Harmonic functions on $\mathbb{R}$-covered foliations

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Abstract

Let $(M, \mathcal{F})$ be a compact codimension one foliated manifold whose leaves are endowed with Riemannian metrics and consider continuous functions on M that are harmonic along the leaves of $\mathcal{F}$. If every such function is constant on leaves we say that $(M, \mathcal{F})$ has the *Liouville property*. Our main result is that codimension-one foliated bundles over compact negatively curved manifolds satisfy the Liouville property. A related result for $\mathbb{R}$-covered foliations is also established.

1 Introduction

Let M be a compact manifold and $\mathcal{F}$ a continuous foliation of M whose leaves are C^r Riemannian manifolds, $r \geq 2$. It is assumed throughout the article that the boundary of M, if non-empty, is a union of (compact) leaves of $\mathcal{F}$. This implies that all compact leaves of $\mathcal{F}$ are closed manifolds. The Riemannian metrics on leaves, as well as their derivatives up to order r, are assumed to vary continuously on M. The pair $(M, \mathcal{F})$ refers here to foliations with the given choice of Riemannian metrics even if the metrics are not always explicitly mentioned. The metrics yield Laplace-Beltrami operators on leaves varying continuously on M.

Let $H(M, \mathcal{F})$ denote the set of real-valued functions on M that are continuous on M, C^2 on leaves, and harmonic on leaves. We call such functions *leafwise harmonic*. If the leaves of $\mathcal{F}$ are Riemann surfaces, or more generally Kähler manifolds, we can similarly consider the subset of $H(M, \mathcal{F})$ consisting of the real part of *leafwise holomorphic* functions. The continuous functions that are constant on leaves, or *leafwise constant* functions, form a subset of $H(M, \mathcal{F})$. If all leafwise harmonic (resp., holomorphic) functions are leafwise constant we say that $(M, \mathcal{F})$ has the *Liouville* (resp., *holomorphic Liouville*) *property.* The goal of this article is to study the Liouville property for certain classes of foliations.

The problem of understanding which foliations have the Liouville property was first considered in [FZ1, FZ2]. A fairly detailed description of the structure of $H(M, \mathcal{F})$ in codimension 1 under C^1 transversal regularity and in the absence of transverse invariant measures is obtained in [DK]. In order to provide some background for what will be proved here, we briefly list below a few pertinent results from these three papers.

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1. For (real) codimension one foliations by Kähler manifolds (or more generally, foliations whose leaves are complex manifolds) the holomorphic Liouville property holds. ([FZ1], Theorem 1.15.)

2. In [DK] an example is given of a codimension one foliation of a 3-manifold by Riemann surfaces for which the Liouville property does not hold. The following is also shown in [DK] (see Theorem 1.1 of [DK] for the full statement): Let $\mathcal{F}$ be a C^1 codimension one foliation of a compact manifold M having no transverse invariant measures (in particular, no compact leaves). Then there exists a finite number of minimal sets $\mathcal{M}_1, \ldots, \mathcal{M}_k$ equipped with probability measures $\mu_1, \ldots, \mu_k$ such that each $f \in H(M, \mathcal{F})$ can be written uniquely as a linear combination:

$$f = \sum_{i=1}^{k} \mu_i(f)\eta_i$$

   where the following notation is used: $\mu_i(f) = \int f d\mu_i$ and η_i is a continuous, leafwise harmonic function on M which gives the probability $\eta_i(p)$ that leafwise Brownian motion starting at p converges towards $\mathcal{M}_i$, for each $p \in M$.

3. It is shown in Theorem 4.1 of [FZ2] that there exists a foliated S^2-bundle over a compact Riemann surface, $(M, \mathcal{F})$, such that:

   (a) The Liouville property does not hold for $(M, \mathcal{F})$;

   (b) $(M, \mathcal{F})$ has exactly two minimal sets, S_1 and S_2, which are compact leaves homeomorphic to the base Riemann surface. In the complement of $S_1 \cup S_2$ the foliation and leafwise harmonic functions are smooth;

   (c) The foliation is ergodic with respect to the smooth measure class. In particular, almost all leaves are dense in M.

The results of [DK], in particular item 2 above, are based on a study of the Lyapunov exponent for holonomy contraction along Brownian paths. They show that for a C^1 foliation with a minimal set A, then either there is a transversely invariant measure supported on A or the harmonic measure of A is unique and the Lyapunov exponent of this measure is negative. If the Lyapunov exponent is negative, then points in A have strong contraction and attraction properties which imply the results of item 2 above. These results depend on the foliation being C^1 or else one cannot even define the Lyapunov exponents. The hypothesis that there are no transverse invariant measures is also crucial or one does not get negative Lyapunov exponents and contraction.

Hence it is natural to ask what can be said about $H(M, \mathcal{F})$ in codimension one when the results of [DK] do not apply, namely when there are compact leaves present or, more generally, transverse invariant measures, and/or the foliation is only C^0. In particular, we want to know under what natural hypothesis codimension one foliations have the Liouville property.

In this article we restrict attention to $\mathbb{R}$-*covered* or I-*covered* foliations. They are defined by the property that the space of leaves of the induced foliation $(\widetilde{M}, \widetilde{F})$ on the universal cover of M is Hausdorff. Equivalently, this leaf space is homeomorphic to the real line or to the interval $I = [0, 1]$, respectively, hence the terminology. These are the simplest situations in terms of the topology of the foliation. In addition, as seen below, they exhibit a difficulty which is not covered by the results in [DK]. Foliated circle bundles are $\mathbb{R}$-covered foliations. Other examples of $\mathbb{R}$-covered foliations can be seen in [Fe]. Based on what we prove below it is natural to ask whether all such foliations have the Liouville property. As an initial support of an affirmative answer, we mention the following easy consequence of the topological structure of $\mathbb{R}$-covered foliations described in [Fe].

Proposition 1.1 *The Liouville property holds for $\mathbb{R}$-covered foliations without compact leaves.*

In fact, for foliations satisfying the conditions of proposition 1.1, we prove that every leafwise harmonic function is constant on M. If there are no compact leaves, one shows that there is only one minimal set, which then easily implies the Liouville property. Compare with results in [DK], where one requires more than one minimal set to produce non trivial leafwise harmonic functions.

If there are compact leaves, the situation is much more interesting. First of all it is clearly possible to have functions that are constant on leaves but not constant on M: when $\mathcal{F}$ is a fibration over the circle, any non-constant function on the circle pulls back to a leafwise constant, non-constant function on M. This also happens to certain more general $\mathbb{R}$-covered foliations with compact leaves.

Given proposition 1.1, our problem is reduced to understanding what happens when there are compact leaves. In order to study leafwise harmonic functions or asymptotic behavior of holonomy (which is relevant here as well), it turns out that compact leaves are much trickier to understand. For example, the results of Deroin and Kleptsyn [DK] do not apply when there are compact leaves (even if one has the additional strong condition that holonomy is C^1). The same restriction holds for the results of Thurston [Th] on asymptotic behavior of holonomy.

At this point we are not able to deal with the most general $\mathbb{R}$-covered foliations. For our main results we assume that the leaves of $\mathcal{F}$ have negative curvature $-$ this is the condition under which the Liouville property might be expected not to hold with greatest likelihood. Clearly, if the leaves of a foliation individually do not admit bounded, non-constant harmonic functions, then the foliated Liouville property holds. This is the case, for example, when the leaves are nilpotent covers of recurrent (in particular, compact) Riemannian manifolds [LS], or the Ricci curvature of leaves is non-negative [SY]. In dimension 3, results of Plante and Sullivan [Pl, Su] show that some form of negative curvature is the generic situation, at least in the large scale: if for example the leaves are π_1-injective and M is atoroidal and closed, then the leaves have negative curvature in the large, that is, they are Gromov hyperbolic. In negative curvature, non-constant harmonic functions are plentiful, so if the Liouville property does hold it must be due to features pertaining to the transversal dynamics.

Our main result is the following:

Theorem 1.2 *Let $(M, \mathcal{F})$ be a continuous codimension-1 foliated bundle (with either circle or interval fibers) over a compact Riemannian manifold of negative sectional curvature. Then the Liouville property holds for $(M, \mathcal{F})$.*

By a *foliated bundle* $(M, \mathcal{F})$ we mean a foliation of the total space M of a fiber bundle whose fibers are everywhere transverse to the leaves of $\mathcal{F}$ and the local holonomy maps of the fiber foliation are Riemannian isometries relative to the metric on the leaves of $\mathcal{F}$.

