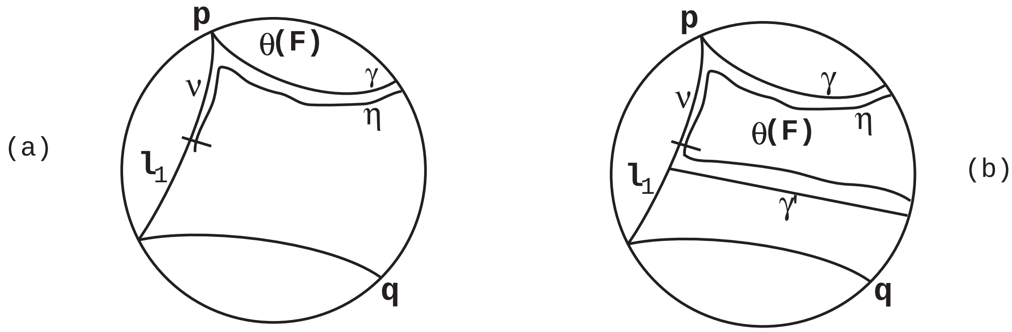

Figure 29: *a. The leaves η close to l_1 have to intersect $\Theta(F)$, b. $Theta(F)$ cannot get close to q.*

$c_F(z) = c_F(p) = e_0$. If on the other hand $l_1 \subset \Theta(F)$ then there is leaf α_1 of $\widetilde{\Lambda}^s_F$ with $\Theta(\alpha) = l_1$.

What is left to prove is the following: suppose l_i, l_j with $i < j$ are the first two l_k contained in $\Theta(F)$ and let α_1, α_2 with $\Theta(\alpha_1) = l_i$, $\Theta(\alpha_2) = l_j$. Then we need to prove that α_1, α_2 are connected by a sequence of perfect fits in F. If l_i, l_j are consecutive, that is, $j = i+1$ then the perfect fits in $\mathcal{O}$ produces a perfect fit in F. So suppose that $j > i+1$.

First of all notice that since l_{i+1} does not intersect $\Theta(F)$ then l_{i+1} cannot separate l_i from l_j in $\mathcal{O}$. Consider the sequence of perfect fits in $\mathcal{O}$ from l_i to l_{i+1}. Since l_i is contained in $\Theta(F)$ and l_{i+1} is contained in $\partial\Theta(F)$ then all these perfect fits are contained in $\Theta(F)$ – except for the corresponding ray of l_{i+1}. The next perfect fit from l_{i+1} to l_j has one side in another ray of l_{i+1} on the same side that l_i is – since l_{i+1} does not separate l_i from l_j. Since l_j is contained in $\Theta(F)$ then this perfect fit is contained in $\Theta(F)$. The two consecutive perfect fits in $\mathcal{O}$ with sides in rays of l_{i+1} coalesce in $\Theta(F)$ producing a single perfect fit. In this way one produces a perfect fit in F from α_1 to α_2.

To conclude the proof, note that the map $c_F : B_F \to \partial_\infty F$ is at most two to one and the map σ is finite to one. By theorem 8.2 this implies that the identifications of φ_F are finite to one.

This finishes the proof of the theorem. □

Much more can be said in certain circumstances, for example when the leaf spaces of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ are Hausdorff or then there are no freely homotopic closed orbits of Φ.

Corollary 10.5. *Suppose there are no freely homotopic closed orbits of Φ. Then the only identifications come from distinct rays of leaves of $\mathcal{O}^s$ or $\mathcal{O}^u$*

Proof. By theorem 2.6, the condition implies that $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ have Hausdorff leaf space. It also implies that there are no perfect fits, for otherwise one gets one orbit freely homotopic to another – by the argument in the proof of theorem 10.1 or [Fe4]. Hence the only possible identifications come from different rays in the same leaf of $\mathcal{O}^s$ or $\mathcal{O}^u$. □