Theorem 1.2 is mainly a result about foliated interval bundles. The claim for circle-bundles is an easy corollary given proposition 1.1. Just as easily, theorem 1.2 implies the following:

Theorem 1.3 *Let $(M, \mathcal{F})$ be a continuous codimension-one foliation with negatively curved leaves and let $\mathcal{M}$ denote (the closure of) the union of all the minimal sets. Suppose that the metric completion $\widehat{U}$ of each component U of $M \setminus \mathcal{M}$ admits an interval-bundle structure that makes the induced foliation on $\widehat{U}$ a foliated interval-bundle over a compact base manifold. Then the Liouville property holds for $(M, \mathcal{F})$.*

We note that a foliation satisfying the hypothesis of theorem 1.3 is either minimal, $\mathbb{R}$-covered, or I-covered foliation. This can be seen as follows. A classical result of Haefliger states that the union of all compact leaves of a codimension one foliation is compact and there are finitely many compact leaves up to isotopy in M. In addition, there are finitely many minimal sets which are not compact leaves. A proof of this well-known fact for C^2 foliations can be found in [CC1], theorem 8.3.2, and Cantwell and Conlon have a short, unpublished proof for C^0 foliations. So one possibility in theorem 1.3 is that $\mathcal{F}$ is minimal, in which case the Liouville property holds by the maximum principle for harmonic functions. (A continuous, leafwise harmonic function must be constant on every minimal set. In fact, by the maximum principle the function is constant on a leaf where it attains its maximum or minimum value over a given minimal set $\mathcal{A}$, hence it is constant on $\mathcal{A}$.) Suppose now that $\mathcal{F}$ is not minimal and let $\mathcal{A}$ be a minimal set. By the just mentioned result of Haefliger's, if $\mathcal{A}$ is not a compact leaf and B is a boundary leaf of $\mathcal{A}$, then B is at a positive distance from any other minimal set. Hence the hypothesis of theorem 1.3 implies that if U is a complementary component of $\mathcal{M}$ that has B as one of its boundary leaves, then B, and hence $\mathcal{A}$, is a compact leaf. Therefore, we conclude that the only minimal sets are compact leaves. Given the finite number of isotopy classes of compact leaves, it follows that by cutting M along a compact leaf we obtain an I-bundle, and we can adjust the foliation to be transverse to the I-fibers. In particular, the resulting foliation is I-covered. So the original foliation (prior to cutting along a compact leaf) is either I-covered or $\mathbb{R}$-covered.

As an example to which theorem 1.3 applies, start with a foliated interval-bundle and glue the boundary leaves with an arbitrary homeomorphism. This gives a foliation of a closed manifold satisfying the conclusion of theorem 1.2. Foliated circle bundles with compact leaves can be described in this way using a periodic map as the gluing map so that all points have the same period. To put things in perspective, consider the situation in dimension 3: the hypothesis of theorem 1.2 implies that M is Seifert fibered, the Seifert fibration given by circle fibers. In particular, it has a normal Z subgroup. (See [He], chapter 12, for standard definitions.) Theorem 1.3, after cutting along a compact leaf, allows for any gluing between top and bottom. The vast majority of such gluings yields hyperbolic 3-manifolds. So this is much more general than theorem 1.2.

Although foliated bundles may seem too restrictive a setting, they are a very common type of foliation and the source of a large variety of examples and counter-examples in foliation theory. They are exactly the foliations that are associated with group actions on the fiber space (the circle or interval, in theorem 1.2).

Here is a brief sketch of the proof of theorem 1.2.

- If $\mathcal{F}$ has no compact leaves, then $\mathcal{F}$ is a foliated circle bundle and it is $\mathbb{R}$-covered. Then the Liouville property is easily derived from the topological properties of such foliations discussed in section 4, and properties of harmonic functions with respect to harmonic measures.

- If there are compact leaves, we restrict attention to a connected component U of $M \backslash \mathcal{K}$, the complement of the union of all compact leaves. (Leafwise harmonic functions must be constant along leaves in $\mathcal{K}$.) The metric completion of U is an interval bundle with compact boundary leaves and no compact leaf in the interior. This reduces the proof of the theorem to foliated interval bundles over a compact manifold and no interior compact leaves. These first 2 steps are done under the much more general condition of $\mathcal{F}$ being $\mathbb{R}$-covered or I-covered.

- The proof of the theorem for interval bundles proceeds by contradiction. We suppose that a nontrivial leafwise harmonic continuous function f exists, and normalize it so that it takes values 0 and 1 on the compact boundary leaves of $\mathcal{F}$. Using the relationship between harmonic functions and properties of the foliated Brownian motion (under the assumption that leaves are negatively curved) we derive that f is a monotone function on fibers of the interval bundle (lemma 6.1). After blowing down interval bundles in $(M, \mathcal{F})$ where f is constant along fibers, it can be assumed that f is strictly monotone on fibers (proposition 6.2). Both of these results make full use of the hypothesis of theorem 1.2: we need the foliated bundle property to directly relate Brownian motion in different leaves. We also need negative curvature on the leaves to relate the harmonic function on the leaf with the behavior at infinity (this is done in the universal cover of the leaf).

- Using the strict monotonicity of f it is possible to define a new foliated interval bundle topologically equivalent to the original one that is now *harmonic* in the following sense: leaves of the new foliation are locally graphs of harmonic functions on the base manifold. This is shown at the beginning of section 7. Although the initial foliation was possibly only C^0, we prove in section 7 that the new foliation is, in fact, Lipschitz continuous.

- In section 9 we prove the following general fact: If $M = K \times I$, where K is a compact Riemannian manifold (no curvature assumption) and I is the interval $[0, 1]$, and $(M, \mathcal{F})$ is a Lipschitz continuous harmonic foliation, then $\mathcal{F}$ is the product foliation. This result leads to a contradiction, since the original foliated interval bundle did not have compact leaves other than the boundary leaves.

The results of this article generate one obvious question: if $\mathcal{F}$ is $\mathbb{R}$-covered or I-covered, does $\mathcal{F}$ have the Liouville property? A key step to answering this question affirmatively is to find some form of transversal monotonicity of leafwise harmonic functions as obtained in section 7 with the additional foliated I-bundle hypothesis. Another very important question is whether the curvature condition can be weakened. In particular what happens when the leaves are Gromov hyperbolic or negatively curved in the large, but not necessarily (Riemannian) negatively curved?

2 Harmonic functions and Brownian motion

We begin by recalling some background material on harmonic functions and Brownian motion on Riemannian manifolds, with special attention to manifolds of negative sectional curvature. More details about Brownian motion can be found, for example, in [Hsu] or [Em]. Brownian motion on foliated spaces is discussed in [Ca] as well as chapter 2 of [CC2]. Information specific to negative curvature can be found in [An] and the other references to be cited below.

A few key facts about harmonic functions are listed below. Let N be a Riemannian manifold and Δ the Laplace-Beltrami operator on N. A real-valued function $f \in C^2(U)$, where U is an open set in N, is *harmonic* on U if $\Delta f = 0$ on U.

1. The *maximum principle*: if f is harmonic on a connected open set U and attains a maximum (or minimum) value in U, then f is constant on U.

2. The principle of *unique continuation* (see [Ar] for a more general fact): if f is harmonic on a connected open set U and constant on a neighborhood of some point in U, then f is constant on U.

3. The *Harnack inequality* [Mo]: If U is open with compact closure and V is a subset whose closure is contained in U, then there exists a constant $C > 0$ depending only on U and V such that $\sup h|_V \leq C \inf h|_V$ for any positive harmonic function h on U.

4. The *Harnack principle* (see [An], p. 6): if U is an open connected set in N and $p \in U$, then the set of non-negative harmonic functions f on U such that $f(p) = 1$ is compact in the topology of uniform convergence on compact subsets of U.

The standard probability setting for manifold-valued stochastic processes is assumed: we fix throughout a probability space $(\Omega, \mathcal{B}, P)$ and a *filtration* $\mathcal{B}_* = \{\mathcal{B}_t : t \geq 0\}$ of σ-algebras contained in $\mathcal{B}$. That is, $(\mathcal{B}_t)$ is an increasing family of σ-algebras $\mathcal{B}_t$ containing all sets of measure 0 in $\mathcal{B}$. If Y is an integrable real valued function on $(\Omega, \mathcal{B}, P)$, its expectation is denoted $E[Y]$, and if $\mathcal{A}$ a σ-algebra contained in $\mathcal{B}$, the conditional expectation of Y given $\mathcal{A}$ is denoted $E[Y|\mathcal{A}]$. Recall that a real valued stochastic process $\{Y_t : t \geq 0\}$ is a *martingale* if Y_t is integrable and $\mathcal{B}_t$-measurable for each t and for every pair $s, t \in [0, \infty)$, $s \leq t$, the equality $Y_s = E[Y_t|\mathcal{B}_s]$ holds. A *Brownian motion* on a Riemannian manifold

N with Laplace-Beltrami operator Δ is an N-valued stochastic process, B_t, $t \geq 0$, which is continuous (i.e., sample paths $t \mapsto B_t(\omega)$ are continuous for a.e. $\omega \in \Omega$), adapted to the filtration (i.e., B_t is $\mathcal{B}_t$-measurable for each $t \geq 0$), and for every smooth function f on N the process

$$M_t^f := f(B_t) - f(B_0) - \frac{1}{2}\int_0^t \Delta f(B_t)dt$$

is a martingale. (This definition does not account for the possibility of explosions since we will only deal with stochastically complete metrics later on.) If B_t is a Brownian motion on N and if $\gamma : N \to N'$ is a local Riemannian isometry, then $\gamma \circ B_t$ is a Brownian motion on N'. For $p \in N$, Brownian motion conditional on $B_0 = p$ will be written B_t^p. The corresponding conditional probability on Ω and expectation will be written P^p and E^p, respectively. Thus, for any bounded f on N, $E^p[f \circ B_t] := \int_\Omega f(B_t(\omega))dP^p(\omega)$.

Let the Riemannian manifold N be geodesically complete, simply connected, of sectional curvature K bounded by constants $-b^2 \leq K \leq -a^2 < 0$. Let $S(\infty)$ be the sphere at infinity of N, which consists of equivalence classes of asymptotic geodesics. Then $\overline{N} = N \cup S(\infty)$ has a natural topology (the *cone topology*) that makes $\overline{N}$ compact and $S(\infty)$ its boundary. The latter is known as the *ideal boundary* of N.

We collect some of the main properties of Brownian motion on N in the following list. (See [Ki] in addition to the references cited in each item.)

1. For any initial point $p \in N$, B_t^p converges in the cone topology, as $t \to \infty$, to a random point B_∞^p of $S(\infty)$. (I.e., the path $B_t^p(\omega)$ converges to a point $B_\infty^p(\omega)$ for P^p-a.e. $\omega \in \Omega$.)

2. The probability distribution of B_∞^p is a Borel probability measure μ_p on $S(\infty)$. Thus $\mu_p(A)$ is the probability of the event $B_\infty^p \in A$, for a Borel $A \subset S(\infty)$. Its main property is that $p \mapsto \mu_p(A)$ is a harmonic function on N. The measure μ_p is known as the *harmonic measure* at p. This should not be confused with harmonic measures for foliations as defined by Lucy Garnett in [Ga]. The latter, which also plays a role in this paper, will be referred to either as *stationary measures* for the foliated Brownian motion or as harmonic measures *in the sense of Garnett.*

3. The measures μ_p are all equivalent among themselves. This is a simple consequence of the maximum principle and that $p \mapsto \mu_p(A)$ is harmonic. By the Harnack inequality, given any pair of points $p, q \in N$ there exists $C > 0$ depending only on p and q such that $C^{-1}\mu_q(A) \leq \mu_p(A) \leq C\mu_q(A)$ for all A. The associated measure class defines the *harmonic measure class* of $S(\infty)$;

4. For any bounded function g on $S(\infty)$, measurable relative to the harmonic measure class, the function

$$H_g(p) := \int_{S(\infty)} g(\xi)d\mu_p(\xi)$$

is harmonic on N. Conversely, if H is a bounded harmonic function on N, there exists a bounded measurable g on $S(\infty)$, uniquely defined up to a set of harmonic measure zero, such that $H = H_g$;

5. If $H = H_g$ is a bounded harmonic function on N with boundary values g, then P^p-almost surely $H(B_t^p)$ converges to $g(B_\infty^p)$ as $t \to \infty$;

6. If $H = H_g$ is a bounded harmonic function on N with boundary value g, then the non-tangential limit of H exists almost everywhere on $S(\infty)$ with respect to the harmonic measure class. More precisely, for $\xi \in S(\infty)$, $a > 0$, and $t \mapsto r(t)$ a geodesic ray limiting at ξ, denote by $C_a(\xi)$ the set of all $p \in N$ such that the distance $d(p, r) < a$. Such a set is called a *non-tangential cone* at ξ. Then, for a.e. $\xi \in S(\infty)$ with respect to the harmonic measure class, and any non-tangential cone $C_a(\xi)$, $H(p)$ converges to $g(\xi)$ as $p \to \xi$ within $C_a(\xi)$. (See [AS].)

A note of caution: there is another natural probability measure on $S(\infty)$ obtained by pushing forward the Lebesgue measure on the unit sphere $T_p^1 N$ to $S(\infty)$ under the map that assigns to each $v \in T_p^1 N$ the asymptotic class of the geodesic with initial condition (p, v). These measures are known to be mutually equivalent for all p and define the *geodesic measure class* on $S(\infty)$. Even though the harmonic measures can be shown to be positive on non-empty open sets and to not have atoms [KL], the geodesic and the harmonic measure classes are in general mutually singular. In fact, by a result of A. Katok this is always the case for $N = \widetilde{K}$ and K a closed surface of non-constant negative curvature. If the sectional curvature is constant, the two measure classes coincide.

3 Leafwise harmonic functions

Let M be a compact manifold and $\mathcal{F}$ a foliation of M. Unless a stronger regularity assumption is explicitly stated, $\mathcal{F}$ is a continuous foliation with C^2 leaves. The tangent bundle of $\mathcal{F}$ is given a Riemannian metric smooth along leaves, and the metric together with its derivatives of any order in the leaf direction are continuous in M. We refer to this setting simply by saying that $\mathcal{F}$ is a foliation of M with Riemannian leaves.

The metric induces a Laplacian on each leaf of $\mathcal{F}$. A continuous real valued function on M is *leafwise harmonic* if its restriction to each leaf is (smooth and) harmonic. Clearly, a leafwise harmonic function f is constant on any compact leaf, or on any leaf containing a point of maximum or minimum value of f, due the the maximum principle. We say that a leafwise harmonic function is *non-trivial* if it is not constant on at least one leaf of $\mathcal{F}$.

Brownian motion on leaves of $\mathcal{F}$ will still be denoted B_t. Thus, for a probability space $(\Omega, \mathcal{B}, P)$ and each $t \in [0, \infty)$, B_t is a random variable with values in M and for P-a.e. $\omega \in \Omega$ the path $t \mapsto B_t(\omega)$ is continuous and lies in the leaf of $\mathcal{F}$ through $B_0(\omega)$. The process and probability, conditional on beginning at $p \in M$, will be written as B_t^p and P^p, respectively.

Proposition 3.1 *Let $\mathcal{F}$ be a foliation of a compact manifold M with Riemannian leaves, as defined above. Let L be a leaf of $\mathcal{F}$ with sectional curvature K_L satisfying at all points $-b^2 \leq K_L \leq -a^2 < 0$. Let $S(\infty)$ be the ideal boundary of the universal cover, $\widetilde{L}$, of L. Suppose that f is a leafwise harmonic function on $(M, \mathcal{F})$ and that the boundary values of the natural lift, $\tilde{f}$, of $f|_L$ to $\widetilde{L}$ are given by the Borel measurable function g on $S(\infty)$. Then,* *for almost every $\xi \in S(\infty)$ with respect to the harmonic class, there exists a leaf of $\mathcal{F}$ on which f is constant and equal to $g(\xi)$.*

Proof. Let $\xi \in S(\infty)$ be a point of non-tangential convergence of $\tilde{f}$ and consider a sequence $p_n \in \widetilde{L}$ converging to ξ along a geodesic ray. For a fixed constant $c > 0$, consider the sequence of balls $D(p_n, c)$ of radius c and center p_n. Then for each n and all $q_n \in D(p_n, c)$ we have $\lim_{n\to\infty} \tilde{f}(q_n) = g(\xi)$. After passing to a subsequence, the projection of p_n to M converges in M to a point p and the balls converge to $D(p, c)$ as sets. Since f is continuous on M, the value of f on $D(p, c)$ is equal to the limit value $g(\xi)$. By the principle of unique continuation of harmonic functions (see section 2) f must be constant, equal to $g(\xi)$, on that leaf. □

4 Foliations without compact leaves and I-covered foliations

If $\mathcal{F}$ is a foliation of a manifold M, let $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to the universal cover $\widetilde{M}$. The *space of leaves* of $\widetilde{\mathcal{F}}$ is the quotient topological space $\mathcal{H}\ \widetilde{M}/\widetilde{\mathcal{F}}$ under the equivalence relation that identifies points of $\widetilde{M}$ lying on the same leaf. A codimension one foliation $\mathcal{F}$ of a closed manifold M is said to be $\mathbb{R}$-*covered* (respectively, I-*covered*) if the space of leaves, $\mathcal{H}$, of the foliation $(\widetilde{M}, \widetilde{\mathcal{F}})$ on the universal cover of M is homeomorphic to $\mathbb{R}$ (respectively, to the closed interval $I = [0, 1]$.)