Corollary 10.6. *Let $\mathcal{F}$ be an $\mathbf{R}$-covered foliation almost transverse to a quasigeodesic pseudo-Anosov flow Φ in M^3 closed with negatively curved fundamental group. Then the maps $\sigma : \partial\mathcal{O} \to S^2_\infty$ and $\varphi_F : \partial_\infty F \to \mathcal{O}$ (for F in $\widetilde{\mathcal{F}}$) are finite to one and identifications come only from rays of leaves of $\mathcal{O}^s$ or $\mathcal{O}^u$, or $\widetilde{\Lambda}^s_F, \widetilde{\Lambda}^u_F$.*

Proof. Since $\mathcal{F}$ is $\mathbf{R}$-covered, there cannot be non trivial free homotopies between closed orbits [Fe11]. Therefore the the previous corollary applies. □

11 Foliations and Kleinian groups

There are many similarities between foliations in hyperbolic 3-manifolds and Kleinian groups. We refer to [Mi, Can, Mar] for basic definitions concerning degenerate and non degenerate Kleinian groups, in particular singly and doubly degenerate groups.

If the foliation is **R**-covered then the limit set of any leaf in $\widetilde{M}$ is the whole sphere. This corresponds to doubly degenerate surface Kleinian groups [Th1, Mi, Can, Mar]. There is always a pseudo-Anosov flow which is transverse to the foliation [Fe9, Cal1]. If the flow is quasigeodesic then the results of this article imply that the foliation has the continuous extension property.

If the foliation has one sided branching, say branching down, then limit sets of leaves can only have domain of discontinuity "above" [Fe5]. Let F in $\widetilde{\mathcal{F}}$ and Λ_F its limit set. If p is not in Λ_F, the p is said to be *above* F if there is a neighborhood V of p in $\widetilde{M} \cup S^2_\infty$, so that $V \cap \widetilde{M}$ is on the positive side of F. This corresponds to simply degenerate surface Kleinian groups [Th1, Mi, Can]. There are examples of foliations with one sided branching transverse to suspension pseudo-Anosov flows provided by Meigniez [Me]. Suspension flows are always quasigeodesic flows [Ze]. The results of this article show the continuous extension property for such foliations. Under these conditions, the limit sets are locally connected, the continuous extension provides parametrizations of these limit sets.

Finally if there is branching in both directions, then there can be domain of discontinuity above and below leaves. This corresponds to non degenerate Kleinian groups [Th1, Mi, Can]. These occur for example in the case of finite depth foliations, where the depth 0 leaves are not virtual fibers [Fe1].

There are many interesting questions:

Question 1 – Given a foliation $\mathcal{F}$, is it **R**-covered if and only if for every $F \in \widetilde{\mathcal{F}}$ then the limit set Γ_F is S^2_∞?

The forward direction is true. The backwards direction is true if there is a compact leaf [Fe5]. In addition if there is one leaf with limit set the whole sphere then all leaves have limit set the whole sphere [Fe5] – whether $\mathcal{F}$ is **R**-covered or not.

Question 2 – Given $\mathcal{F}$ an **R**-covered foliation, is there a quasigeodesic transverse pseudo-Anosov flow?

This is true in the case of slitherings or uniform foliations as defined by Thurston [Th5]. Examples are fibrations, **R**-covered Anosov flows and many others. There is always a transverse pseudo-Anosov flow, the question is whether it is quasigeodesic.

Question 3 – Is there domain of discontinuity of Λ_F only above F if and only if $\mathcal{F}$ has one sided branching in the negative direction?

This occurs for the examples constructed by Meigniez [Me].

Question 4 – Are the pseudo-Anosov flows constructed by Calegari [Cal2] and transverse to one sided branching foliations quasigeodesic?

Question 5 – If $\mathcal{F}$ has 2 sided branching is there always domain of discontinuity above and below? Is there a quasigeodesic pseudo-Anosov flow almost transverse to $\mathcal{F}$?

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Florida State University
Tallahassee, FL 32306-4510