Proposition 4.1 *Let $\mathcal{F}$ be an $\mathbb{R}$-covered foliation of a manifold M. Then one of the three following cases happens:*

1. *There is a compact leaf;*
2. *$\mathcal{F}$ is minimal;*
3. *$\mathcal{F}$ is not minimal and there is a unique minimal set.*

Notice that items 1 and 3 in proposition 4.1 are not mutually exclusive.

The proof for 3-manifolds is in [Fe]. Using the Dippolito decomposition (see description below and [Di, CC1]) the same proof works in general. Dippolito proved the following: Let U be an open, $\mathcal{F}$-saturated connected subset of M. Let $\widehat{U}$ be its metric completion. Then $\widehat{U}$ has an *octopus decomposition* (proposition 5.2.14 of [CC1]): $\widehat{U} = C \cup A_1 \cup \cdots \cup A_l$, where C is compact, $C \cap A_i$ is both the transverse boundary of A_i and a connected component of the transverse boundary of C, and the A_i are I-bundles over non-compact manifolds B_i and the foliation restricted to A_i is transverse to the I-fibers. The B_i have boundary and the thickness of the bundle goes to 0 as distance from the boundary of B_i grows to infinity.

In Dippolito's work the A_i are called foliated I-bundles. Here we restrict that terminology to foliations by Riemannian leaves so that local holonomy along the I-fibers are Riemannian isometries. (See the next section.) Given the decomposition above of $\widehat{U}$, then C is the *core* and the A_i are the *arms* of the decomposition.

Proposition 4.1 implies the Liouville property for $\mathbb{R}$-covered foliations without compact leaves:

Corollary 4.2 *Let $(M, \mathcal{F})$ be a compact foliated space with Riemannian leaves. Suppose that the foliation is $\mathbb{R}$-covered without compact leaves. Then continuous, leafwise harmonic functions are constant on M.*

Proof. Let g be continuous leafwise harmonic. By continuity, the closure of a leaf on which $g = c$, for some constant c, contains a minimal set where $g = c$. By the maximum principle, the maximum and minimum values of g must be attained at points contained in leaves where g is constant. If there are no compact leaves, the previous proposition says that there is only one minimal set, therefore the maximum and minimum values of g coincide. □

There is more that can be said about $\mathbb{R}$-covered foliations of a compact M when there are compact leaves:

Proposition 4.3 *Let the $\mathbb{R}$-covered foliation $\mathcal{F}$ be transversely orientable and have a compact leaf K. Let T be the manifold obtained by cutting M along K and letting $\mathcal{F}_1$ be the induced foliation on T. Then $\pi_1(K)$ surjects in $\pi_1(T)$. If, in addition, $\dim M = 3$ and M is not doubly covered by $S^2 \times S^1$, then T is an I-bundle and $\mathcal{F}_1$ is isotopic to a foliation transverse to the I-fibers.*

This is proved in [Fe] for 3-manifolds and the proof generalizes directly to n dimensions.

Proposition 4.4 *Let $\mathcal{F}$ be a codimension-1 foliation of a compact manifold M with boundary $\partial M = A_0 \cup A_1$, where A_0 and A_1 are leaves of $\mathcal{F}$. Suppose that no leaf of $\mathcal{F}$ other than A_0 and A_1 is compact and that the space of leaves of $\widetilde{\mathcal{F}}$ is Hausdorff. Then the leaf space $\mathcal{H}$ of $\widetilde{\mathcal{F}}$ is homeomorphic to a closed interval whose endpoints correspond to the unique lifts of A_0 and A_1, and every interior leaf limits on both A_0 and A_1.*

Proof. We first show that $\mathcal{H}$ is homeomorphic to $[0, 1]$. Suppose that there is a transversal arc in $\widetilde{M}$ connecting a leaf of $\widetilde{\mathcal{F}}$ to itself. Join the endpoints by a path in the leaf to produce a closed curve. Since $\widetilde{\mathcal{F}}$ is transversely orientable, this path can be perturbed to produce a closed transversal, γ, to $\widetilde{\mathcal{F}}$. As $\widetilde{M}$ is simply connected, γ bounds a singular disc, D, which can be assumed to be in general position with respect to $\widetilde{\mathcal{F}}$. (See corollary 7.1.12 of [CC1].) Hence $\widetilde{F}$ induces on D a singular 1-dimensional foliation $\mathcal{F}^*$, which is transverse to the boundary of D and has isolated singularities. By a standard argument there must be a limit cycle, γ, in D, and the germ of holonomy of $\mathcal{F}^*$ is contracting on at least one side of γ. (See, for example, proposition 7.3.2 of [CC1]; it is known that this argument, which is related to the Poincaré-Bendixson theorem, can be carried out for C^0 foliations; see [So, GO].)

This closed curve lies on a leaf, B, of $\widetilde{\mathcal{F}}$ having contracting holonomy germ along γ. But then, there are many leaves of $\widetilde{\mathcal{F}}$ near B which cannot be separated from B, contradicting the assumption that $\mathcal{H}$ is Hausdorff.

Hence any transversal to $\widetilde{\mathcal{F}}$ intersects a given leaf at most once, and so $\mathcal{H}$ is a 1-manifold. It is clearly simply connected. In addition, it has a countable basis and is Hausdorff by assumption. Therefore, $\mathcal{H}$ is homeomorphic to either $(0, 1)$, $[0, 1)$, or $[0, 1]$. But $\mathcal{H}$ has at least two boundary points, which must be lifts of A_0 and A_1. In particular, A_0 and A_1 have unique lifts to $\widetilde{M}$, denoted A'_0 and A'_1 and $\mathcal{H}$ is homeomorphic to $[0,1]$, where 0 and 1 are identified with A'_0 and A'_1, respectively.

We now show that the interior leaves of $\mathcal{F}$ must limit on both A_0 and A_1. First observe that $\mathcal{F}$ is transversely orientable. If not, some element of the fundamental group of M would switch the leaves A'_0 and A'_1 in $\mathcal{H}$, and these would project to a single leaf in M, which is not the case. Suppose that an interior leaf L does not limit on one of the boundary leaves, say A_0. Consider all the lifts of L to $\widetilde{M}$. Each of them separates A'_0 from A'_1. This is because $\mathcal{H}$ is homeomorphic to $[0,1]$ and the projection from $\widetilde{M}$ to $\mathcal{H}$ is continuous, so a path from A'_0 to A'_1 produces a path from 0 to 1.

Let $\mathcal{T}$ denote the subset of $\mathcal{H}$ corresponding to leaves of $\widetilde{\mathcal{F}}$ that are separated from A'_0 by some lift of L. The above properties show that $\mathcal{T}$ is connected and homeomorphic to an interval $(c,1]$ or $[c,1]$. Clearly $c<1$ since any lift of L separates A'_0 from A'_1 and also $c>0$, due to the assumption that L does not limit on A_0. Let C be the leaf of $\widetilde{\mathcal{F}}$ corresponding to c. In particular C is not a lift of A_0 or A_1.

We claim that any covering translation of $\widetilde{M}$ must map C to itself. Covering transformations induce an action by orientation preserving homeomorphisms of $[0,1]$. If there is h covering translation so that h does not leave C invariant, then up to taking an inverse we may assume that $h(c)<c$. This contradicts the definition of c as the infimum of $\Theta(V)$ where V is a lift of L. Here Θ is the the projection map from $\widetilde{M}$ to $\mathcal{H}$.

Let $\pi : \widetilde{M} \to M$ be the universal covering map. If $\pi(C)$ is not compact, then there is a foliation box Z in M in which a sequence of distinct sheets contained in $\pi(C)$ limit on a sheet of $\mathcal{F}$ in Z. Lifting coherently to $\widetilde{M}$, we obtain a sheet B' of $\widetilde{\mathcal{F}}$ and a sequence of distinct sheets in translates of C that converge to B'. This was disallowed by the previous paragraph and shows that $\pi(C)$ is compact, contradicting the hypothesis on $\mathcal{F}$. $\square$

As an example to which proposition 4.4 applies, let $M = K \times I$, where K is a compact Riemannian manifold, $I=[0,1]$ and $\mathcal{F}$ is a continuous foliation everywhere transverse to the fibers of the fibration $\pi_2 : K \times I \to I$, so that $A_i = K \times \{i\}$, $i=0,1$, are leaves of $\mathcal{F}$. The proof of the previous lemma is much simpler for this special case.

5 Harmonic functions on I-covered foliations

We describe here some basic consequences of assuming that an I-covered foliation carries a non-trivial leafwise harmonic function. Our goal is to show that under certain additional hypothesis there are no nontrivial such functions.

Lemma 5.1 *Consider the same setting and assumptions of proposition 4.4. If $(M,\mathcal{F})$ admits a non-trivial leafwise harmonic function, then there exists a unique such function, f, with the properties: the range of values of f is the interval $[0,1]$; f equals 0 on A_0 and 1 on A_1; and the restriction of f to any leaf other than A_0 and A_1 has the range of values $(0,1)$. Any other leafwise harmonic function g is of the form $g=af+b$ for constants a,b.*

Proof. Let g be a nontrivial leafwise harmonic function. As already remarked, g is constant on each compact leaf, hence let a_0, a_1 be the constant values of g on A_0 and A_1, respectively.

Without loss of generality we assume $a_0 < a_1$. Note that a_1 and a_0 are the maximum and minimum values of g on M. In fact, suppose that a maximum value, c, of g was attained at an interior point, q. By the maximum principle the restriction of g to the leaf through q would be constant, equal to c. Since an interior leaf must limit on both A_0 and A_1 by proposition 4.4, then $a_0 = a_1 = c$. This forces the maximum and minimum values of g to coincide, a contradiction. (Similarly, if c is a minimum value.) The range of g on each interior leaf is the full open interval (a_0, a_1) due, again, to interior leaves limiting on A_0 and A_1. By composing g with an affine function of the line we obtain f with the claimed properties. Uniqueness follows from the observation that if a leafwise harmonic function h is 0 on A_0 and A_1, then the above argument shows that h is identically zero on all other leaves. □

For the next 2 results we assume that leaves of $\mathcal{F}$ have pinched negative curvature, so we can use the facts of section 2.

Lemma 5.2 *Assume, as in proposition 4.4, that: $\mathcal{F}$ is a codimension-1 foliation of a compact manifold M with boundary $\partial M = A_0 \cup A_1$, where A_0 and A_1 are leaves of $\mathcal{F}$; no leaf of $\mathcal{F}$ other than A_0 and A_1 is compact; and the space of leaves of $\widetilde{\mathcal{F}}$ is Hausdorff. In addition, suppose that the leaves of $\mathcal{F}$ have negative sectional curvature and that $(M, \mathcal{F})$ admits a non-trivial leafwise harmonic function. Let f be the unique such function taking value i on A_i, $i = 0, 1$. Then the following properties hold:*

1. *B_t is transient in $M \setminus \partial M$; that is, for any interior point p of M, and any compact set $V \subset M \setminus \partial M$ containing p, then for P^p-a.e. $\omega \in \Omega$, there is $\tau(\omega) < \infty$ such that $B_t^p(\omega)$ lies in the complement of V for all $t \geq \tau(\omega)$. In other words, with probability one, B_t converges towards A_0 or A_1;*

2. *Let L be the leaf through $p \in M$, $\widetilde{L}$ the leaf through a lift p' of p, $\tilde{f}$ the lift of f to $\widetilde{L}$, and $S(\infty)$ the ideal boundary of $\widetilde{L}$. Then there exists a measurable set $S_1 \subset S(\infty)$ for which the following holds: (i) almost surely, Brownian motion $B_t^{p'}$ (in $\widetilde{L}$) converges to a point in S_1 if and only if B_t^p converges to A_1; (ii) the probability that B_t^p converges to A_1 equals $f(p) = \mu_{p'}(S_1)$;*

3. *For every p and a.e. unit vector $v \in T_p^1\mathcal{F}$ with respect to the harmonic class, viewed here as a measure class on $T_p^1\mathcal{F}$, the geodesic ray with initial conditions (p, v) converges to either A_1, if v corresponds to ξ in S_1, or A_0 otherwise. (We make no similar claim for the visual measure on $T_p^1\mathcal{F}$.)*

Proof. These assertions are consequences of proposition 3.1, lemma 5.1, and the various facts about Brownian motion and boundary values of harmonic functions enumerated in section 2. The curvature pinching $-b^2 \leq K \leq -a^2 < 0$ holds since M is compact. It is convenient to pass to the universal cover $(\widetilde{M}, \widetilde{\mathcal{F}})$. The lifts of the two compact leaves are denoted A_i', $i = 0, 1$. Then the above properties follow from the corresponding assertions for the lifted Brownian motion.

A key point to note is that, as $t \to \infty$, the distance between $B_t^p(\omega)$ and A_i goes to zero if and only if $\tilde{f}(B_t^{p'}(\omega))$ converges to i since f is continuous, equals i on A_i, maps interior points of M into $(0,1)$, and $\tilde{f}(B_t^{p'}(\omega)) = f(B_t^p(\omega))$. This occurs because the corresponding fact holds in M, since M is compact.

Now the limit $\tilde{f}(B_t^{p'})$ exists with $P^{p'}$-probability 1 and equals $g(B_\infty^{p'})$, where g is a function on $S(\infty)$ such that $\tilde{f} = H_g$. But by proposition 3.1, and since $\tilde{f}$ is not constant on any leaf except for A_0' and A_1', it follows that, almost surely, g only takes the values 0 and 1. Therefore, g can be taken to be the indicator function of a subset of $S(\infty)$, denoted S_1. This shows assertions 1 and 2. The last statement of assertion 2 follows from property 4 of Brownian motion. Assertion 3 is a consequence of the existence of non-tangential limits of harmonic functions on $S(\infty)$. (Property 6 of section 2.) □

The main conclusion of lemma 5.2 (parts 1 and 2) is summarized in the next corollary.

Corollary 5.3 *Let $\mathcal{F}$ be a codimension one foliation of a compact manifold M with boundary $\partial M = A_0 \cup A_1$, where A_0 and A_1 are leaves of $\mathcal{F}$; no leaf of $\mathcal{F}$ other than A_0 and A_1 is compact; the space of leaves of $\widetilde{\mathcal{F}}$ is Hausdorff; and leaves have negative sectional curvature. If $(M, \mathcal{F})$ admits a non-trivial leafwise harmonic function, then the unique such function f taking values i on the boundary leaves A_i, $i = 0, 1$, satisfies: $f(p)$ is the probability that the foliated Brownian motion starting at p converges to A_1.*

We refer to the function f as the *normalized* leafwise harmonic function on $(M, \mathcal{F})$.

6 Foliated bundles and monotonicity of f

It is natural to ask whether the normalized leafwise harmonic function f, which varies from 0 to 1 in the way from A_0 to A_1, is in some sense transversely monotone. It is not clear how such a property should be defined for general I-covered foliations, where the manifold may not even have an I-bundle structure. Here we make the additional restriction that $(M, \mathcal{F})$ be a foliated I-bundle, as defined below.

We first recall some definitions. Let K be a compact $n-1$-dimensional manifold and $\pi : M \to K$ a fiber bundle whose fibers are everywhere transverse to a foliation $\mathcal{F}$. We assume that the restriction of π to any leaf of $\mathcal{F}$ is a Riemannian covering of K. We say in this case that $(M, \mathcal{F})$ (together with the map π) is a *foliated bundle* with base manifold K.

A foliated bundle also has the following description. Let $X = \pi^{-1}(q)$, $q \in K$, represent a typical fiber of $\pi : M \to K$ (a compact topological space) and let $\rho : \pi_1(K, q) \to \mathrm{Hom}(X)$ denote the holonomy representation of the fundamental group of K acting on X by homeomorphisms (or C^r diffeomorphisms, if the foliation is C^r). Let $\widetilde{K}$ be the universal covering of K. Then it can be shown that the quotient space $(\widetilde{K} \times X)/\Gamma$ for the natural action of $\Gamma = \pi_1(K, q)$ on the product is isomorphic as a foliated bundle to $(M, \mathcal{F})$.

We are especially interested in the case where K has negative sectional curvature and the fibers of the foliated bundle are homeomorphic to the interval $X = I = [0, 1]$, where 0 and 1 are fixed points of ρ. We refer to this setting as a *foliated I-bundle* with negatively curved leaves. For these I-bundles, M has two boundary leaves, which are isometric to K. As already noted, the foliation is transversely orientable since an orientation reversing transformation would have 0 and 1 in the same orbit of ρ, and M would have only one boundary component rather than two.

On $\widetilde{M}$, the map along I-fibers from $\widetilde{K}$ to the lift of any leaf is a global isometry. Also M is diffeomorphic to the product $K \times I$, so we can introduce a global *height function* $\eta : M \to [0,1]$ corresponding to the projection on the second component of the product. This is a smooth function on M. Let A_i be the boundary leaf of M corresponding to $\eta = i$, $i = 0, 1$. Let q be any point in K and fix a lift $q' \in \widetilde{K}$. For any $p \in \pi^{-1}(q)$, let L be the leaf of $\mathcal{F}$ through p. Then there is a unique local isometry

$$\Phi_p : \widetilde{K} \to L, \text{ with } \Phi_p(q') = p \text{ and } \pi \circ \Phi_p : \widetilde{K} \to K$$

is the universal covering map of K.

Let $B_t^{q'}$ denote Brownian motion on on $\widetilde{K}$ with initial point q'. This is the same as the lift of Brownian motion, B^q, on K with initial point q. Then Brownian motion B_t^p on L, for any p in the fiber above q, has a version given by $\Phi_p \circ B_t^{q'}$, which is also the lift to L of B_t^q. This is because the restriction of π to any leaf of $\mathcal{F}$ is a Riemannian covering. The fact that Brownian motion along leaves can be, in this sense, "synchronized" along the I-fibers is the main feature of the Brownian motion on $(M, \mathcal{F})$ that we need here to deduce the property that if a non-trivial leafwise harmonic function existed, then it would be monotone. This observation is the content of the next lemma.

Lemma 6.1 *Suppose that the foliated I-bundle has no compact leaves other than A_0 and A_1, leaves have negative sectional curvature, and there exists a non-trivial leafwise harmonic continuous function. For any $p \in M$, let $f(p)$ be the probability that the foliated Brownian motion starting at p will converge towards the boundary leaf A_1. Then for each $q \in K$, the restriction of f to the fiber $\pi^{-1}(q)$ is a weakly monotone increasing function.*

Proof. Recall that, if a non-trivial leafwise harmonic function exists, the unique such function taking values i on A_i, $i = 0, 1$, is f. This is due to corollary 5.3. As $\mathcal{F}$ is transversely orientable, given any two points p_1, p_2 in the fiber above q and any continuous curve $B_t^{q'}$ on $\widetilde{K}$ starting at q', with q' a lift of q to $\widetilde{K}$, we have

$$\eta(p_1) < \eta(p_2) \Rightarrow \eta(\Phi_{p_1} \circ B_t^{q'}) < \eta(\Phi_{p_2} \circ B_t^{q'})$$

for all $t \geq 0$. It follows that the event $\Omega^1_{p_1}$ that $B_t^{p_1}$ limits on A_1 can be regarded as a subset of the event $\Omega^1_{p_2}$ that $B_t^{p_2}$ limits on A_1. Since the probabilities of these events are $f(p_1)$ and $f(p_2)$, respectively, we must have $f(p_1) \leq f(p_2)$. □

Proposition 6.2 *Let $(M, \mathcal{F})$ be a foliated I-bundle, $I = [0,1]$, with base manifold K, where K is a compact Riemannian manifold of negative sectional curvature. We assume that there are no compact leaves in the interior of M and that there exists a non-trivial leafwise harmonic function. Let f be the normalized such function. Then, after possibly blowing down interval-bundles in $(M, \mathcal{F})$, the restriction of f to each I-fiber is a strictly increasing function onto I.*

Proof. Let A_0 and A_1 be as in lemma 6.1, and $U = M \setminus (A_0 \cup A_1)$. We first make the following general observation. Let W be a noncompact foliated interval bundle in U. The lower and upper boundary leaves of W, denoted L_0 and L_1, respectively, are allowed to be the same. Let $p_i \in L_i$, $i = 0, 1$, be points in the same I-fiber. There is an isometry from L_0 to L_1 which sends Brownian motion in L_0 starting at p_0 to Brownian motion in L_1 starting at p_1. We consider the octopus decomposition of W. By lemma 5.2, Brownian motion $B_t^{p_0}$ in L_0 converges to either A_0 or A_1 almost surely. In particular, it escapes into the arms of W almost surely. Since the thickness of these arms converges to 0, then $B_t^{p_0}$ converges A_1 if and only if $B_t^{p_1}$ converges to A_1, and similarly for A_0. (Note that the index i of the A_i to which both $B_t^{p_0}$ and $B_t^{p_1}$ converge is a random variable, that is, a measurable function of the sample path.) Notice that this does not work in general if L_0 and L_1 are not contained in a foliated I-bundle of $\mathcal{F}$. We now lift all these sets to the universal cover $\widetilde{M}$ of M and let $\tilde{f}$ be the pull-back of f to $\widetilde{M}$. Let $\widetilde{L}_i$, A_i', denote the lifts of L_i, A_i, respectively, for $i = 0, 1$, where the $\widetilde{L}_i$ are boundary leaves of a connected lift of W. Consider the isometry $\Phi : \widetilde{L}_0 \to \widetilde{L}_1$ defined via the holonomy map along I-fibers and fix $p_i' \in \widetilde{L}_i$ such that $\Phi(p_0') = p_1'$. Denote by $S(\widetilde{L}_i)$ the set in the ideal boundary of $\widetilde{L}_i$ consisting of limit points, $B_\infty^{p_i'}$, of Brownian paths converging to A_1'. Then $S(\widetilde{L}_0)$ and $S(\widetilde{L}_1)$ are identified under the map induced by Φ on the ideal boundaries. Therefore, $\tilde{f}|_{\widetilde{L}_0}$ and $(\tilde{f}|_{\widetilde{L}_1}) \circ \Phi$ have almost surely the same boundary values at infinity and thus define the same harmonic function on $\widetilde{L}_0$. This remark clearly also applies to any pair of leaves between $\widetilde{L}_0$ and $\widetilde{L}_1$. This shows that f is constant along subsegments of I-fibers contained in W.

First assume that some leaf L in U accumulates only on A_0 and A_1. Let $\widehat{W}$ be the metric completion of $W = U \setminus L$. For every p in L, the path starting at p moving upwards along the I-fiber of p will hit L again. But L does not limit in U (that is, L is properly embedded in U), so it makes sense to consider the first hit point from p back in L. We obtain in this way a function from L to itself that is easily seen to be an isometry and is given by the holonomy map of an I-bundle structure on $\widehat{W}$. By the argument of the previous paragraph the restrictions of f to the lifts of the boundary leaves of $\widehat{W}$ are equal on endpoints of I-fibers of $\widehat{W}$. But the top and bottom boundaries of $\widehat{W}$ map to the the same leaf L. Therefore, the lifts of W cover $\widetilde{M} \setminus (A_0' \cup A_1')$. This shows that the restriction of f to each I-fiber is constant, contradicting that $f = i$ on A_i, $i = 0, 1$.

It will be assumed from now on that every leaf in U limits on U. Suppose now that there exist distinct points p_0, p_1 on the fiber I_q of q in the base manifold K such that $f(p_0) = f(p_1)$. We want to show that these points lie in the closure of a foliated I-bundle in U. Take the interval $J \subset I_q$ with endpoints p_0, p_1 to be maximal, i.e., p_0 is the lowest point in I_q such that $f(p_0) = f(p_1)$, and p_1 is the highest. Notice that J is contained in the interior of I_q because $f(0) = 0$ and $f(1) = 1$. Pass to the universal cover $\widetilde{M}$ and consider the harmonic function $g = (\tilde{f}|_{E_1}) \circ \Phi - \tilde{f}|_{E_0}$, where E_i stands for the leaf of $\widetilde{\mathcal{F}}$ through lifts p_i' of p_i on the fiber $I_{q'}$ of a lift q' of q, for $i = 0, 1$. Here, Φ is the fiber-respecting Riemannian isometry from E_0 to E_1 such that $\Phi(p_0) = p_1$. By lemma 6.1, g is a non-negative harmonic function on E_0 such that $g(p_0') = 0$. The maximum principle now implies that g is identically 0. Let J' be the interval of $I_{q'}$ with endpoints p_0', p_1'.

Now consider the returns of J to I_q under the foliation holonomy. From what has been shown, on any such interval return the function f is constant and the interval is maximal relative to this property. Therefore, the returns are either equal to J or disjoint from J. In other words, the leaves of $\mathcal{F}$ through p_0, p_1 are the boundaries of an I-bundle in U. To see this, consider the set W of leaves of $\widetilde{\mathcal{F}}$ through J. For any element γ of $\pi_1(M)$, consider $\gamma(J)$. Move it along by holonomy of $\widetilde{\mathcal{F}}$ to a subinterval J_1 of I_q. What has been shown above is that either J_1 equals J or it is disjoint from J. This shows that $\gamma(W)$ is either equal to W or disjoint from W. Hence W projects to a foliated I-bundle in M whose boundary leaves are the leaves through p_0 and p_1.

So far we have proved that whenever f takes the same value on two distinct points of any I-fiber, there is an I-bundle containing the two points such that f is constant on each fiber of it, and the I-bundle is maximal relative to this property.

Now let $\mathcal{U}$ be the union of the interiors of all these I-bundles. The complement of $\mathcal{U}$ is a closed $\mathcal{F}$-saturated set in M and its intersection with I_q is a closed subset $V \subset I_q$ invariant under holonomy of $\mathcal{F}$. V does not have isolated points: if v in V is isolated then the two open intervals in $I_q \setminus V$ abutting v would have the same value of f, contradicting the maximality property above. Hence V is a Cantor set.

We can now collapse every I-bundle to one of its boundary leaves. This operation is done at most countably many times. There is an induced collapsed foliation and induced continuous function which is harmonic on leaves of the new foliation. The new function is now clearly strictly monotone along fibers. $\square$

7 The Lipschitz property

Although we have assumed that the foliation and harmonic functions are only continuous transversely, it turns out that more regularity can be deduced in the case of foliated I-bundles. This fact will be essential in proving the main result.

Let $(M, \mathcal{F})$ be as in proposition 6.2 and suppose that a non-trivial, continuous leafwise harmonic function exists. Let f be, as above, the unique such function taking values 0 and 1 on the boundary components of M. We assume that we have done the collapsing operation of proposition 6.2 so that f is strictly increasing on I-fibers. Let $\pi : M \to K$ be the bundle map and define

$$\Psi : M \to K \times I, \quad \Psi(p) = (\pi(p), f(p)).$$

Then Ψ is a bijection. (Recall that if a fiber in M is identified with $I = [0,1]$ then the restriction of f to that fiber is a bijection from I to itself.) Since f is continuous on M and smooth along leaves, Ψ is continuous on M and smooth along leaves of $\mathcal{F}$. It can also be shown, using the strict monotonicity of f, that Ψ maps $\mathcal{F}$ to a continuous foliation, $\mathcal{F}'$, of $K \times I$ whose leaves are smooth and transverse to the fibers of the product fibration $\pi_1 : K \times I \to K$. We remark that $\mathcal{F}'$ is, like $\mathcal{F}$, a foliated bundle over the same base K, but it has the following additional property: sheets of $\mathcal{F}'$ in any foliation box of the form $\pi_1^{-1}(D)$ are graphs of harmonic functions from a sufficiently small disc D in K to $[0,1]$. In particular, if S is an interval in a fiber $I_{q_0} = \pi_1^{-1}(q_0)$ and S_q is the image of S under local holonomy of $\mathcal{F}$ from I_{q_0} to I_q, then by fixing q_0 and letting q vary, the length of S_q is a harmonic function of q, since it is the difference between two locally defined harmonic functions corresponding to two different sheets of $\mathcal{F}'$.

Similarly, the height function $\pi_2 : K \times I \to I$ is a non-trivial leafwise harmonic function for $\mathcal{F}'$. A foliated I-bundle in $K \times I$ having the property that π_2 is leafwise harmonic will be called a *harmonic foliation*, and $\mathcal{F}'$ may be viewed as a "harmonic straightening" of $(M, \mathcal{F})$. (No assumption on the curvature or topology of leaves is made in this definition.) It should be emphasized that the concept of a harmonic foliation is rather restrictive. In fact, under the fairly general assumptions of proposition 9.1, we show that a harmonic foliation is a product. This will contradict the existence of non-trivial leafwise harmonic functions under the conditions of theorem 1.2.

Lemma 7.1 *Suppose that the compact manifold K has negative sectional curvature and let $\mathcal{F}$ be a foliation of $M = K \times I$ having the properties: (i) $(M, \mathcal{F})$ is a harmonic foliation and (ii) no interior leaf is compact. Then $\mathcal{F}$ is transversely Lipschitz.*

Proof. It is convenient to pass to the universal cover $\widetilde{M} = \widetilde{K} \times I$. The leaves of the lifted foliation, $\widetilde{\mathcal{F}}$, are now isometric to $\widetilde{K}$ under the natural projection. We fix throughout the proof two points $q_1, q_2 \in \widetilde{K}$ and use the natural parameter $t \in [0, 1]$ to represent a point on the fiber $I_{q_i} = \{q_i\} \times [0, 1]$, $i = 1, 2$. The holonomy map from I_{q_1} to I_{q_2} is then given by a strictly increasing function $t \mapsto H(t)$ onto $[0, 1]$. Our goal is to show that this function is Lipschitz.

Due to (i) and (ii), Brownian motion starting at $p = (q, t) \in \widetilde{M}$ converges to A_1 with probability t. This follows from corollary 5.3 and from the fact that the height function, $h = \pi_2$, is leafwise harmonic. (Clearly, this also holds for $t = 0$ and 1.) Let L be the leaf of $\widetilde{\mathcal{F}}$ through p and let $\mathcal{A}_1(L)$ be the measurable subset of the ideal boundary of L where $h|_L$ has boundary values equal to 1. (Recall lemma 5.2, part 2.) Then

$$t = h(p) = \mu_p(\mathcal{A}_1(L)).$$

(This is due to the same lemma; μ_p is the harmonic probability measure described in section 2.) It will be convenient to be more explicit and denote $\mathcal{A}_1(q, t) := \mathcal{A}_1(L)$. Note, however, that this set only depends on the leaf L, so the following relation holds, by definition:

$$\mathcal{A}_1(q_1, t) = \mathcal{A}_1(q_2, H(t))$$

for all $t \in [0, 1]$. Using the Riemannian isometry $\pi_1|_L : L \to \widetilde{K}$ we may identify the ideal boundary of any leaf L with the ideal boundary, $S(\infty)$, of $\widetilde{K}$ and the harmonic measure μ_p with μ_q, $q = \pi_1(p)$. From now on, we regard each $\mathcal{A}_1(q, t)$ as a subset of $S(\infty)$ and write $t = \mu_q(\mathcal{A}_1(q, t))$. The same argument used to prove monotonicity of the leafwise harmonic function in lemma 6.1 also shows that $\mathcal{A}_1(q, s) \subset \mathcal{A}_1(q, t)$ whenever $s < t$. This shows that, for any fixed $q \in \widetilde{K}$,

$$t - s = \mu_q(\mathcal{A}_1(q, t) \setminus \mathcal{A}_1(q, s)).$$

Now, for any given Borel set $U \subset S(\infty)$, the function $q \mapsto \mu_q(U)$ defined on $\widetilde{K}$ is harmonic. By the Harnack inequality there exists a constant $C = C(q_1, q_2) > 0$ so that

$$\mu_{q_2}(U) \leq C\mu_{q_1}(U)$$

independent of U. It follows that

$$\begin{aligned}H(t) - H(s) &= \mu_{q_2}(\mathcal{A}_1(q_2, H(t)) \setminus \mathcal{A}_1(q_2, H(s))) \\ &= \mu_{q_2}(\mathcal{A}_1(q_1, t) \setminus \mathcal{A}_1(q_1, s)) \\ &\leq C\mu_{q_1}(\mathcal{A}_1(q_1, t) \setminus \mathcal{A}_1(q_1, s)) \\ &= C(t-s).\end{aligned}$$

Therefore, H is locally Lipschitz. Now, the Harnack inequality shows that the corresponding $C(q, q')$ is bounded for (q, q') in a compact neighborhood of any (q_1, q_2), hence $\mathcal{F}$ can be covered by foliation charts with Lipschitz transition functions. By compactness of M it follows that $\mathcal{F}$ is Lipschitz. □

8 Stationary measures

Stationary measures under foliated Brownian motion were introduced and studied in [Ga], where they were named *harmonic measures.* Since we have been using the term to designate the measures μ_p on the Poisson boundary (see section 2; this is the more traditional terminology from probability theory) we will refer to Garnett's measures as *stationary* (for the foliated Brownian motion) or harmonic *in the sense of Garnett.* See [Ca] and chapter 2 of [CC2] for a comprehensive overview of the subject.

The definition is as follows. Let $(M, \mathcal{F})$ be a foliation by Riemannian leaves and Δ the Laplace-Beltrami operator on leaves. A Borel measure m on M is harmonic in the sense of Garnett if $\Delta m = 0$. By the duality between measures and functions, this is interpreted by $\int_M \Delta\phi(x)dm(x) = 0$, for all compactly supported smooth functions, ϕ, on M. (By general measure theory, m must be a regular Borel measure. See [Ru], theorem 2.18.)

It is shown in [Ga] (see also proposition 2.4.10 of [CC2]) that m is harmonic in the sense of Garnett if and only if, on any given foliated chart $U = D \times Z$ with transversal Z, m can be disintegrated as $dm = h(q,t)d\sigma(q)d\nu(t)$, where σ is the measure on sheets induced by the Riemannian volume form, ν is a measure on Z, and $q \mapsto h(q,t)$ is a non-negative harmonic function on $D \times \{t\}$ for ν-a.e. $t \in Z$.

Proposition 8.1 (Garnett) *Let m be a harmonic probability measure on the foliated manifold $(M, \mathcal{F})$ and f a measurable, m-integrable, leafwise harmonic function on M. Then f is constant on m-a.e. leaf.*

Proof. We refer the reader to [Ga] or [CC2]. This corresponds to proposition 2.5.6 of [CC2] and the fact that leafwise harmonic functions are precisely the functions which are invariant under the diffusion semi-group, denoted D_t in [CC2]. □

In the present paper, the function f to which this proposition is applied is continuous, so the conclusion is that f is constant on any leaf in the support of any harmonic measure in the sense of Garnett. (Recall: a point p of a compact Hausdorff space lies in the support of a finite regular Borel measure if, by definition, every neighborhood of p has positive measure, so arbitrarily close to p in the support of m there are leaves on which f is constant.)

9 Harmonic foliations and stationary measures

In this section we use stationary measures to prove that if $\mathcal{F}$ is a Lipschitz harmonic foliation, then $\mathcal{F}$ is the product foliation. This is then used in the next section to prove the main theorem. The result of this section is very general in that we do not assume that leaves of $\mathcal{F}$ have negative curvature nor that $\mathcal{F}$ does not have compact leaves in the interior of M. These further conditions will be imposed in the next section.

Recall the setup of section 7: Let $M = K \times I$, $I = [0, 1]$, and let $\mathcal{F}$ be a foliated I-bundle of M. Let $h := \pi_2 : M \to I$ be the projection map. We assume that h is a leafwise harmonic function. In other words, the leaves of $\mathcal{F}$ are locally graphs of harmonic functions. This $(M, \mathcal{F})$ is called a harmonic foliation. In addition we assume that $\mathcal{F}$ is Lipschitz continuous, that is, its holonomy satisfies the Lipschitz property.

Proposition 9.1 *Let $(M, \mathcal{F})$ be a Lipschitz continuous harmonic foliation of $M = K \times I$, where K is a compact Riemannian manifold. Then $\mathcal{F}$ is the product foliation.*

Proof. Let ω denote the normalized Riemannian volume form on K, so that the total volume is 1, and let m denote the probability measure associated to the product volume form $\nu = \omega \wedge dt$ on M. We claim that m is a harmonic measure in the sense of Garnett. As remarked in section 8, to show the claim it suffices to verify that the density functions for the disintegration of m on a foliation box are harmonic on sheets.

Let $\mathcal{W}$ be a foliation box of the form $\pi_1^{-1}(D)$, where D is a small enough Riemannian ball in K with center q_0, and define the the map

$$\Phi : D \times I \to \mathcal{W} \;\text{ by }\; \Phi(q, t) = (q, \varphi(q, t)),$$

where for each fixed t, the graph of $q \mapsto \varphi(q, t)$ is the sheet of $\mathcal{F}$ in $\mathcal{W}$ through the point (q_0, t). In particular, $\varphi(q_0, t) = t$. As $\mathcal{F}$ is a foliated bundle, the restriction of π_1 to each leaf is a local isometry onto K with local inverse $q \mapsto (q, \varphi(q, t))$ for some t and q_0. So for a fixed q_0, we have that $t \mapsto \varphi(\cdot, t)$ is a one-parameter family of isometries from D to sheets of $\mathcal{W}$. Since the holonomy of $\mathcal{F}$ is Lipschitz, the function φ is jointly Lipschitz in q and t and smooth in q. Denoting by $I_q = \{q\} \times I$ the fiber $\pi_1^{-1}(q)$ above $q \in K$, note that the map $H_q(t) = \varphi(q, t)$ is the holonomy map over D from I_{q_0} to I_q. Thus Φ is a Lipschitz homeomorphism from $D \times I$ onto $\mathcal{W}$.

As Φ is Lipschitz, the pullback $\nu' = \Phi^*\nu$ is a measurable, bounded form on $D \times I$ by Rademacher's theorem on Lebesgue a.e. differentiability of Lipschitz functions. (Theorem 3.1.6 of [Fed].) As ν' and ν are top-degree forms, there is a bounded measurable function F on $D \times I$ such that

$$\nu'_{q,t} = F(q, t)\omega_q \wedge dt.$$

An elementary Jacobian determinant calculation gives that $F(q, t) = \varphi_t(q, t)$, whenever $\varphi_t(q, t)$ exists, where φ_t denotes partial differentiation with respect to t. In fact, for a.e. (q, t) and all $u \in T_qK \times \{0\} \subset T_{q,t}(K \times I)$, denoting $\tau = \frac{d}{dt}$, then $d\Phi_{q,t}u = u + c\tau$ for some scalar c and $d\Phi_{q,t}\tau = \varphi_t(q, t)\tau$, so the determinant of $d\Phi_{q,t}$ with respect to a basis adapted to the product $K \times I$ is $\varphi_t(q, t)$. Furthermore, the change of coordinates formula holds:

$$\int_M g\, dm = \int_0^1 \int_K (g \circ \Phi)(q, t)F(q, t)d\sigma(q)dt$$

for any continuous function g with compact support in $\mathcal{W}$. (This is easily derived from theorem 3.2.12 of [Fed].) Here σ is the normalized measure associated to the Riemannian volume form ω. In particular, this shows that the $F(q,t)$, when they exist, are the density functions for the disintegration formula in the foliation boxes.

Therefore, to prove the claim that m is harmonic we need to verify that $\varphi_t(q,t)$ is a harmonic function of q on D for almost every t. To see that this is the case we first define:

$$L(q,t,s) = \frac{\varphi(q,t+s) - \varphi(q,t)}{s}.$$

Note that $q \mapsto L(q,t,s)$ is a positive harmonic function for each fixed t and s. Let U denote the set of $(q,t) \in D \times I$ where the limit of L as $s \to 0$ exists. This set is easily seen to be measurable. Since φ is Lipschitz, U has full measure with respect to the product measure on $D \times I$. Now apply the standard Fubini theorem on product measure spaces to the characteristic function of U to obtain that, since U has full measure, the slice

$$U_t := U \cap (D \times \{t\})$$

has full measure for a.e. t. Therefore, for a.e. $t \in I$, $\lim_{s \to 0} L(q,t,s)$ exists (and is bounded) for a.e. $q \in D$. If for a given t the limit is 0 for a.e. q, then $\varphi_t(\cdot,t)$ agrees a.e. with a (constant) harmonic function and the claim holds. Now fix a t for which U_t has full measure in D and suppose that for some $q' \in U_t$ the limit is positive. Then the family

$$l(\cdot,t,s) = L(\cdot,t,s)/L(q',t,s)$$

of positive harmonic functions satisfies $l(q',t,s) = 1$ for each s. By the Harnack principle (see section 2), there is a subsequence $s_n \to 0$ such that $l(\cdot,t,s_n)$ converges to a positive harmonic function on U_t. Therefore $\varphi_t(\cdot,t)$ agrees a.e. in D with a positive harmonic function. This concludes the proof that m is a harmonic measure in the sense of Garnett.

We can now apply proposition 8.1 to obtain that the height function h must be constant a.e. on the support of m. But m has full support, so h is constant on leaves everywhere. Therefore, $\mathcal{F}$ is the trivial foliation. □

10 End of proof of the main theorem

We have finally obtained the desired contradiction to the existence of non-trivial harmonic functions on foliated I-bundles.

Lemma 10.1 *Let $(M,\mathcal{F})$ be a foliated I-bundle with base manifold K, where K is a compact Riemannian manifold of negative sectional curvature. Suppose that no interior leaf is compact. Then $\mathcal{F}$ does not admit continuous, non-trivial, leafwise harmonic functions.*

Proof. This is now an immediate consequence of propositions 6.2, 9.1 and lemma 7.1. Assume by way of contradiction that $(M,\mathcal{F})$ has a nontrivial leafwise harmonic function f, which we take to be normalized. Proposition 6.2 shows that after a blow down of I-bundles of $\mathcal{F}$ we can assume that f is strictly increasing along I-fibers. The map $\Psi(p)$ defined prior to lemma 7.1 transforms this into a harmonic foliation in $K \times I$. Lemma 7.1 shows that this foliation is Lipschitz. Proposition 9.1 shows that the new foliation is a product foliation, contradicting the fact that the original foliation did not have compact leaves in the interior. □

Theorem 10.2 *Let $(M, \mathcal{F})$ be a continuous foliated S^1-bundle with base manifold K, where K is a compact Riemannian manifold of negative sectional curvature. Then $\mathcal{F}$ does not admit a non-trivial, continuous, leafwise harmonic function.*

Proof. If there are no compact leaves, the theorem reduces to corollary 4.2, since a foliated S^1-bundle is $\mathbb{R}$-covered. Otherwise, let $\mathcal{K}$ be the union of all compact leaves. By a well-known theorem of Haefliger $\mathcal{K}$ is a compact set. The maximum principle implies that any leafwise harmonic function on M is leafwise constant on $\mathcal{K}$, so we may assume that $M \setminus \mathcal{K}$ is non-empty. Let U be a component of the complement of $\mathcal{K}$. Then the metric completion of U is an interval bundle with compact boundary leaves and no compact leaf in the interior. We can now apply lemma 10.1 to conclude the proof. □

Essentially the same argument shows the more general theorem 1.3. If there are no compact leaves, the result follows from corollary 4.2. Otherwise, the proof reduces to the foliated interval bundle case just as was done above for foliated circle bundles.

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